ZERO-HOPF BIFURCATION OF A $5D$ HYPERCHAOTIC QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. A zero-Hopf equilibrium of a 5-dimensional autonomous differential system is an equilibrium point for which the Jacobian matrix of the system evaluated at that equilibrium has three zero eigenvalues and a pair of purely imaginary eigenvalues.

Using the averaging theory we provide sufficient conditions for the existence of at least two families periodic orbits bifurcating from a zero-Hopf equilibrium point of a 5-dimensional hyperchaotic quadratic polynomial differential system studied in [14]. We also provide analytical expressions for the initial conditions of these two families. This zero-Hopf equilibrium point is very special because it is contained in a straight line filled with zero-Hopf equilibria.

1. Introduction and statement of the main results

Studying and analyzing the periodic orbits is a primary objective of the qualitative theory of differential equations. Researchers usually rely on numerical methods to study these diuretics due to the complexity of analytical methods. Hopf bifurcations and zero-Hopf bifurcations, which are studied through the averaging theory, are effective methods for detecting analytically periodic orbits. The literature includes many articles dealing with the study of zero-Hopf bifurcations in dimension 3 and 4 see for instance [1, 3, 4, 9, 10, 11, 13] and [5, 8] respectively.

A way of finding periodic orbits is through a *zero-Hopf bifurcation*, which in dimension 5 takes place when a periodic orbit bifurcates from an equilibrium point whose linear part has eigenvalues $\lambda_k = 0$ for $k = 1, 2, 3$ and $\pm\mu i$ with $\mu > 0$, i.e. from a zero-Hopf equilibrium. Currently, there are few results on zero-Hopf bifurcations in systems of dimension $n \geq 4$.

In this paper we will study the existence of periodic orbits bifurcating from the zero-Hopf equilibria of the following $5D$ hyperchaotic quadratic polynomial differential system

$$
\begin{aligned}
\dot{x} &= a(y - x),\\
\dot{y} &= (c - a)x - xz + cy + w,\\
\dot{z} &= xy - bz,\\
\dot{v} &= mw,\\
\dot{w} &= -y - hv,
\end{aligned}
\tag{1}
$$

where a, b, c, d, m, h are real parameters of system (1).

System (1) was analyzed by Niu in [14], who explored its synchronization with a $3D$ chaotic system with variable coefficients. More precisely, for studying the chaotic synchronization or hyperchaotic synchronization of heterogeneous systems, as shown in the literature [7, 15], structural compensation is still a relatively common design. The function of structural compensation is to compensate for the structural differences between the response system and the

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drive system so that the two systems can be synchronized under the action of the synchronous controller. It is worth studying how to design a suitable structural compensator. The author of the paper [14] proposed the center translation synchronization method of [16] and arrived to the polynomial differential system (1).

The origin of coordinates $O = (0, 0, 0, 0, 0)$ is an equilibrium point of system (1) for all values of the parameters. But system (1) has two additional equilbria p_+ and p_- when the parameters satisfy that $b(2c - a) > 0$ and $h \neq 0$, namely,

$$p_\pm = \left(\pm\sqrt{b(2c-a)}, \pm\sqrt{b(2c-a)}, 2c - a, \mp\frac{\sqrt{b(2c-a)}}{h}, 0 \right).$$

Moreover, system (1) with $b = 0$ has the straight line $(0, 0, z, 0, 0)$ for all $z \in \mathbb{R}$ filled with equilibria.

We note that the differential system (1) is invariant under the symmetry $(x, y, z, v, w) \mapsto (-x, -y, z, -v, -w)$. So if $(x(t), y(t), z(t), v(t), w(t))$ is a solution of system (1), then $(-x(t), -y(t), z(t), -v(t), -w(t))$ is also a solution, eventually they can coincide.

Our main results related with the zero-Hopf equilibria and the zero-Hopf bifurcations of the differential system (1) are the following.

