**The method of judging satisfactory consistency of linguistic judgment matrix based on adjacency matrix and 3-loop matrix**

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**Abstract:** As an effective way of expressing uncertain information, language phrases are more conform to the language habits of decision-makers to describe the evaluation of things. The consistency judgment of language judgment matrix is the key to analytic hierarchy process(AHP). If the linguistic judgment matrix has satisfactory consistency, the rank of the decision schemes can be determined. In this study, first, the comparison relation between the decision schemes is represented by a directed graph. The preference relation matrix of the linguistic judgment matrix is the adjacency matrix of the directed graph. We can use thest power of the preference relation to judge the linguistic judgment matrix whether has satisfactory consistency. The method is if there is one and only one element in thest power of the preference relation, and the element 1 is not on the main diagonal. Then the linguistic judgment matrix has satisfactory consistency. If there are illogical judgments in the judgment, the decision schemes that form a 3-loop can be identified and expressed through the second-order sub-matrix of the preference relation matrix. The feasibility of this theory can be verified through examples. The corresponding schemes for illogical judgments are represented in a spatial coordinate system.

**Keyword：**adjacency matrix, linguistic judgment matrix, satisfactory consistency, 3-loop matrix

**1. Introduction**

Decision analysis is the process in which decision-makers select one or more decision options by analyzing alternative decision schemes based on their values, individual preferences, and cognitive structure. In general, a basic decision-making process includes identifying decision problems, determining decision objectives, and selecting decision schemes. At first, certain mathematical models are used to solve simple decision-making problems. With the complexity of social life, the uncertainty of economic activities, and the increase of unknown factors in the decision-making process, it has become increasingly difficult for decision experts to rely solely on simple mathematical models to solve a problem. Based on this situation and the universality of decision-making problems in people's daily lives, finding more effective ways to express decision-making information is an inevitable requirement for the development of decision-making science and human activities.

In the process of developing decision theory, people's understanding of problems has broken down their either or cognition. There are many uncertain concepts in people's thinking, such as the weather should be good, the plan should be good at present, and the mood should be good. The objects described by these uncertain concepts cannot be simply described by "yes" or "no". Because of the ambiguity of the membership objects of these concepts and their importance in the decision-making process, Zadeh[1] put forward the concept of fuzzy set in 1965 and a series of theoretical, which well solved the problem of ambiguous membership relations. In fact, when expressing uncertain information, people tend to prefer language expression. Although language variables are not as precise as traditional numerical variables, they are relatively close to natural language and human cognitive habits. Therefore, this natural language variable that expresses decision information through qualitative means is an effective tool for expressing uncertain information. In some complex decision-making environments, decision experts may exhibit hesitation between several decision values, so relying on a single linguistic term cannot accurately express the expert's hesitation. In view of this, Torra[2] proposed hesitant fuzzy set (HFSs), which allowed decision experts to have multiple membership degrees when evaluating decision information. However, in the actual decision-making process, the personal knowledge and experience of decision experts make different membership degrees have different degrees of importance. The proposal of probability language technical terminology(PLTSs) solved this problem well. In the decision-making process, the probability information corresponding to each membership degree was given, which effectively expressed the importance between different membership degrees[3]. Because of the uncertainty of things, it is difficult for experts to express their views in certain terms. The uncertainty language technical terminology[4](ULTs), hesitant fuzzy language set[5](HFLTSs), and extended hesitant fuzzy language set(EHFLTSs) can all be effective tools for decision experts to express uncertain information. However, in the process of expressing uncertain information, decision makers cannot clearly express the range of decision information. In fact, makers are accustomed to using language modifiers to express, such as "very good weather, very good mood, possible effective scheme, etc.", in which "very" and "possibly" are language modifiers. Zadeh[6] proposed the concept of a language correction set, and transformed the initial fuzzy set by modifying the shape of the membership function.

