ARTICLE TEMPLATE

Tracking control for a class of fractional order uncertain systems with time-delay based on composite nonlinear feedback control

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ABSTRACT
This paper deals with the tracking control problem of a class of fractional-order uncertain systems with time delays. In order to handle the effects brought by the uncertainties, external disturbances, time-delay terms and to overcome the obstacles caused by inputs saturation, the tracking controller, which consist of linear control law, nonlinear law and robust control law proposed in this paper, is designed by combining the composite nonlinear feedback control method and the properties of fractional order operator. Furthermore, the validation of this tracking controller is proved.

KEYWORDS
Fractional-order uncertain systems; Composite nonlinear feedback controller; Saturation constraints

1. Introduction

There exist many literatures on fractional calculus and related topics [1, 2, 3, 4, 5, 6, 7, 8], such as Podlubny [2] talked about several classical definitions of fractional order operators; Miller [5] introduced the general theory of fractional differential equations; a new fractional derivatives with nonlocal and non-singular kernel due to Atangana and Baleanu [8], to name but a few. In recent years, relying on the fact that many complex phenomenon can be simplified and accurately described by fractional-order operators, fractional-order systems have attracted great attention in applied sciences [9, 10, 11].. The control problem is one of the important issues in theory and applications of fractional order systems. Recently, varieties of fractional-order control methods have been designed, such as sliding mode control [12, 13], adaptive control[14], feedback control [15] and so on. It is mentioned that the sliding mode control method can effectively

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ensure the stability and robustness of nonlinear fractional order system, alternatively, it can switch the motion to the sliding mode surface through the switching control law, so as to ensure rapid response and robustness. In addition, the combinations of several controllers are effective ways to achieve better control effects by taking the advantages of different control methods [16, 17, 18, 19, 20, 21]. However, to our best knowledge, there are hardly exists result on the tracking control of fractional-order systems based on the composite nonlinear feedback (CNF) control method, particularly for systems with time delays and actuator saturation constraints. On the other hand, due to the presence of uncertainties and external disturbances in the system, it is necessary to identify unknown nonlinear terms which should be compensated in the process of designing the controller. Furthermore, the time delays bring some obstacles in designing the controller and proving the stability.

The systems with time delays are basic mathematical models to describe the practical problems, for example, chemical reaction, mechanical vibration, power system, and so on, for more detailed, one can refer to Ref [22]. When the control problems for systems with time delays are considered. The time delays lead to the complex of designing control and the proof for the system controlled, for more detailed, see [23, 24, 25, 26]. In addition, the phenomenon of actuator saturation usually happens in systems controlled. Usually, the input saturations restrict the system's performance, which result in the inaccuracies and instabilities of the system considered. To deal with control problems for the time-delay system with actuator saturation, many control methods have been developed [27, 28, 29]. In Ref[30], a class of linear systems with input saturation constraints and time delay is studied, and Lyapunov-Razumihkin and Lyapunov-Krasovskii functional approach are used to analyze the domain of attraction problem and stability problem of the system. In [31], a state feedback controller design method was proposed for a class of uncertain discrete time-delay systems with control input saturation and bounded external disturbances, which guarantee the trajectories of system converge to the desired state.

In the above control methods, most of the control inputs depend on the sign function, which results in that the control law is not smooth. In order to improve the transient performance of the tracking ability of the closed-loop system, the composite nonlinear feedback control method was established in [32], and developed by Mobayen and Tchier [33], Chen et al [34], Lin et al [35], He et al [36] and so on. CNF control method is often used to deal with tracking control problems of systems with input saturation, and it can improve the transient performance of the closed-loop system, while maintain a small overshoot or even no overshoot. Jafari et al [37] designed a CNF controller based on disturbance observer, which can effectively guarantee the tracking performance of the system. Based on CNF control method, a discrete integral sliding mode controller which can produce the superior transient performance was proposed by Mondal S. et al [38]. In Ref [39], employing CNF control method, Jafari et al considered the control problem for the system with a singular time delay. In term of CNF control method, a novel controller for nonlinear time-delay systems with saturation constraints was given by Ghaffari et al [40]. For more detailed, one can refer to [41, 42, 43] and the references therein. It must be mentioned that most investigations which considered control problem for differential systems by CNF control method were focused on the integer order differential systems with time delay, it is necessary to develop composite nonlinear feedback control to deal with the control problem for fractional-order systems.

Relying on CNF control methods, this paper considers the control problems for fractional-order uncertain systems with time delay and external disturbances, the rest of paper is organized as follows. In Section 2, we describe the fractional-order system investigated in this paper. Section 3 is devoted to give main results and the associated proofs.

2. Preliminaries and system formulation

The following are the definitions of Caputo-fractional order derivative adopted in this paper.

Definition 2.1. [2] For a continuous function $x(t) : [0, \infty) \to R$, the Caputo-type fractional order derivative with the order α of the function $x(t)$ is defined as

$$ {}_0^C D_t^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} x'(s) ds, 0 < \alpha < 1. $$

Definition 2.2. [2] The Caputo-type fractional integral with the order α of function $x(t)$ is defined as

$$ {}_0 I_t^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, 0 < \alpha < 1. $$

Some properties of fractional calculus operators are introduced as follows.

Proposition 2.3. *[16] Let $x \in C^k[a, b]$ for some $a < b$ and some $k \in N$. Moreover, let $n, \varepsilon > 0$ such that there exists some $\ell \in N$ with $\ell \leq k$ and $n, n+\varepsilon \in [\ell-1, \ell]$. Then,*

$$ {}_0^C D_t^{\varepsilon} ({}_0^C D_t^{n} x(t)) = {}_0^C D_t^{\varepsilon+n} x(t). $$

Proposition 2.4. *[2] If the Caputo fractional differential ${}_0^C D_t^{\alpha} x(t)$ is integrable, then*

$$ {}_0 I_t^{\alpha} ({}_0^C D_t^{\alpha} x(t)) = x(t) - x(0), $$

if the function $x(t) \in C^1[0, t]$, and $0 < \alpha < 1$.