Proposition 1. *All the equilibria of the straight line $(0, 0, z, 0, 0)$ are zero-Hopf equilibria of the $5D$ hyperchaotic polynomial differential system (1) when $a = b = c = 0$ and $hm > -1$. The eigenvalues at these zero-Hopf equilibria are $0, 0, 0$ and $\pm\sqrt{1 + hm}\, i$.*

Theorem 2. *Let $a = \varepsilon a_1$, $b = \varepsilon b_1$, $c = \varepsilon c_1$ and $hm > -1$, being ε a sufficiently small parameter. Then the $5D$ hyperchaotic quadratic polynomial differential systems (1) has a zero-Hopf bifurcation at the equilibrium point O when $\varepsilon = 0$, and using the averaging theory of first order at least two periodic orbits, $(x_k(t,\varepsilon), y_k(t,\varepsilon), z_k(t,\varepsilon), v_k(t,\varepsilon), w_k(t,\varepsilon))$ for $k = 1, 2$, bifurcate from this equilibrium when $\varepsilon = 0$ if $b_1(2c_1 - a_1) > 0$ and $a_1 c_1 hm \neq 0$. These two periodic orbits at time $t = 0$ satisfy*

$$(x_1(0,\varepsilon), y_1(0,\varepsilon), z_1(0,\varepsilon), v_1(0,\varepsilon), w_1(0,\varepsilon)) = \left(\varepsilon w_0, -\varepsilon\frac{hz_0}{1+hm}, \varepsilon v_0, \varepsilon\frac{z_0}{1+hm}, 0 \right) + O(\varepsilon^2),$$

and

$$(x_2(0,\varepsilon), y_2(0,\varepsilon), z_2(0,\varepsilon), v_2(0,\varepsilon), w_2(0,\varepsilon)) = \left(-\varepsilon w_0, -\varepsilon\frac{hz_0}{1+hm}, -\varepsilon v_0, \varepsilon\frac{z_0}{1+hm}, 0 \right) + O(\varepsilon^2),$$

where $z_0 = -\dfrac{(hm+1)\sqrt{b_1(2c_1 - a_1)}}{h}$, $v_0 = 2c_1 - a_1$ and $w_0 = \sqrt{b_1(2c_1 - a_1)}$. Moreover, if $a_1 c_1 hm < 0$ these two periodic orbits are unstable.

Proposition 1 and Theorem 2 are proved in section 3. In section 2 we present the results on the averaging theory that we need for proving Theorem 2.

Note that when the zero-Hopf bifurcation described in Theorem 2 takes place at the equilibrium O for $a = b = c = 0$ and $hm > -1$, this equilibrium is not isolated because it is contained in the straight line $(0, 0, z, 0, 0)$ for all $z \in \mathbb{R}$ filled with equilibria,

We also have studied if from the zero-Hopf equilibria $(0, 0, z, 0, 0)$ with $z \neq 0$ of the differential system (1) with $a = b = c = 0$ and $hm > -1$ bifurcates some periodic orbit, and unfortunately the averaging theory used for proving Theorem 2 does not provide any information about the existence or non-existence of such periodic orbits.

2. The averaging theory of first order

We recall the averaging theory of first order for studying the periodic orbits of nonlinear differential systems. The averaging theory of higher order for studying the periodic orbits was developed in [12]. For a proof of the averaging theory of first order, one can refer to Theorems 11.5 and 11.6 in [18], see improvements in [2]. For a broader perspective on averaging theory see the book [17].

We deal with the following two initial value problems

$$\dot{\mathbf{x}} = \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{2}$$

and

$$\dot{\mathbf{y}} = \varepsilon f(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \quad \text{with } f(\mathbf{y}) = \frac{1}{T}\int_0^T F_1(t,\mathbf{y})dt, \tag{3}$$

where $\mathbf{x}, \mathbf{x}_0, \mathbf{y} \in D$, being D an open subset of $\mathbb{R}^n$, ε is a sufficiently small parameter and $t \geq 0$. Here $F_1(t,\mathbf{x})$, $F_2(t,\mathbf{x},\varepsilon)$ be periodic functions of period T in the variable t. The function $f_1(\mathbf{y})$ is known as the averaged function of system (2).

Theorem 3. *Assume:*

(i) *The functions F_1, $D_{\mathbf{x}}F_1$, $D_{\mathbf{xx}}F_1$, F_2 and $D_{\mathbf{xx}}F_2$ are defined, continuous and bounded by a constant independent of ε in $[0,\infty) \times D$ and $\varepsilon \in (0, \varepsilon_0]$ for some $\varepsilon_0 > 0$.*
(ii) *F_1 and F_2 are T-periodic in t (T independent of ε).*
(iii) *$\mathbf{y}(t)$ belongs to D on the interval of time $[0, 1/\varepsilon]$.*
Then the following statements hold,
(a) *If $\mathbf{p}$ is an equilibrium point of the differential system (3) and*

$$\det\left(D_{\mathbf{y}} f_1(\mathbf{p})\right) \neq 0, \tag{4}$$

then there exists a T-periodic solution $\mathbf{x}(t,\varepsilon)$ of system (2) such that $\mathbf{x}(0,\varepsilon) \to \mathbf{p}$ as $\varepsilon \to 0$.
(b) *The stability or instability of the limit cycle $\mathbf{x}(t,\varepsilon)$ is given by the stability or instability of the equilibrium point $\mathbf{p}$ of the differential system (3). In fact, the equilibrium point $\mathbf{p}$ has the stability behavior of the Poincaré map associated to the limit cycle $\mathbf{x}(t,\varepsilon)$.*