Many achievements have been made in determining the satisfactory consistency of linguistic judgment matrices[7-13]. From the current research results, the definition of satisfactory consistency of linguistic judgment matrices is mature, but there are relatively few definitions that directly determine satisfactory consistency. Lian Y. Y. et al. [7] proposed a multi-granularity language reasoning method, which mainly dealt with incomplete information. Chen Z. S. et al. [8] interpreted PHFLTSs from the perspective of T2 fuzzy logic and proposed a new PHFLTS encoding method based on CON. Zhang S. et al. [9] proposed a solution method for group consensus decision-making problems based on multiplication language and an adaptive consistency model of fuzzy information granulation. Dong et. al.[10] decision makers are comfortable to provide the evaluation information with imprecision or linguistic evaluation variables, which are intuitive and flexible approaches to describe decision maker’s fuzziness and qualitative evaluation information. Liu, Song, and Yang [11] proposed a prospect cross-efficiency (PCE) model to determine the sorting order of decision making units(DMUs). Jin et al. [12] decision makers not only focus on the utility results derived by the alternatives they choose, but also focus on the utility results that may be rewarded if they choose other alternatives, and they avoid choosing alternatives that they will regret. Therefore, as an important behavioral decision-making theory, regret theory can be utilized to deal with the behavior of regret aversion. Wang et al. [13] investigated an interval type-2 fuzzy multi-attribute decision making method by combining the regret theory and projection model.

In recent years, other consistency determination methods for linguistic judgment matrices have also achieved some results [14-22]. Based on the generalized distance measure of hesitant fuzzy language, Guo X. J. et al. [14] defined a compatibility measure of hesitant fuzzy language preference relationships to handle consistency problems. Grogelj P. et al. [15] discussed the acceptability of the judgment matrix based on the decision matrix provided by each decision-maker. Wu et al[17] discover a method that derives an acceptable consistency pairwise comparison matrix(PCM) (consistent ratio (CR) CR<0.1), eliminates all logical errors, and makes the departure between the modified PCM and the original PCM even smaller than existed studies. J A et al.[18] presented a new multi criteria group decision-making method suitable for non-static frameworks, which allowed experts to use the preferred terms of the preferred language label set through the use of multi granularity fuzzy language modeling. Zhang Y. et al. [19] introduced the definition of IFLPR and its transitive properties, and provided a group decision-making method based on genetic algorithm and incomplete information. Li C. et al. [20] proposed a consistency driven method using PISS to manage distributed language preference relationships(DLPRS). This method can estimate unknown elements in incomplete DLPRS and obtain personalized numerical meanings of language expression for decision-makers, ensuring optimal consistency between incomplete DLPRS and unknown elements. A.R et al. [21] gave a WASPAS based judgment method on the basis of IVIFS. In the process of calculating weights, decision experts' weights and criteria weights were calculated on the basis of interval valued intuitionistic fuzzy information measures. Su S. C. et al. [22] studied the use of Pairwise comparison algorithm to build a multi criteria Decision model with incomplete verbal preference relations. On the basis of the concept of interval valued hesitant fermatean fuzzy set (IVHFFS), A. R et al. [23] proposed the aggregation operator (AO) to aggregate interval valued hesitant fermatean fuzzy information, and discussed some properties of the operator in detail. M.D et al. [24] proposed a new entropy based weighted aggregate product assessment (WASPAS) method and a multi criteria decision-making method combining interval type 2 hesitant Fuzzy set (IT2HFS), and tested it with a specific case of an all service operator in Türkiye. M.D et al. [25] used the Delphi method based on interval type 2 Fuzzy set to rank the indicators that affect the location of vehicle crushing facilities. Wang L. et al. [26] extended the BM operator to the LTWH environment and proposes a language term with a weakening hedging BM (LTWHBM) operator to describe uncertainty in HCW management. Wu X. L. et al.[27] introduced a geometric language scale characterized by the proportional relationship between progressive levels and their corresponding fuzzy meanings. An aggregation method based on the geometric linguistic scale is proposed to deal with decision matrices with direct linguistic evaluation, complete Pairwise comparison, and partial Pairwise comparison, respectively. Yang Q. et al. [28] proposed an enhanced large-scale group decision-making method combining proportional hesitant fuzzy language technical terminology (PHFLTS) and cumulative prospect theory (CPT). G.S et al [29] proposed a model based on the HFLTS-QFD method to support the formulation of SSDP, which combines the hesitant fuzzy language Technical terminology with quality function deployment technology. Fan X. et al. [30] explored a consensus model based on limited trust propagation, which considers individuals' attitudes towards modifying preference relationships in a social network environment with uncertain preference information.

WU S. H. [17] Note that logical consistency is quite similar to ordinal consistency (or transitivity) in [31] and weak consistency in[32]. If , thenshould be satisfied. If however, when, then the preference judgments are called logically inconsistent, or intransitivity. Therefore, the logical inconsistency can be defined as , which represents a directed circuit. Actually, logical consistency requires the DM to be consistent for each pairwise comparison, which may be unrealistic in real-life applications, especially for high order PCM. Some researchers [33] consider logical consistency as the acceptable consistent level, while some others [34,35] consider numerical consistency as the acceptable consistent level. In our opinion, both ideas are one-sided. Generally speaking, logical consistency and acceptable numerical consistency do not have necessary relationships. However, in most cases, a logically inconsistent PCM is usually unacceptable in numerical consistency.