Consider the following multi-input and multi-output fractional-order uncertain system with actuator saturation

$$ \begin{cases} {}_0^C D_t^{\alpha} x(t) = (A + \Delta A(\nu(t)))x(t) + \bar{A}(\varsigma(t))x(t-\tau(t)) + (B \\ \qquad\qquad + \Delta B(\sigma(t)))sat(u(t)) + D(\theta(t)), \\ y(t) = Cx(t), 0 < t < +\infty, \end{cases} \tag{1} $$

where $x(t) \in R^n$, $y(t) \in R^m$, $m < n$ and $u(t) \in R^n$ are the system state vector, the system output vector and the control input vector respectively. The matrix A denotes the system matrix, B is the input matrix and C represents the output matrix, they are both the constant matrices with the appropriate dimensions. $\tau(t) \in R^+$ is the time delay. The terms $\Delta A(\cdot)$ and $\Delta B(\cdot)$ represent the uncertainties of the system, and $D(\cdot)$ denotes the perturbation, the uncertain terms $\nu(\cdot) : R^+ \to \mathbb{D}$, $\sigma(\cdot) : R^+ \to \mathbb{D}$ and $\theta(\cdot) : R^+ \to \mathbb{D}$ are Lebesgue measurable functions, where $\mathbb{D}$ is a compact bounded set.

The control input vector is constrained by a saturation function $sat : R^n \to R^n$

with the following form

$$sat(u(t)) = \begin{bmatrix} sat(u_1(t)) \\ sat(u_2(t)) \\ \vdots \\ sat(u_n(t)) \end{bmatrix}, \tag{2}$$

where the operator

$$sat(u_i(t)) = sign(u_i(t)) min(|u_i|, \bar{u}_i), i = 1, 2, \cdots, n, \tag{3}$$

and $\bar{u}_i$ represents the saturation level of the i-th control channel.

The objective in this paper is to derive the composite controller $u(t)$, which leads to the output vector $y(t)$ of the system (1) can track the output vector $y_r(t)$ of the reference system rapidly and smoothly. The reference system is defined as following

$$\begin{cases} {}_0^C D_t^\alpha x_r(t) = A_r x_r(t), \\ y_r(t) = C_r x_r(t), \end{cases} \tag{4}$$

where $A_r \in R^{n\times n}$ and $C_r \in R^{n\times n}$ are both constant matrices. $x_r(t) \in R^n$ denotes the reference state vector and $y_r(t) \in R^m$ is the reference output vector. For the purposes of the tracking control, it is required that there exists a constant $d > 0$ such that $||x_r(t)|| \leqslant d$ for all $t \geqslant 0$.

It is turned to list some hypothesises about the system (1) and system (4).

Hypothesis 1. There exist two constant matrices G and H which satisfy

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} GA_r \\ C_r \end{bmatrix}. \tag{5}$$

Moreover, for any positive-definite matrix $Q \in R^{n\times n}$, there exists an unique positive-definite matrix $P \in R^{n\times n}$ satisfying the following Riccati algebraic equation [44]

$$A^T P + PA - \eta PBB^T P = -Q. \tag{6}$$

Hypothesis 2. The fractional derivative of the unknown time delay $\tau(t)$ is bounded, which means there is a positive constant ϑ such that $|{}_0^C D_t^\alpha \tau| \leqslant \vartheta$. Furthermore, suppose $\vartheta < 1$.

Hypothesis 3. The matrices $\Delta A(\nu(t))$, $\Delta B(\sigma(t))$ and $D(\theta(t))$ are matched, and there exist continuous and bounded functions $N_1(\cdot)$, $N_2(\cdot)$ and $N_3(\cdot)$ with the boundary

$$\begin{aligned} \rho_1 &= \max_{\nu\in\mathbb{D}} \|N_1(\nu)\|, \\ \rho_2 &= \max_{\sigma\in\mathbb{D}} \|N_2(\sigma)\|, \\ \rho_3 &= \max_{\theta\in\mathbb{D}} \|N_3(\theta)\|, \end{aligned} \tag{7}$$

such that

$$\begin{aligned} \Delta A(\nu(t)) &= BN_1(\nu), \\ \Delta B(\sigma(t)) &= BN_2(\sigma), \\ D(\theta(t)) &= BN_3(\theta). \end{aligned} \tag{8}$$

Moreover, assume the time-delay matrix $\bar{A}$ is matched and

$$\bar{A}(\varsigma) = B\bar{N}. \tag{9}$$

Hypothesis 4. The pair $\{A, B\}$ from the system (1) is completely controllable.

The next lemma is very important in deriving the main results of this paper.

Lemma 2.5. *[45](Schur Complement) The following LMI condition*

$$\begin{bmatrix} F_{11}(t) & F_{12}(t) \\ F_{21}(t) & F_{22}(t) \end{bmatrix} < 0 \tag{10}$$

holds if and only if

$$\begin{cases} F_{22}(t) < 0, \\ F_{11}(t) - F_{12}(t)F_{22}^{-1}(t)F_{21}^{T}(t) < 0, \end{cases}$$

or is equivalent to

$$\begin{cases} F_{11}(t) < 0, \\ F_{22}(t) - F_{21}(t)F_{11}^{-1}(t)F_{12}^{T}(t) < 0, \end{cases}$$

where $F_{11}(t) = F_{11}^{T}(t)$, $F_{12}(t) = F_{21}^{T}(t)$ *and* $F_{22}(t) = F_{22}^{T}(t)$.

3. Main results

This section is devoted to obtain the main results and the proof associated. Firstly, we transform the system (1) to the error system.

3.1. Model transformation and associated stability results

Consider the following tracking error vector $e(t)$ and the auxiliary state vector defined by

$$e(t) = y(t) - y_r(t), \tag{11}$$

and

$$\tilde{x}(t) = x(t) - Gx_r(t), \tag{12}$$

where the matrix G satisfies the **Hypothesis 1**. Thus, combining the system (1) with the reference system (4) gives

$$e(t) = C(x(t) - Gx_r(t)) = C\tilde{x}(t), \tag{13}$$

then

$$\|e(t)\| = \|C\tilde{x}(t)\| \leqslant \|C\|\|\tilde{x}(t)\|, \tag{14}$$

which implies that

$$\lim_{t\to+\infty}\|e(t)\| \leqslant \lim_{t\to+\infty}\|\tilde{x}(t)\|.$$

Thus, we obtain $\lim_{t\to+\infty}\|e(t)\| = 0$ when $\lim_{t\to+\infty}\|\tilde{x}(t)\| = 0$, which means that $\|\tilde{x}(t)\| \to 0$ with $t \to \infty$ can guarantee the output $y(t)$ can be forced to track the reference output $y_r(t)$ asymptotically.