3. Proof of the results

Proof of Proposition 1. The Jacobian matrix of $5D$ hyperchaotic polynomial differential systems (1) at the equilibrium point O is

$$\mathrm{J} = \begin{pmatrix} -a & a & 0 & 0 & 0 \\ c-a & c & 0 & 0 & 1 \\ 0 & 0 & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & m \\ 0 & -1 & 0 & -h & 0 \end{pmatrix}.$$

The characteristic polynomial $P(\lambda)$ for the matrix J is

$$-(b+\lambda)\left(ahm(a-2c) + (ahm + a - chm)\lambda + \left(a^2 - 2ac + hm + 1\right)\lambda^2 + (a-c)\lambda^3 + \lambda^4\right).$$

In order that the equilibrium point O be a zero-Hopf equilibrium the polynomial $P(\lambda)$ must be of the form $\lambda^3(\lambda^2+\omega^2)$ with $\omega \neq 0$. Then we obtain $a = b = c = 0$, $hm > -1$ and $\omega = \sqrt{1+hm}$.

If we compute the characteristic polynomial of the Jacobian matrix of the differential system (1) at the equilibrium points p_- and p_+ and force that such polynomials be the polynomial $\lambda^3(\lambda^2+\omega^2)$, we again obtain that $a=b=c=0$, $hm>-1$ and $\omega=\sqrt{1+hm}$. But if $a=b=c=0$ the equilibrium points p_- and p_+ become the equilibrium O. Hence the equilibrium points p_- and p_+ never are zero-Hopf equilibria.

If we compute the characteristic polynomial of the Jacobian matrix of the differential system (1) with $b=0$ at the equilibrium point $(0,0,k,0,0)$ with $k\in\mathbb{R}$ and force that this polynomial be the polynomial $\lambda^3(\lambda^2+\omega^2)$, we again obtain that $a=c=0$, $hm>-1$ and $\omega=\sqrt{1+hm}$. Hence all the equilibria of the straight line $(0,0,z,0,0)$ with $z\in\mathbb{R}$ are zero-Hopf equilibria when $a=b=c=0$ and $hm>-1$.

An easy computation shows that the eigenvalues at the equilibria $(0,0,z,0,0)$ for all $z\in\mathbb{R}$ are $0,0,0$ and $\pm\sqrt{1+hm}\,i$. This completes the proof of the proposition. $\square$

Proof of Theorem 2. Take $a=\varepsilon a_1$, $b=\varepsilon b_1$, $c=\varepsilon c_1$ and $hm>-1$, where ε is sufficiently small. In order to apply the averaging theory to the differential system (1) we must write system (1) in the normal form (2) of the averaging theory. For doing this we start writting the linear part of system (1) at the origin O in its real Jordan normal form, i.e. as the matrix

$$\begin{pmatrix} 0 & -\sqrt{1+hm} & 0 & 0 & 0\\ \sqrt{1+hm} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This is done changing to the coordinates (X,Y,Z,V,W) through

$$x=W,\quad y=\frac{Y\sqrt{1+hm}-hZ}{1+hm},\quad z=V,\quad v=\frac{mY\sqrt{1+hm}+Z}{1+hm},\quad w=X. \tag{5}$$

Then system (1) becomes

$$\begin{aligned}
\dot X &= -\sqrt{1+hm}Y,\\
\dot Y &= \frac{(1+hm)X-VW}{\sqrt{1+hm}}+\varepsilon\frac{\left(c_1\sqrt{1+hm}Y-c_1hZ-(a_1-c_1)(1+hm)W\right)}{(1+hm)^{3/2}},\\
\dot Z &= mVW+\varepsilon m\left(a_1W+c_1\left(\frac{hZ}{1+hm}-\frac{Y}{\sqrt{1+hm}}-W\right)\right),\\
\dot V &= \frac{\left(Y\sqrt{1+hm}-hZ\right)W}{1+hm}-\varepsilon b_1V,\\
\dot W &= \varepsilon a_1\left(-\frac{hZ}{1+hm}+\frac{Y}{\sqrt{1+hm}}-W\right).
\end{aligned}\tag{6}$$