Gou X. J.[14]gave an adaptive consensus model based on fuzzy information granulation (fuzzy IG). Firstly, a granular representation of linguistic terms is concerned with the triangular fuzzy formation of a family of information granules over given Analytical Hierarchy Process (AHP) numerical scales. On this basis, the individual consistency and group consensus measure indices using the fuzzy granulation technique are constructed, respectively. Then, the optimal cut-off points of fuzzy information granules are obtained by establishing a multi-objective optimization model together with a multi-objective particle swarm optimization (MOPSO) algorithm. A novel group consensus decision-making approach where the consensus reaching process (CRP) is achieved by adaptively adjusting individual preferences through the optimization of the cut-off points is proposed. After conflict elimination, the obtained group preference gives the ranking of the alternatives.

However, most of these studies use consistent ratio(CR) or directed graphs to determine the satisfactory consistency of the language judgment matrix, or directly study methods for improving consistency. Few researchers have expressed illogical decision-making schemes. The logical consistency and acceptable numerical consistency are discussed. The acceptable consistency level is the criterion for judgment[17]. An adaptive consensus model is established to judge consistency[14]. The consistency discussed in this paper mainly manifests in the transitivity of the superiority and inferiority relationships between decision schemes, and a judgment matrix with satisfactory consistency can determine the ranking of schemes. That is, if ,then.

This paper mainly discusses the method of judging the satisfactory consistency of the linguistic judgment matrix based on the adjacency matrix. The preference relation matrix of the linguistic judgment matrix is regarded as a directed graph. The preference relation matrix st power is used to judge whether the linguistic judgment matrix has satisfactory consistency. If only one element in the preference relation matrix st power which is not in the main diagonal is 1, the linguistic judgment matrix has satisfactory consistency. If the linguistic judgment matrix does not have satisfactory consistency, we can determine whether the second-order sub-matrices of the preference relationship matrix are 3-loop matrices to find the illogical judgment. The illogical decision schemes can be represented by software.

The remainder of this paper is organized as follows.

Firstly, we start with notations, definitions, theorems of the preference matrix, and adjacency matrix (Section 2). In Section 3, we present the idea and process of the consistency method, including satisfactory consistency based on the adjacency matrix, and the representation of illogical judgments. Numerical examples are provided in Section 4 to illustrate and compare the proposed model. The paper is summarized and concluded in Section 5.

**2 Basic concepts**

and are two sets, whereis an even number. The preference information of pairwise comparison given by decision makers can be described by a matrix. The objects in the matrix are selected from the linguistic term set as the evaluation results of and. The number of objects in the matrix is called granularity of the linguistic term set. For example, a linguistic term set with 13 granularity can be described as ={=absolute difference, =quite poor, =very poor, =weak,=Poor, =slightly poor, =equivalent, =slightly better, =better，=good, =very good, =quite good, =absolutely good}.

,,,are four linguistic term sets.

**Definition 1**[36-42] A linguistic judgment matrix on the finite object setfor the linguistic term setis defined as,where

 ; (1)

for all.

 **Definition 2** In the linguistic judgment matrix, if

,

thenandis called an equivalent object and denoted as～.

 Table.1 mathematical symbol

|  |
| --- |
| **mathematical symbol significance** equivalent to superior to superior to |

In this paper, we mainly discuss the situation where there are no equivalent between decision schemes.

**Definition** **3** In the linguistic judgment matrix, if the dominance relation of the decision schemes is transitive and there is no loop phenomenon except for the equivalent schemes, then the linguistic judgment matrixis said to have satisfactory consistency.

**Definition 4** In the linguistic judgment matrix, if the degree of superiority between decision schemes is also transitive, that is, the degree to whichis superior tois equal to the degree to whichis superior toplus the degree to whichis superior to, then the linguistic judgment matrixis said to have complete consistency.

**Definition 5** If there is a phenomenon ofin the comparison results between decision schemes, the comparison result ofis called an illogical judgment is called a 3-loop formed by.