The following Lemmas and Definitions are very important to obtain the main results in this paper.

Lemma 3.1. *[46] Suppose $x(t)$ is continuously differentiable function, then, for any time variable $t \geqslant 0$, the following inequality holds*

$$\frac{1}{2}{}_0^C D_t^{\alpha} x^2(t) \leqslant x(t)({}_0^C D_t^{\alpha} x(t)), 0 < \alpha < 1.$$

Lemma 3.2. *[47] Let $x(t)$ be a vector and $x^T(t)Px(t)$ is continuously differentiable function for any symmetric matrix P, then, for each time $t \geqslant 0$, the following can be obtained.*

$$\frac{1}{2}{}_0^C D_t^{\alpha}(x^T(t)Px(t)) \leqslant x^T(t)P({}_0^C D_t^{\alpha} x(t)), \forall \alpha \in (0,1], \forall t \geqslant 0,$$

Definition 3.3. [48] If the continuous function $\alpha(\cdot) : [0,t) \to [0,\infty)$ is strictly increasing and $\alpha(0) = 0$, then, it belongs to $K-$class function.

Lemma 3.4 (Fractional Order Mittag-Leffer asymptotical stability)**.** *[49] Let $x = 0$ be an equilibrium point of the fractional system (1). Assume that there exist a Lyapunov function $V(x(t))$ and $K-$class functions $\alpha_i(\cdot)(i = 1,2,3)$ satisfying*

$$\alpha_1(\|x(t)\|) \leqslant V(x(t)) \leqslant \alpha_2(\|x(t)\|),$$

$${}_0^C D_t^{\alpha} V(x(t)) \leqslant -\alpha_3(\|x(t)\|),$$

where $0 < \alpha \leqslant 1$. Then, the equilibrium point of system (1) is asymptotically stable.

Lemma 3.5 (Integer-order Barbalat's Lemma)**.** *[50] If $\eta : R \to R$ is a uniformly continuous function for $t \geqslant 0$ and $\lim_{t\to\infty} \int_0^t \eta(\omega)d\omega$, $0 < q < 1$ exists and is finite, then $\lim_{t\to\infty} \eta(t) = 0$.*

3.2. The design of composite nonlinear tracking control

The objective in this part is to design a tracking control law based on the CNF control approach without large overshoot and unfavorable actuator saturation effect.

The process of the controller design can be divided into the following four steps.

Step 1: The design of a linear state feedback controller.
Step 2: The design of a nonlinear feedback controller,
Step 3: The design of a robust tracking controller.

Step 4: The design for the CNF controller needed.

The exact process is as following.
Step 1: The linear feedback controller is designed as

$$u_L(t) = Fx(t) + (H - FG)x_r(t) \qquad (15)$$
$$= F\tilde{x}(t) + Hx_r(t),$$

where F represents a gain matrix which is determined later. The linear part can ensure the closed-loop system possesses the properties of fast response and enough small damping ratio.
Step 2:) The nonlinear feedback controller is expressed as

$$u_N(t) = \mu(t)B^T P\tilde{x}(t), \qquad (16)$$

where P is a positive definite matrix, and

$$\mu(t) = -\frac{\kappa^2(t)}{\kappa(t)\|B^T P\tilde{x}(t)\| + \varrho(t)}, \qquad (17)$$

where $\kappa(t) > 0$ is a function which is needed to be designed and the bounded function $\varrho(t)$ is an any non-negative and uniform continuous function. Moreover, $\varrho(\cdot)$ satisfies

$$\sup_{t\in[0,+\infty)} \int_0^t [\varrho(\tilde{x}, s)]ds \leqslant \bar{\varrho}, \qquad (18)$$

where $\bar{\varrho} > 0$, then one can have

$$\lim_{t\to+\infty} \int_0^t [\varrho(\tilde{x}, s)]ds \leqslant \bar{\varrho} < +\infty. \qquad (19)$$

Obviously, $\mu(t)$ formulated by (17) is non-positive and satisfies the local Lipschitz condition.

Remark 1. The value of $\varrho(t)$ which is depended on the error signal $e(t)$ would increase with the output signal $y(t)$ far away from the reference signal $y_r(t)$, meanwhile, the value of $|\mu(t)|$ would decrease, which can leads to that the effect of the nonlinear part can be eliminated, and vice versa.

Step 3:) Consider a fractional-order sliding mode surface as following

$$s(t) \quad = k_1\tilde{x}(t) + k_2({}_0^C D_t^{\alpha}\tilde{x}(t)) + \cdots + k_n({}_0^C D_t^{(n-1)\alpha}\tilde{x}(t))$$
$$= \textstyle\sum_{i=1}^n k_i({}_0^C D_t^{(i-1)\alpha}\tilde{x}(t)), \qquad (20)$$

where $k_i (i = 1, 2, \cdots, n)$ is a constant row vector. Taking the fractional-order derivative with respect to t in both sides of (20) implies

$$\begin{aligned} {}_0^C D_t^{\alpha} s(t) &= k_1({}_0^C D_t^{\alpha} \tilde{x}(t)) + k_2({}_0^C D_t^{2\alpha} \tilde{x}(t)) + \cdots + k_n({}_0^C D_t^{n\alpha} \tilde{x}(t)) \\ &= \sum_{i=1}^{n} k_i({}_0^C D_t^{i\alpha} \tilde{x}(t)). \end{aligned} \tag{21}$$

On the other hand, when the states of the system arrive the sliding mode surface $s(t)$, then $s(t) = 0$, thus, the robust control law can be constructed as

$$u_s(t) = -\big(k_1 B\big)^{-1}\Big[\sum_{i=2}^{n} k_i({}_0^C D_t^{i\alpha} \tilde{x}(t)) + k_1(A + BF + \mu(t)BB^T P)\tilde{x}(t) + ls(t) + ksgn(s)\Big], \tag{22}$$

where $k_1 B$ is non-vanishing, and l and k are two positive constants. This robust controller can guarantee the process of tracking for the output signal to the reference signal can not be affected by external disturbances and uncertainties, and the tracking ability of the system can be further improved.