Now we change the variables X and Y by the variables R and θ through $X=R\cos\theta$, $Y=R\sin\theta$, and after we rescale the variables as follows $(R,Z,V,W)=(\varepsilon r,\varepsilon\bar z,\varepsilon\bar v,\varepsilon\bar w)$, finally in the new variables system (6) writes

$$
\begin{aligned}
\dot r &= \varepsilon \frac{\sin\theta}{(1+hm)^{3/2}}\left(c_1\sqrt{1+hm}\, r\sin\theta - c_1 h\bar z - (1+hm)(a_1-c_1+\bar v)\bar w\right) + O(\varepsilon^2),\\
\dot\theta &= \sqrt{1+hm} - \varepsilon \frac{\cos\theta\left((1+hm)(a_1-c_1+\bar v)\bar w - c_1\sqrt{1+hm}\, r\sin\theta + c_1 h\bar z\right)}{r(1+hm)^{3/2}} + O(\varepsilon^2),\\
\dot{\bar z} &= \varepsilon m\left((a_1-c_1+\bar v)\bar w - \frac{c_1 r\sin\theta}{\sqrt{1+hm}} + \frac{c_1 h\bar z}{1+hm}\right) + O(\varepsilon^2),\\
\dot{\bar v} &= -\varepsilon \frac{\left(b_1(1+hm)\bar v - \sqrt{1+hm}\, r\bar w\sin\theta + h\bar z\bar w\right)}{1+hm} + O(\varepsilon^2),\\
\dot{\bar w} &= \varepsilon a_1\left(\frac{r\sin\theta}{\sqrt{1+hm}} - \frac{h\bar z}{1+hm} - \bar w\right) + O(\varepsilon^2).
\end{aligned}
\tag{7}
$$

Now we pass from the independent variable t to the new independent variable θ and we obtain the differential system

$$
\begin{aligned}
r' &= \varepsilon \frac{\sin\theta}{(1+hm)^2}\left(c_1\sqrt{1+hm}\, r\sin\theta - c_1 h\bar z - (1+hm)(a_1-c_1+\bar v)\bar w\right) + O(\varepsilon^2),\\
\bar z' &= \varepsilon \frac{m}{\sqrt{1+hm}}\left((a_1-c_1+\bar v)\bar w - \frac{c_1 r\sin\theta}{\sqrt{1+hm}} + \frac{c_1 h\bar z}{1+hm}\right) + O(\varepsilon^2),\\
\bar v' &= -\varepsilon \frac{\left(b_1(1+hm)\bar v - \sqrt{1+hm}\, r\bar w\sin\theta + h\bar z\bar w\right)}{(1+hm)^{3/2}} + O(\varepsilon^2),\\
\bar w' &= \varepsilon \frac{a_1}{\sqrt{1+hm}}\left(\frac{r\sin\theta}{\sqrt{1+hm}} - \frac{h\bar z}{1+hm} - \bar w\right) + O(\varepsilon^2),
\end{aligned}
\tag{8}
$$

where the prime denotes derivative with respect to θ.

Note that differential system (8) is written in the normal form (2) for applying the averaging theory. Using the notation of Theorem 3, we have that $t=\theta$, $\mathbf{x}=(r,\bar z,\bar v,\bar w)$, $T=2\pi$ and $n=4$. Since all the hypotheses of Theorem 3 are satisfied we can apply this theorem to system (8). Computing the first order averaged function $f(r,\bar z,\bar v,\bar w)$ defined in (3) with components (f_1,f_2,f_3,f_4) of the differential system (8) we obtain

$$
\begin{aligned}
f_1 &= \frac{c_1 r}{2(1+hm)^{3/2}},\\
f_2 &= \frac{m((1+hm)(a_1-c_1+v)w + c_1 hz)}{(1+hm)^{3/2}},\\
f_3 &= -\frac{b_1(1+hm)v + hzw}{(1+hm)^{3/2}},\\
f_4 &= -\frac{a_1(hz+(1+hm)w)}{(1+hm)^{3/2}}.
\end{aligned}
$$

Solving the system $f_1=f_2=f_3=f_4=0$ where $r\ge 0$, we obtain the following three solutions for $(r,\bar z,\bar v,\bar w)$:

$$
S_1=(0,0,0,0),\quad S_2=(0,z_0,v_0,w_0),\quad S_3=(0,-z_0,v_0,-w_0),
$$

where $z_0 = -\dfrac{(hm+1)\sqrt{b_1(2c_1-a_1)}}{h}$, $v_0=2c_1-a_1$ and $w_0=\sqrt{b_1(2c_1-a_1)}$.