If the linguistic judgment matrix has satisfactory consistency, the dominance relation of decision schemes is transitive. If the linguistic judgment matrix has complete consistency, the degree of dominance relation is transitive. The linguistic judgment matrix provided by decision makers has a higher requirement of complete consistency than satisfactory consistency. Satisfactory consistency requires that there is a dominance relation between decision schemes, while complete consistency requires that the degree of dominance relation between decision schemes be reflected.

**Definition 6** is called a preference relation matrix of a linguistic judgment matrix, where:

 . (2)

For example, a linguistic judgment matrix given by a decision-maker using a language phrase evaluation set with a granularity of 7 is, the corresponding preference relation matrix is. In the preference relation, 1 represents that the scheme corresponding to the row is superior to the scheme corresponding to the column. 0 represents that the scheme corresponding to the row is inferior to the scheme corresponding to the column, but the degree of dominance relation is not given. The comparison result between the scheme itself and itself is represented by 0, which does not mean that it is inferior to itself, but rather that it does not compare itself with itself. To obtain the ranking of decision schemes and participate in group decision-making, the linguistic judgment matrix provided must have satisfactory consistency.

**Directed graphs**. is a directed graph, whereis a set of vertices andis a set of ordered pairsof distinct vertices called arcs. Note that we do not allow loops and multiple edges or arcs.

**Adjacency matrix.** Given a directed graph,, whereis the number of edges adjacent tofrom,is called the adjacency matrix of.

The comparison relation is represented by a directed graph, where the scheme is a vertex in the directed graph and the comparison relation between schemes is represented by an edge. If schemeis better than scheme, there is an edge starting fromand ending from. This way, the comparison results of all schemes can be represented by a directed graph. The pairwise comparison of all schemes is a single comparison and no comparison is made between itself and itself. The directed graph corresponding to the comparison relationship is a simple graph, without rings and parallel edges. The adjacency matrix corresponding to the directed graph also represents the dominance relation of the schemes. In the adjacency matrix,means thatis better than. Therefore, the elements in the adjacency matrix are only 0 and 1 and the elements on the main diagonal are all 0. From the above analysis, we can see that the preference relation matrix of the linguistic judgment matrix is the corresponding adjacency matrix when the judgment matrix is regarded as a directed graph.

Letbe the adjacency matrix of the directed graph,, then the elementinis the number of paths with lengthfromto, is the total number of paths (including loop) with lengthin, whereis the number of loop with lengthin. According to the definition of satisfactory consistency of the linguistic judgment matrix, if the linguistic judgment matrix has satisfactory consistency, the order of dominance relation of the schemes can be obtained. There is no illogical judgment between the schemes, and the elementinis the number of paths with a length of fromto. If the linguistic judgment matrix has satisfactory consistency,(the order of dominance relation of the schemes) can be obtained. Then, there is only one path fromtoand only one element inis 1 which is not on the main diagonal, all other elements are 0.

Definition 7 [17] For a given pairwise comparison matrix, if CR<0.1, is said to have acceptable numerical consistency.

Definition 8 [17] For a given pairwise comparison matrix, we callhas reached acceptable consistent level, if and only ifhas acceptable numerical consistency and its preference matrixis logically consistent.

Definition 9 If the advantages and disadvantages of each scheme have transitivity, and there is no cyclic phenomenon except for the equivalent schemes, then the judgment matrix is said to have satisfactory consistency.

Some researchers consider logical consistency as the acceptable consistent level, while others consider numerical consistency as the acceptable consistent level. In our opinion, both ideas are one-sided. Generally speaking, logical consistency and acceptable numerical consistency do not have necessary relationships. However, in most cases, a logically inconsistent pairwise comparison matrix is usually unacceptable in numerical consistency. In this article, the satisfactory consistency mainly refers to the transitivity of relationship between the advantages and disadvantages of decision-making schemes. In other words, based on the decision matrix provided by the decision-maker, the order of superiority and inferiority among decision schemes can be determined.

**3. A method for determining satisfactory consistency of linguistic judgment matrix**

**Theorem 2** The sufficient and necessary condition for a linguistic judgment matrix to have satisfactory consistency is that there is one and only one element in its preference relation matrix power, and that element 1 is not on the main diagonal.

**Proof:** **Necessity** According to the definition of satisfactory consistency, if the linguistic judgment matrix has satisfactory consistency, there is a relation between advantages and disadvantages(the order of schemes) all of the schemes.represents that decision schemeis superior to decision scheme. There is no illogical judgment in the comparison results between schemes. In the directed graph corresponding to the linguistic judgment matrix, there exists a directed edge withas the starting point andas the ending point. The comparison relation is represented by a directed graph. Then there is only one path with the length of from the best scheme to the worst scheme in the directed graph. There is only one element is 1 in the preference relation matrix of the linguistic judgment matrix and that element is not on the main diagonal. If the element of the main diagonal is 1, it means that the comparison result is a loop. In the comparison results, the elements of the starting and ending points cannot be the same.