Step 4: The CNF controller is comprised of the linear, nonlinear and robust control laws, which are derived in Step 1, Step 2 and Step 3 respectively, with the following form

$$u(t) = F\tilde{x}(t) + Hx_r(t) + \mu(t)B^T P\tilde{x}(t) + u_s(t), \tag{23}$$

where

$$\mu(t) = -\frac{(\rho_1(\|\tilde{x}(t)\| + \|Gx_r(t)\|) + \rho_3 + \rho_2\bar{u} + 2\bar{u} + \tilde{\rho}(\bar{u}))^2}{(\rho_1(\|\tilde{x}(t)\| + \|Gx_r(t)\|) + \rho_3 + \rho_2\bar{u} + 2\bar{u} + \tilde{\rho}(\bar{u}))\|B^T P\tilde{x}(t)\| + \varrho(\tilde{x}(t))}, \tag{24}$$

here $\tilde{\rho}(\bar{u})$ is a positive constant and satisfies $\|u(t)\| \leq \tilde{\rho}(\bar{u})$.

Remark 2. Because $\tilde{x}(t)$, $x_r(t)$ and $s(t)$ are all bounded, the input of controller formulated by (23) is also bounded.

Set

$$\omega(t) = sat(u(t)) - F\tilde{x}(t) - Hx_r(t), \tag{25}$$

which together with (23) implies

$$\omega(t) = sat(F\tilde{x}(t) + Hx_r(t) + \mu(t)B^T P\tilde{x}(t) + u_s(t)) - F\tilde{x}(t) - Hx_r(t). \tag{26}$$

Taking the fractional-order derivative with respect to t in both sides of (12) along the trajectories of (1) and (4), we can get

$$\begin{aligned} {}_0^C D_t^{\alpha} \tilde{x}(t) &= {}_0^C D_t^{\alpha} x(t) - G({}_0^C D_t^{\alpha} x_r(t)) \\ &= (A + \Delta A)x(t) + \bar{A}x(t-\tau) + (B + \Delta B)sat(u) + D - GA_r x_r(t) \\ &= (A + \Delta A)\tilde{x}(t) + (A + \Delta A)Gx_r(t) + \bar{A}\tilde{x}(t-\tau) + \bar{A}Gx_r(t-\tau) \\ &\quad + (B + \Delta B)sat(u) + D - GA_r x_r(t). \end{aligned} \tag{27}$$

Substituting $\omega(t)$ into (27) yields that

$$\begin{aligned} {}_0^CD_t^\alpha \tilde{x}(t) &= (A + \Delta A + BF)\tilde{x}(t) + BHx_r(t) + B\omega(t) + (A + \Delta A)Gx_r(t) \\ &\quad + \bar{A}\tilde{x}(t-\tau) + \bar{A}Gx_r(t-\tau) + D - GA_rx_r(t) + \Delta Bsat(u) \\ &= (A + \Delta A + BF)\tilde{x}(t) + B\omega(t) + \bar{A}\tilde{x}(t-\tau) + \bar{A}Gx_r(t-\tau) \\ &\quad + D + \Delta AGx_r(t) + \Delta Bsat(u). \end{aligned} \tag{28}$$

Remark 3. The matrix $\mathbf{A}$ is a negative definite matrix if and only if the even order principal sub-formula $D_i > 0$, and the order principal sub-formula of odd order $D_i < 0$. Then, the quadratic $f(x_1, x_2, \cdots, x_n) = X^T\mathbf{A}X$ is a negative quadratic.

The main results of this paper is represented by the coming Theorem 3.6.

Theorem 3.6. *Consider the fractional-order uncertain system* (1) *and the reference system* (4)*. Suppose the Hypothesises 1, 2 and 3 hold, and for any* $\delta_i \in (0,1)(i = 1,2)$*, let* c_δ *is the largest positive scalar such that* $\tilde{x} \in X_\delta$ *with* $X_\delta = \{\tilde{x} : \tilde{x}^TP\tilde{x} \leqslant c_\delta\}$*, the following inequalities hold,*

$$\|F\tilde{x}(t)\| \leqslant (1 - \delta_1 - \delta_2)\bar{u}, \tag{29}$$

$$\|Hx_r(t)\| \leqslant \delta_1\bar{u}, \tag{30}$$

$$\|u_s(t)\| \leqslant \delta_2\bar{u}. \tag{31}$$

If there exist a matric $Z > 0$ *with adequate dimensions, and satisfy the following condition:*

$$\Lambda = \begin{bmatrix} \Lambda_{11} & P\bar{A} \\ * & -(1-\vartheta)Z \end{bmatrix} < 0, \tag{32}$$

where $\Lambda_{11} = (A+BF)^TP + P(A+BF) + (1-\vartheta)^{-1}P^2 + Z + Q + F^TWF$*, and* $Q + F^TWF$ *is a positive definite matrix. Then, under the controller formulated by* (23)*, the error* $e(t)$ *defined by* (11) *converges to zero asymptotically with* $t \to +\infty$*.*

Proof. The whole proof is divided into four situations.