The solution S_1 corresponds to the equilibrium point O and consequently it does not provide any periodic orbit. But the other two solutions provide two isolated periodic orbits $(r_k(\theta,\varepsilon),\bar z_k(\theta,\varepsilon),\bar v_k(\theta,\varepsilon),\bar w_k(\theta,\varepsilon))$ for $k=1,2$, i.e. two limit cycles, for the differential system (8) because for both the determinant (4) is

$$\det=\frac{a_1b_1c_1hm(a_1-2c_1)}{(1+hm)^4}\neq 0,$$

by assumptions.

Also by assumptions $b_1(a_1-2c_1)>0$, so if $a_1c_1hm<0$ then the determinant $\det<0$, and consequently some of the four eigenvalues of the Jacobian matrix of the function (f_1,f_2,f_3,f_4) evaluated at the points S_2 and S_3 is negative. Hence, by the statement (b) of Theorem 3 it follows that these two periodic orbits are unstable if $a_1c_1hm<0$.

Again by Theorem 3 these two limit cycles satisfy

$$(r_1(0,\varepsilon),\bar z_1(0,\varepsilon),\bar v_1(0,\varepsilon),\bar w_1(0,\varepsilon))=(0,z_0,v_0,w_0)+O(\varepsilon),$$
$$(r_2(0,\varepsilon),\bar z_2(0,\varepsilon),\bar v_2(0,\varepsilon),\bar w_2(0,\varepsilon))=(0,-z_0,v_0,-w_0)+O(\varepsilon).$$

These two limit cycles in the differential system (7) are the limit cycles $(r_k(t,\varepsilon),\theta_k(t,\varepsilon),\bar z_k(t,\varepsilon),$ $\bar v_k(t,\varepsilon),\bar w_k(t,\varepsilon))$ for $k=1,2$ such that

$$(r_1(0,\varepsilon),\theta_1(0,\varepsilon),\bar z_1(0,\varepsilon),\bar v_1(0,\varepsilon),\bar w_1(0,\varepsilon))=(0,\sqrt{1+hm}\,t,z_0,v_0,w_0)+O(\varepsilon),$$
$$(r_2(0,\varepsilon),\theta_2(0,\varepsilon),\bar z_2(0,\varepsilon),\bar v_2(0,\varepsilon),\bar w_2(0,\varepsilon))=(0,\sqrt{1+hm}\,t-z_0,v_0,-w_0)+O(\varepsilon).$$

And in the differential system (6) are the limit cycles $(X_k(t,\varepsilon),Y_k(t,\varepsilon),Z_k(t,\varepsilon),V_k(t,\varepsilon),W_k(t,\varepsilon))$ for $k=1,2$ such that

$$(X_1(t,\varepsilon),Y_1(t,\varepsilon),Z_1(t,\varepsilon),V_1(t,\varepsilon),W_1(t,\varepsilon))=(0,0,\varepsilon z_0,\varepsilon v_0,\varepsilon w_0)+O(\varepsilon^2),$$
$$(X_2(t,\varepsilon),Y_2(t,\varepsilon),Z_2(t,\varepsilon),V_2(t,\varepsilon),W_2(t,\varepsilon))=(0,0,-\varepsilon z_0,\varepsilon v_0,-\varepsilon w_0)+O(\varepsilon^2).$$

Finally, taking into account the change of variables (5) we obtain the two limit cycles $(x_k(t,\varepsilon),$ $y_k(t,\varepsilon),z_k(t,\varepsilon),v_k(t,\varepsilon),w_k(t,\varepsilon))$ for $k=1,2$ of the differential system (1) satisfying

$$(x_1(0,\varepsilon),y_1(0,\varepsilon),z_1(0,\varepsilon),v_1(0,\varepsilon),w_1(0,\varepsilon))=\left(\varepsilon w_0,-\varepsilon\frac{hz_0}{1+hm},\varepsilon v_0,\varepsilon\frac{z_0}{1+hm},0\right)+O(\varepsilon^2),$$
$$(x_2(0,\varepsilon),y_2(0,\varepsilon),z_2(0,\varepsilon),v_2(0,\varepsilon),w_2(0,\varepsilon))=\left(-\varepsilon w_0,-\varepsilon\frac{hz_0}{1+hm},-\varepsilon v_0,\varepsilon\frac{z_0}{1+hm},0\right)+O(\varepsilon^2).$$

From these last expressions it is clear that these two limit cycles bifurcate from the equilibrium point O when $\varepsilon=0$. This completes the proof of Theorem 2. □

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