**Sufficiency** If there is one and only one element that is not on the main diagonal is 1 in the and the other elements are 0. Consider the comparison relationship of the decision schemes as a directed graph, then the preference relationship matrix of the linguistic judgment matrix is the adjacency matrix after the comparison relationship of the decision schemes as a directed graph. The elementinis the number of paths with lengthfromto. One and only one element inthat is not on the main diagonal is 1, which means that there is a path with lengthin the directed graph corresponding to the linguistic judgment matrix. In the directed graph, the scheme corresponding to the start point of the edge is better than the scheme corresponding to the end point of the edge. In the path with length, the scheme corresponding to the start point is the best, the scheme corresponding to the end point is the worst, and the middle is the order of the schemes. Then the ranking of all the decision schemes can be obtained.

From the above theorem, if the linguistic judgment matrix has satisfactory consistency, there only one element that is not on the main diagonal is 1 in. If multiple elements are 1, then the judgment matrix does not have satisfactory consistency. The following are the cases where multiple elements are 1.

**1. Multiple 1 appear in positions that are not on the main diagonal**

If there are multiple 1 inof the preference relation matrix, one 1 corresponds to one path and multiple 1 indicate multiple paths. According to the meaning expressed by thepower of the adjacency matrix of a digraph, multiple dominance relation appear in comparison results. Whether there are equivalent schemes, the comparison results of the schemes are not unique. If the linguistic judgment matrix has satisfactory consistency , there is only one order of dominance relation between decision schemes. If Multiple 1 correspond to multiple orders of dominance relation, it indicates that the linguistic judgment matrix does not have satisfactory consistency and there must be illogical phenomena in the judgment process. It is necessary to identify the illogical judgments and correct them to obtain a judgment matrix with satisfactory consistency.

**2. 1 appears on the main diagonal**

1 appears on the main diagonal of , which means that a loop with a length of appears in the directed graph corresponding to the linguistic judgment matrix and one of the schemes in the dominance relation cannot appear twice. A loop means that the starting point and the end point of the comparison are the same, which obviously does not conform to the actual comparison situation. That indicates the judgment is unreasonable in the comparison process. Then the comparison result of schemes is a loop. If 1 appears on the main diagonal of the preference relation matrix that is less thanpower, the corresponding comparison result of schemes is also a loop. For example, if 1 appears on the main diagonal , the comparison result of four schemes is a loop.

Assumingis the preference relation matrix of a certain linguistic judgment matrix,





In this example, There are multiple elements of 1 in. So the judgment matrix does not have satisfactory consistency and cannot obtain the dominance relation of the schemes.









Fig. 1. Directed graph corresponding to comparison relations

In the above example, there are four decision schemes. Using enumeration methods, we can get the following paths with a length of 3: ,, ,,. The starting point and ending point of are the same inand, which are called loop in graph theory. In this way, the element with 1 on the main diagonal of represents a loop. For example, the loop corresponding toinis. It can also be inferred from the result of matrix multiplication that in, the 1 in the second row and second column corresponds toand the element at this position is obtained by multiplying and adding the second row inand the second column in,that is, whereand the other elements are 0, then, so in the comparison relationship, the element beforeis. Fromand, where only is 1 and other elements are 0. So it can be concluded that the element beforein the comparison relationship is, so the comparison relationship corresponding to the loop is .

The elements of the matrix obtained by multiplying may have values greater than 1, such as:

，.

There are two comparison results for the three decision schemes including decision schemes, which can be obtained by multiplying the matrix by ruleand. The comparison including decision schemesare and.

, it can be concluded that the linguistic judgment matrix has satisfactory consistency. In the process of determining whether the linguistic judgment matrix has satisfactory consistency, the following theorem can also be used.

**Theorem 3**: The necessary condition for a linguistic judgment matrixto have satisfactory consistency is that the sum of the elements inis.

**Proof**: If the linguistic judgment matrixhas satisfactory consistency, the comparison relation between schemes can be. The comparison relation formed byelements can be judged from the elements of . the number of comparison relations formed byelements is to selectelements from, that is.