S1: The input signal is unsaturated which means the values of inputs are less than the supremum of saturation function and more than the infimum of saturation function
S2: The values of all input channels of control are more than the supremum of saturation function.
S3: The values of input channels of control are less than he infimum of saturation function.
S4: Some of the inputs channels are unsaturated, and the others are saturated

Proof for S1: In this case, we have

$$|u_i(t)| \leqslant \bar{u}_i, i = 1, 2, \cdots, n, \tag{33}$$

then $sat(u) = u(t)$, therefore, it can be obtained that

$$\begin{aligned}\omega(t) &= sat(F\tilde{x}(t) + Hx_r(t) + \mu(t)B^T P\tilde{x}(t) + u_s(t)) - F\tilde{x}(t) - Hx_r(t) \\ &= \mu(t)B^T P\tilde{x}(t) + u_s(t).\end{aligned} \tag{34}$$

Given the following Lyapunov function

$$V_1(\tilde{x}(t)) = \frac{1}{2}s^2(t). \tag{35}$$

Taking the fractional-order derivative with respect to t in both sides of (35) along the trajectories of the sliding mode surface (20), which together with Lemma 3.1 yields

$$\begin{aligned}{}_0^C D_t^\alpha V_1(t) &\leqslant s(t)({}_0^C D_t^\alpha s(t)) \\ &= s(t)\big[k_1({}_0^C D_t^\alpha \tilde{x}(t)) + \sum_{i=2}^{n} k_i({}_0^C D_t^{i\alpha} \tilde{x}(t))\big].\end{aligned} \tag{36}$$

Substituting (28) into (36) gives

$$\begin{aligned}{}_0^C D_t^\alpha V_1(t) \leqslant{}& s(t)\big[k_1(A + \Delta A + BF)\tilde{x}(t) + k_1 B\omega(t) + k_1\bar{A}\tilde{x}(t-\tau) + k_1 D \\ &+ k_1\bar{A}Gx_r(t-\tau) + k_1\Delta AGx_r(t) + k_1\Delta Bsat(u) + \sum_{i=2}^{n} k_i({}_0^C D_t^\alpha \tilde{x}(t))\big] \\ ={}& s(t)\big[k_1(A + \Delta A + BF + \Delta BF)\tilde{x}(t) + k_1\bar{A}\tilde{x}(t-\tau) + k_1 B\omega(t) \\ &+ k_1\mu(t)\Delta BB^T P\tilde{x}(t) + k_1\chi(t) + \sum_{i=2}^{n} k_i({}_0^C D_t^\alpha \tilde{x}(t))\big],\end{aligned}$$

where

$$\begin{aligned}\chi(t) &= \bar{A}Gx_r(t-\tau) + D + \Delta AGx_r(t) + \Delta BHx_r(t) + \Delta Bu_s(t) \\ &= B\xi(t),\end{aligned} \tag{37}$$

along with Hypothesis 3, we have

$$\chi(t) = B\xi(t), \tag{38}$$

here

$$\xi(t) = \bar{N}Gx_r(t-\tau) + N_3 + N_1 Gx_r(t) + N_2 Hx_r(t) + N_2 u_s(t). \tag{39}$$

With robust control law (22) and Hypothesis 3, from (34), we can get

$$\begin{aligned}{}_0^C D_t^\alpha V_1(t) \leqslant{}& s(t)\big[k_1(\Delta A + \Delta BF)\tilde{x}(t) + k_1\bar{A}\tilde{x}(t-\tau) + k_1\mu(t)\Delta BB^T P\tilde{x}(t) \\ &+ k_1\chi(t)\big] - ls^2(t) - k|s(t)| \\ ={}& s(t)\big[k_1 B(N_1 + N_2 F)\tilde{x}(t) + k_1 B\bar{N}\tilde{x}(t-\tau) + k_1 N_2\mu(t)BB^T P\tilde{x}(t) \\ &+ k_1 B\xi(t)\big] - ls^2(t) - k|s(t)|,\end{aligned}$$

then

$$
{}_0^C D_t^\alpha V_1(t) \leqslant |s(t)|\|k_1 B\|[(\rho_1+\rho_2\|F\|)\|\tilde{x}(t)\| + \|\bar{N}\|\|\tilde{x}(t-\tau)\| \\
+ \rho_2|\mu(t)|\|B^T P\|\|\tilde{x}(t)\| + \rho_\xi] - ls^2(t) - k|s(t)|,
$$

where $\rho_\xi = max\|\xi(t)\|$.

Thus, when the system parameters satisfy the following switching condition

$$
k \geqslant \|k_1 B\|[(\rho_1+\rho_2\|F\|)\|\tilde{x}(t)\| + \|\bar{N}\|\|\tilde{x}(t-\tau)\| + \rho_2|\mu(t)|\|B^T P\|\|\tilde{x}(t)\| + \rho_\xi],
$$

it can be asserted that

$$
{}_0^C D_t^\alpha V_1(t) \leqslant -ls^2(t).
$$

Therefore, using Lemma 3.4, we can derive the equilibrium point of the system (1) is asymptotically stable and the trajectories converge to the sliding surface.

Conducting the following discussion requires an alternative approach, thus, we need another Lyapunov functional candidate as follows

$$
V_2(\tilde{x}(t), x_r(t)) = {}_0I_t^{1-\alpha}[\tilde{x}^T(t)P\tilde{x}(t)] + \int_{t-\tau}^{t} \tilde{x}^T(\beta)Z\tilde{x}(\beta)d\beta \\
+ {}_0I_t^{1-\alpha}[x_r^T(t)P_r x_r(t)] + \int_{t-\tau}^{t} x_r^T(\beta)G^T\bar{A}^T\bar{A}Gx_r(\beta)d\beta, \tag{40}
$$

where the matrix Z and P_r are positive definite which can be determined later. Taking derivative in both sides of (40), along with Hypothesis 2, we can find

$$
\dot{V}_2(t) \leqslant [{}_0^C D_t^\alpha \tilde{x}(t)]^T P\tilde{x}(t) + \tilde{x}^T(t)P({}_0^C D_t^\alpha \tilde{x}(t)) + \tilde{x}^T(t)Z\tilde{x}(t) + [{}_0^C D_t^\alpha x_r(t)]^T P_r x_r(t) \\
- (1-\vartheta)\tilde{x}^T(t-\tau)Z\tilde{x}(t-\tau) + x_r^T(t)P_r({}_0^C D_t^\alpha x_r(t)) + x_r^T(t)G^T\bar{A}^T\bar{A}Gx_r(t) \\
- (1-\vartheta)x_r^T(t-\tau)G^T\bar{A}^T\bar{A}Gx_r(t-\tau).
$$

According to (4) and (28), we have

$$
\begin{aligned}
\dot{V}_2(t) \leqslant\; & \tilde{x}^T(t)[(A+\Delta A+BF)^T P + P(A+\Delta A+BF) + Z]\tilde{x}(t) \\
& +\tilde{x}^T(t-\tau)\bar{A}^T P\tilde{x}(t) + \tilde{x}^T(t)P\bar{A}\tilde{x}(t-\tau) + x_r^T(t-\tau)G^T\bar{A}^T P\tilde{x}(t) \\
& +\tilde{x}^T(t)P\bar{A}Gx_r(t-\tau) + x_r^T(t)G^T\Delta A^T P\tilde{x}(t) + \tilde{x}^T(t)P\Delta AGx_r(t) \\
& +\omega^T(t)B^T P\tilde{x}(t) + \tilde{x}^T(t)PB\omega(t) + D^T P\tilde{x}(t) + [sat(u)]^T\Delta B^T P\tilde{x}(t) \\
& +\tilde{x}^T(t)P\Delta Bsat(u) + \tilde{x}^T(t)PD - (1-\vartheta)\tilde{x}^T(t-\tau)Z\tilde{x}(t-\tau) \\
& +x_r^T(t)P_r A_r x_r(t) - (1-\vartheta)x_r^T(t-\tau)G^T\bar{A}^T\bar{A}Gx_r(t-\tau) \\
& +[A_r x_r(t)]^T P_r x_r(t) + x_r^T(t)G^T\bar{A}^T\bar{A}Gx_r(t),
\end{aligned} \tag{41}
$$

together with the Hypothesis 3, we get

$$
\begin{aligned}
h(t) &= D + \Delta AGx_r(t) + \Delta Bsat(u) \\
&= B\gamma(t),
\end{aligned} \tag{42}
$$

where

$$\gamma(t) = N_1 G x_r(t) + N_2 sat(u) + N_3.$$

Since, for any given $\varepsilon > 0$, the following holds

$$\mathcal{M}^T \mathcal{N} + \mathcal{N}^T \mathcal{M} \leqslant \varepsilon \mathcal{M}^T \mathcal{M} + \varepsilon^{-1} \mathcal{N}^T \mathcal{N},$$

where $\mathcal{M}$ and $\mathcal{N}$ are any matrices with the appropriate dimensions, then we have

$$\begin{aligned} & x_r^T(t-\tau) G^T \bar{A}^T P \tilde{x}(t) + \tilde{x}^T(t) P \bar{A} G x_r(t-\tau) \\ & \qquad \leqslant \varepsilon \tilde{x}^T(t) P^2 \tilde{x}(t) + \varepsilon^{-1} x_r^T(t-\tau) G^T \bar{A}^T \bar{A} G x_r(t-\tau). \end{aligned} \tag{43}$$

Employing the inequality (43), the inequality (41) can be written as

$$\begin{aligned} \dot{V}_2(t) \leqslant\ & \tilde{x}^T(t)[(A+BF)^T P + P(A+BF) + \varepsilon P^2 + Z]\tilde{x}(t) + \tilde{x}^T(t) P B \omega(t) \\ & + \tilde{x}^T(t) P \bar{A} \tilde{x}(t-\tau) + \tilde{x}^T(t-\tau) \bar{A}^T P \tilde{x}(t) - (1-\vartheta)\tilde{x}^T(t-\tau) Z \tilde{x}(t \\ & - \tau) + \varepsilon^{-1} x_r^T(t-\tau) G^T \bar{A}^T \bar{A} G x_r(t-\tau) + x_r^T(t)(A_r^T P_r + P_r A_r \\ & + G^T \bar{A}^T \bar{A} G) x_r(t) - (1-\vartheta) x_r^T(t-\tau) G^T \bar{A}^T \bar{A} G x_r(t-\tau) \\ & + \tilde{x}^T(t)[\Delta A^T P + P \Delta A]\tilde{x}(t) + \tilde{x}^T(t) P h(t) + h^T(t) P \tilde{x}(t) + \omega^T(t) B^T P \tilde{x}(t). \end{aligned}$$

Let $\varepsilon = (1-\vartheta)^{-1}$, we get

$$\begin{aligned} & \dot{V}_2(t) + \tilde{x}^T(t)(Q + F^T W F)\tilde{x}(t) \\ & \quad \leqslant \tilde{x}^T(t)[(A+BF)^T P + P(A+BF) + (1-\vartheta)^{-1} P^2 + Z + Q \\ & \qquad + F^T W F]\tilde{x}(t) + \tilde{x}^T(t) P \bar{A} \tilde{x}(t-\tau) + \tilde{x}^T(t-\tau) \bar{A}^T P \tilde{x}(t) - (1-\vartheta)\tilde{x}^T(t \\ & \qquad - \tau) Z \tilde{x}(t-\tau) + x_r^T(t)(P_r A_r + A_r^T P_r + G^T \bar{A}^T \bar{A} G) x_r(t) + \tilde{x}^T(t)[\Delta A^T P \\ & \qquad + P \Delta A]\tilde{x}(t) + \tilde{x}^T(t) P h(t) + h^T(t) P \tilde{x}(t) + \omega^T(t) B^T P \tilde{x}(t) + \tilde{x}^T(t) P B \omega(t), \end{aligned} \tag{44}$$

where Q and W are positive definite matrixes, and the matrix P_r satisfies the following Riccati algebraic equation

$$G^T \bar{A}^T \bar{A} G + P_r A_r + A_r^T P_r \leqslant 0.$$

223 By using the matrix inequality (32), the inequality (44) can be simplified as

$$\begin{aligned} & \dot{V}_2(t) + \tilde{x}^T(t)(Q + F^T W F)\tilde{x}(t) \\ \leqslant\ & \Psi^T \Lambda \Psi + \tilde{x}^T(t)[\Delta A^T P + P \Delta A]\tilde{x}(t) + \tilde{x}^T(t) P h(t) + h^T(t) P \tilde{x}(t) \\ & + \omega^T(t) B^T P \tilde{x}(t) + \tilde{x}^T(t) P B \omega(t), \end{aligned} \tag{45}$$