If the sum of elements is notduring the calculation of , the linguistic judgment matrixmust not have satisfactory consistency.

The following is a general method for finding the corresponding elements of a cyclic.

**Step 1**: In, if and,then there is one loop, and if is not equal to 1, then there are multiple loops;

**Step 2**: ,

(1) Assumingand all other parts are 0, it can be concluded that the comparison scheme beforeis, and;

(2) If and part of is;

If ,then at least one of andis greater than 1, thenmust be greater than 1 and, go to step 3;

 (3) If and the sum of several parts is, the part with a value of 1 shall be calculated according to step 2 (1), and the part with a value of not 1 shall be calculated according to step 2 (2);

**Step 3**: (1) Assuming that,, and all other terms are 0, the comparison scheme beforeis, and;

 (2) If a portion of andis. If , then at least one of andis greater than 1, thenmust be greater than 1 and , go to step 4;

 (3) If the sum of several parts is 1, the part with a value of 1 shall be calculated according to step 2 (1), and the part with a value of not 1 shall be calculated according to step 2 (2);

**Step 4**: Repeat the above process until: . Assuming thatand all other terms are 0, the comparison scheme beforeis,and, and the final cycle obtained is.

The method of the dominance relation of decision schemes can also derive in.

**Step 1**: If , then the optimal scheme isand the worst scheme is;

**Step 2**: , then there is only one option that is not 0 and is 1. Assuming, then the decision scheme that ranks beforeis;

**Step 3**: Assumingin Step 2, it can be inferred that neithernoris 0, and. Assuming, the decision scheme in front of is;

**Step 4**: Assumingin Step 3, it can be inferred that neithernoris 0, and. Assuming, the decision scheme in front of is;

**Step 5**: Repeat the above process until, and the decision scheme afteris;

**Step 6**: Obtain the final decision plan in the following order:.

The following is a method for identifying illogical decision schemes using a cyclic matrix.

**Definition 10** In the second-order sub-matrix of the preference relationship matrixof the linguistic judgment matrix, as follows:

;;;

are called a 3-loop matrix, which corresponds to the loopformed by the schemes of.

The loop matrix is a sub matrix of the preference relationship matrix, but the opposite may not necessarily hold true. The comparison relation is same in the loop matrix and the original linguistic judgment matrix The comparison result of the decision schemes corresponding to the loop matrix is an illogical judgment. Although there are four formal forms mentioned above, in fact, it corresponds to the comparison result of the same three decision schemes.

For example, in the 3-loop matrix, the result of comparing withis, the result of comparingwithis, and the result of comparing withis, the result comparison of is. In the 3-loop matrix, the result of comparing withis, the result of comparingwithis, and the result of comparingwithis, the result comparison of  is. Similarly, in and, the result of comparingis also. This indicates that the illogical judgments obtained by the same three decision schemes have different representations. The reason for the different representations is that a 3-loop matrix can exchang rows and columns. After exchanging rows and columns, the corresponding decision schemes are also exchanged. Although the form has changed, the essence has not changed.

**Theorem 4** The necessary and sufficient condition for the linguistic judgment matrixto not have satisfactory consistency is that there at least exists a loop matrix in the second-order sub matrix of the preference relation matrixof the linguistic judgment matrix.

**Prove** **sufficient** If there is a loop matrix in the sub-matrix of the preference relation matrix, the dominance relation of the decision scheme corresponding to the loop matrix is a loop, which is an illogical judgment. The ranking decision schemes cannot be obtained. That is, the linguistic judgment matrixdoes not have satisfactory consistency.

**Necessity:** If the linguistic judgment matrixdoes not have satisfactory consistency, according to the definition of satisfactory consistency, there is a phenomenon of illogical judgment in the linguistic judgment matrix. The illogical judgment is represented by a loop matrix. No matter how many decision schemes are formed, the illogical judgment must include the illogical judgment formed by three decision schemes. The sub-matrix of the preference relationship matrixat least has a 3-loop matrix.

The theorem has been proven. Theorem 4 not only provides a method for determining the satisfactory consistency of the linguistic judgment matrix, but more importantly, illogical decision schemes can be represented by second-order sub-matrix. The feasibility of this method has been theoretically proven. The key to this method is how to use a second-order sub-matrix to identify illogical decision solutions. Finding illogical decision-making solutions can not only determine whether the linguistic judgment matrix has satisfactory consistency, but also prepare for improving consistency.

the linguistic judgment matrix

preference relationship matrix

Have satisfactory consistency

Yes

Whether its preference relation matrix power, and that element 1 is not on the main diagonal.

No

Finding the illogical decision schemes

Return to decision maker for modification

Improving

consistency

Fig. 2. Flow chart of judgment method

**4. Example Analysis**

**Example 1** A teacher gave the following evaluation comparison to his four graduates:

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Determine whether the linguistic judgment matrix provided by this teacher has satisfactory consistency. If there is satisfactory consistency, provide the rank of the four graduates.

The preference relation matrix can be obtained from formula (2) is:

.

According to theorem 4, it can be determined whether the second-order sub-matrix of the preference relationship matrixis a 3-loop matrix. For a 4-order matrix, 36 second-order sub matrices need to be judged. Therefore, manually determining whether there is a 3-loop matrix in the second-order sub-matrix is difficult. Therefore, we can use Matlab to find a second-order sub matrix whether is a 3-loop matrix, the output result is as follows:

Primitive matrix

A =

 1 0 1 0

 1 1 1 1

 0 0 1 0

 1 0 1 1

Sub matrix

B =

 1 0

 0 1

There are 0 sub-matrices in the original matrix.

There are 0 sub-matrices with elements located on the main diagonal of the original matrix.

The output results indicate that there is no 3-loop matrix in the second-order sub-matrix of this preference relation matrix, indicating that the linguistic judgment matrix has satisfactory consistency. We can calculate CR=0.0784<0.1 [17]. However, this method requires a large amount of computation. Using adjacency matrix judgment only requires observing the characteristics of the adjacency matrix. Representing illogical solutions using second-order sub-matrixes only requires observing the output results.

**Example 2**． Let the decision maker give the linguistic judgment matrix of five alternativesas follows:

,

judge whether *P* is consistent or not.

The preference relation matrixofis:

.

Using the Matlab, we can judge a second-order sub-matrix whether is a 3-loop matrix. The output result is as follows:

Primitive matrix

A =

 0 1 1 0 1

 0 0 0 1 0

 0 1 0 1 1

 1 0 0 0 0

 0 1 0 1 0

Sub matrix

B =

 1 0

 0 1

There are 6 sub matrices of the original matrix

There are 3 sub matrixes with elements located on the main diagonal of the original matrix Position of the first sub matrix:

a11:(1,2)

a12:(1,4)

a21:(2,2)

a22:(2,4)

Position of the second sub matrix:

a11:(1,3)

a12:(1,4)

a21:(3,3)

a22:(3,4)

Position of the second sub matrix:

a11:(4,1)

a12:(4,4)

a21:(5,1)

a22:(5,4)

From the output results, it can be seen that there are three 3-loop matrices in the second-order sub-matrix, represented as follows:

，，，。

The illogical judgment corresponding to four 3-loop matrices are:

；；.

The illogical judgments are expressed by three-dimensional coordinates as follows:



Fig. 3. The corresponding schemes for illogical judgment

From Figure 3, it can be seen that there are three 3-loop matrix. The first 3-loop matrix is formed by, the second 3-loop matrix is formed by, the third 3-loop matrix is formed by. The loop depends on the comparison relation in the original judgment matrix. However, as long as points appear in the three-dimensional coordinate map, it indicates that there are illogical judgments in the judgments given by the decision-maker. The linguistic judgment matrix does not have satisfactory consistency and cannot determine the dominance of the decision schemes.

By definition of the 3- loop matrix, the form of a 3-loop matrix is:

、and、;

Although their representations are different, the loop they represent are essentially the same. Let's use to find the 3-loop matrix to find illogical judgments.

Primitive matrix

A =

 0 1 1 0 1

 0 0 0 1 0

 0 1 0 1 1

 1 0 0 0 0

 0 1 0 1 0

Sub matrix

B =

 0 1

 1 0

There are 9 sub matrices of the original matrix

There are 6 sub matrices with elements located on the main diagonal of the original matrix

Position of the first sub matrix:

a11:(1,1)

a12:(1,2)

a21:(4,1)

a22:(4,2)

Position of the second sub matrix:

a11:(1,1)

a12:(1,3)

a21:(4,1)

a22:(4,3)

Position of the third sub matrix:

a11:(1,1)

a12:(1,5)

a21:(4,1)

a22:(4,5)

4th sub matrix position:

a11:(1,4)

a12:(1,5)

a21:(5,4)

a22:(5,5)

5th sub matrix position:

a11:(2,1)

a12:(2,4)

a21:(4,1)

a22:(4,4)