here $\Psi = [\tilde{x}(t) \quad \tilde{x}(t-\tau)]^T$, and

$$\Lambda = \begin{bmatrix} \Lambda_{11} & P\bar{A} \\ * & -(1-\vartheta) Z \end{bmatrix},$$

where $\Lambda_{11} = (A+BF)^T P + P(A+BF) + (1-\vartheta)^{-1}P^2 + Z + Q + F^T W F$.
Proof for S2: When the values of control input $u_i(t)$ of all input channels overbear their upper boundaries, which means $u_i(t) \geq \bar{u}_i$, then we have $sat(u_i) = \bar{u}_i$ and

$$\tilde{\rho}_i(\bar{u}_i) \geqslant u_i(t) = F_i\tilde{x}(t) + H_i x_r(t) + \mu(t){B_i}^T P\tilde{x}(t) + u_s^i(t) \geqslant \bar{u}_i,$$

where $\tilde{\rho}_i(\bar{u}_i)$ is the maximum value of $u_i(t)$. By (3) and (26), we find

$$\omega_i(t) = \bar{u}_i - F_i\tilde{x}(t) - H_i x_r(t). \tag{46}$$

Using (29), (30) and (31), we get

$$\begin{aligned} F_i\tilde{x}(t) + H_i x_r(t) + u_s^i(t) &\leqslant |F_i\tilde{x}(t) + H_i x_r(t) + u_s^i(t)| \\ &\leqslant |F_i\tilde{x}(t)| + |H_i x_r(t)| + |u_s^i(t)| \\ &\leqslant (1-\delta_1-\delta_2)\bar{u}_i + \delta_1\bar{u}_i + \delta_2\bar{u}_i \\ &\leqslant \bar{u}_i. \end{aligned} \tag{47}$$

From (3), (26) and (47), we have

$$\omega_i(t) = \bar{u}_i - F_i\tilde{x}(t) - H_i x_r(t) \geqslant 0. \tag{48}$$

According to the equation (23), we can obtain

$$F_i\tilde{x}(t) + H_i x_r(t) = u_i(t) - \mu(t)B_i^T P\tilde{x}(t) - u_s^i(t). \tag{49}$$

Therefore, applying (48) and (49), we get

$$\omega_i(t) = \bar{u}_i - u_i(t) + \mu(t)B_i^T P\tilde{x}(t) + u_s^i(t). \tag{50}$$

Since the $\mu(t) \leqslant 0$ and $\mu(t)B_i^T P\tilde{x}(t) \geqslant 0$, it can be asserted that

$${B_i}^T P\tilde{x}(t) = \tilde{x}^T(t)PB_i \leqslant 0.$$

Proof for S3: When the control input $u_i(t)$ of all input channels are less than the lower bounds, alternatively,

$$-\tilde{\rho}_i(\bar{u}_i) \leqslant u_i(t) = F_i\tilde{x}(t) + H_i x_r(t) + \mu(t){B_i}^T P\tilde{x}(t) + u_s^i(t) \leqslant -\bar{u}_i,$$

which implies $sat(u_i) = -\bar{u}_i$. From (3) and (26), we have

$$\omega_i(t) = -\bar{u}_i - F_i\tilde{x}(t) - H_i x_r(t) \leqslant 0. \tag{51}$$

Following the similar manner of obtaining (50), we find

$$\omega_i(t) = -\bar{u}_i - u_i(t) + \mu(t)B_i^T P\tilde{x}(t) + u_s^i(t).$$

Since $\mu(t) \leqslant 0$ and $\mu(t)B_i^T P\tilde{x}(t) \leqslant 0$, we get

$${B_i}^T P\tilde{x}(t) = \tilde{x}^T(t)PB_i \geqslant 0.$$

Proof for S4: When values of some control input $u_i(t)$ are unsaturated, but the others are saturated. As for the unsaturated inputs, we can obtain $\tilde{x}^T(t)PB_i\omega_i(t) \leqslant 0$, and

$$\omega_i(t) = \mu(t)B_i^T P\tilde{x}(t) + u_s^i(t).$$

With respect to saturated inputs the values of which are more than the supremum of saturation function, the results in *S2* imply $\omega_i(t) \geqslant 0$ and $\tilde{x}^T(t)PB_i \leqslant 0$, then we have $\tilde{x}^T(t)PB_i\omega_i(t) \leqslant 0$, thus

$$\omega_i(t) = \bar{u}_i - u_i(t) + \mu(t)B_i^T P\tilde{x}(t) + u_s^i(t).$$

As for the saturated inputs the values of which are less than the infimum of saturation function, the assertions of *S3* indicate $\omega_i(t) \leqslant 0$ and $\tilde{x}^T(t)PB_i \geqslant 0$, then we can get $\tilde{x}^T(t)PB_i\omega_i(t) \leqslant 0$, and

$$\omega_i(t) = -\bar{u}_i - u_i(t) + \mu(t)B_i^T P\tilde{x}(t) + u_s^i(t).$$

As indicated above, together with the inequality (45), we can assert

$$\begin{aligned}&\dot{V}_2(t) + \tilde{x}^T(t)(Q + F^T WF)\tilde{x}(t)\\ &\quad\leqslant \Psi^T\Lambda\Psi + \tilde{x}^T(t)Ph(t) + h^T(t)P\tilde{x}(t) + \tilde{x}^T(t)(\Delta A^T P + P\Delta A)\tilde{x}(t)\\ &\qquad + 2\tilde{x}^T(t)PB(\bar{u} - u(t) + \mu(t)B^T P\tilde{x}(t) + u_s(t)),\end{aligned} \tag{52}$$

combining with hypothesis 3, we can obtain

$$\begin{aligned}&\dot{V}_2(t) + \tilde{x}^T(t)(Q + F^T WF)\tilde{x}(t)\\ &\quad\leqslant \Psi^T\Lambda\Psi + 2\|B^T P\tilde{x}(t)\|[\rho_1(\|\tilde{x}(t)\| + \|Gx_r(t)\|) + \rho_3 + \rho_2\bar{u} + 2\bar{u} + \tilde{\rho}(\bar{u})]\\ &\qquad + 2\|B^T P\tilde{x}(t)\|^2\mu(t).\end{aligned} \tag{53}$$

By (24) and (53), we can get

$$\begin{aligned}&\dot{V}_2(t) + \tilde{x}^T(t)(Q + F^T WF)\tilde{x}(t)\\ \leqslant\ &\Psi^T\Lambda\Psi\\ &+\frac{2(\rho_1(\|\tilde{x}(t)\| + \|Gx_r(t)\|) + \rho_3 + \rho_2\bar{u} + 2\bar{u} + \tilde{\rho}(\bar{u}))\|B^T P\tilde{x}(t)\|\varrho(\tilde{x}(t))}{(\rho_1(\|\tilde{x}(t)\| + \|Gx_r(t)\|) + \rho_3 + \rho_2\bar{u} + 2\bar{u} + \tilde{\rho}(\bar{u}))\|B^T P\tilde{x}(t)\| + \varrho(\tilde{x}(t))}.\end{aligned} \tag{54}$$

Obviously, the following inequality holds

$$0 \leqslant \frac{\varrho(\tilde{x}(t))\phi}{\varrho(\tilde{x}(t)) + \phi} \leqslant \varrho(\tilde{x}(t)), \forall \varrho(\tilde{x}(t)) > 0, \phi > 0. \tag{55}$$

Then, it can be obtained that

$$\frac{(\rho_1(\|\tilde{x}(t)\| + \|Gx_r(t)\|) + \rho_3 + \rho_2\bar{u} + 2\bar{u} + \tilde{\rho}(\bar{u}))\|B^T P\tilde{x}(t)\|\varrho(\tilde{x}(t))}{(\rho_1(\|\tilde{x}(t)\| + \|Gx_r(t)\|) + \rho_3 + \rho_2\bar{u} + 2\bar{u} + \tilde{\rho}(\bar{u}))\|B^T P\tilde{x}(t)\| + \varrho(\tilde{x}(t))} \leqslant \varrho(\tilde{x}(t)). \tag{56}$$

Combined (54) and (56), it's obtained that

$$\dot{V}_2(t) + \tilde{x}^T(t)(Q + F^T W F)\tilde{x}(t) \leqslant \Psi^T \Lambda \Psi + 2\varrho(\tilde{x}(t)).$$

If there exist some matrices $X > 0$ and $Z > 0$ such that

$$\Lambda = \begin{bmatrix} \Lambda_{11} & P\bar{A} \\ * & -(1-\vartheta)Z \end{bmatrix} < 0,$$

then, $\lambda(\Lambda) < 0$. Thus

$$\dot{V}_2(t) + \tilde{x}^T(t)(Q + F^T W F)\tilde{x}(t) \leqslant \lambda_{min}(\Lambda)\|\Psi(t)\|^2 + 2\varrho(\tilde{x}(t)).$$

Here, we choose

$$\varrho(\tilde{x}(t)) \leqslant \frac{1}{2}\tilde{x}^T(t)(Q + F^T W F)\tilde{x}(t) \leqslant \frac{1}{2}\lambda_{max}(Q + F^T W F)\|\tilde{x}(t)\|^2.$$

Moreover, according to the representation of the Lyapunov function $V_2(t)$, there exist two $K-$class functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ such that

$$\alpha_1(\|\tilde{x}(t)\|) \leqslant V_2(\tilde{x}(t)) \leqslant \alpha_2(\|\tilde{x}(t)\|), \tag{57}$$

which implies

$$\begin{aligned} \alpha_1(\|\tilde{x}(t)\|) &= \int_0^t \dot{V}_2(\tilde{x}(s))ds + V_2(\tilde{x}(0)) \\ &\leqslant \alpha_2(\|\tilde{x}(0)\|) + \int_0^t \lambda_{min}(\Lambda)\|\Psi(s)\|^2 ds + 2\int_0^t \varrho(\tilde{x}(s))ds, \end{aligned} \tag{58}$$

which together with (18) gives

$$\begin{aligned} \alpha_1(\|\tilde{x}(t)\|) &\leqslant \alpha_2(\|\tilde{x}(0)\|) + 2\int_0^t \varrho(\tilde{x}(s))ds \\ &\leqslant \alpha_2(\|\tilde{x}(0)\|) + 2\bar{\varrho}. \end{aligned} \tag{59}$$

Then, we can conclude that for any $t > 0$,

$$\begin{aligned} -\int_0^t \lambda_{min}(\Lambda)\|\Psi(s)\|^2 ds &\leqslant \alpha_2(\|\tilde{x}(0)\|) + 2\int_0^t \varrho(\tilde{x}(s))ds \\ &\leqslant \alpha_2(\|\tilde{x}(0)\|) + 2\bar{\varrho}, \end{aligned} \tag{60}$$

which implies that

$$-\lim_{t\to+\infty}\Big[\int_0^t \lambda_{min}(\Lambda)\|\Psi(s)\|^2 ds\Big] \leqslant \alpha_2(\|\tilde{x}(0)\|) + 2\bar{\varrho} < +\infty. \tag{61}$$

Hence, it follows from Barbalat's Lemma that

$$\lim_{t\to+\infty}\Big[\int_0^t \lambda_{min}(\Lambda)\|\Psi(t)\|^2 ds\Big] = 0,$$

furthermore

$$\lim_{t\to+\infty}\|\Psi(t)\| = 0.$$

As indicated above, the auxiliary state $\tilde{x}(t)$ converges to zero asymptotically. Thus, based on the relationship of $\tilde{x}(t)$ and $e(t)$, it can be asserted that the system output $y(t)$ can be forced to track the reference state $y_r(t)$ asymptotically. □

4. Conclusion

Compared with the results in literatures [33, 35, 42, 51, 52, 53], the system considered in this paper is fractional-order uncertain system with time delays and saturation function, which is very complex. The tracking controller is designed by CNF control approach. Furthermore, based on fractional-order Mittag-Leffer asymptotical stability theorem, the asymptotical tracking and stability of the controller proposed is proved by designing a fractional-order Lyapunov function and the fractional Barbalat's Lemma.

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