6th sub matrix position:

a11:(3,1)

a12:(3,4)

a21:(4,1)

a22:(4,4)



Fig. 4. The corresponding schemes for illogical judgment

The illogical judgments corresponding to the 3-loop matrix usingare as follows: ，，, and then usingto find the illogical judgments corresponding to the 3-loop matrix, as shown in the figure above. The first and fifth points are the same, representing the illogical judgments formed by. The second and sixth points represent the illogical judgments formed by. The third and fourth points represent the illogical judgments formed by, and the illogical judgments present in the linguistic judgment matrix are also：，，. This also verifies that the forms、and、given in the definition of the loop matrix are different, but represent the same illogical judgments.

We can calculate CR=0.1351>0.1 [17]. This linguistic judgment matrix don’t have satisfactory consistency. The satisfactory consistency of the language judgment matrix in example 2 can also be determined using definitions 8, 9 [9]. The fuzzy gross consistency degree associated and COG[9] can be obtained. However, it was not determined which judgments were illogical. The method used in this paper not only assessed satisfactory consistency, but also provided illogical schemes. illogical schemes can be returned to decision-makers or improve consistency.

 **Example 3** The decision maker makes the following evaluation for the four printers, and gives the following preference information:



According to the above steps, judge or revise the satisfactory consistency of .

The preference relation matrix obtained from definition 6 is as follows:

.

Using matlab to edit the program and search for the second-order sub-matrix of a 3-loop matrix, the output result is as follows:

Primitive matrix

A =

 0 0 1 0

 1 0 0 0

 0 1 0 0

 1 1 1 0

Submatrix

B =

 1 0

 0 1

All sub matrices of the original matrix have 1

There is 1 sub matrix with elements located in the Main diagonal of the original matrix

Position of the first sub-matrix:

a11:(2,1)

a12:(2,2)

a21:(3,1)

a22:(3,2)



Fig. 5. Schemes corresponding to illogical judgment

Next, we will use the 3-loop matrix  to find illogical judgments, and the output results are as follows:

Primitive matrix

A =

 0 0 1 0

 1 0 0 0

 0 1 0 0

 1 1 1 0

Sub matrix

B =

 0 1

 1 0

All sub matrices of the original matrix have 2

There are 2 sub matrixes with elements located on the Main diagonal of the original matrix

Position of the first sub matrix:

a11:(1,1)

a12:(1,3)

a21:(2,1)

a22:(2,3)

Position of the second sub matrix:

a11:(1,2)

a12:(1,3)

a21:(3,2)

a22:(3,3)



Fig. 6. Schemes corresponding to illogical judgments

From the output results, it can be seen that there are two 3-loop matrices in the second-order sub matrix, which are:

and.

The illogical judgments represented are all, so using different forms of 3-cyclic matrices to find illogical judgments is the same.

The linguistic judgment matrix in example 3 does not have satisfactory consistency. Compared with the judgment methods in references [9], [14], [17], this method does not require a large amount of computation and does not establish a complex model. Satisfactory consistency can be obtained through simple calculation and observation. When there is no satisfactory consistency, the second-order sub-matrix can represent illogical judgments.

**5. Conclusions**

This article mainly uses a directed graph to represent the comparison relation in the linguistic judgment matrix, and uses thepower of the preference relation to determine whether the linguistic judgment matrix has satisfactory consistency. If the linguistic judgment matrix does not have satisfactory consistency, use the sub-matrix of the linguistic judgment matrix to determine whether it is a 3-loop matrix to find the corresponding decision scheme for illogical judgments, and use a program to implement the search process. How to find the illogical judgments formed by three schemes is studied. The sub-matrix of the linguistic judgment matrix is used to represent the three schemes that are judged to be illogical, and the characteristics of the sub-matrix are analyzed. This method is simple and intuitive. By observing and comparing, the solution that judges to be illogical can be found. This method can be applied to comprehensive evaluation. The decision-maker give for pairwise comparisons between decision schemes according to a set of language phrases. A language judgment matrix can be formed by pairwise comparison. If the language judgment matrix has satisfactory consistency, the ranking relation between decision options can be obtained. If the language judgment matrix does not have satisfactory consistency, the second order sub-matrix can be used to represent illogical judgments. Illogical judgments are returned to decision-makers or consistency is improved.

Although this article provides a method for determining the satisfactory consistency of the language judgment matrix and identifying illogical decision schemes, it does not provide a method for determining the complete consistency of the linguistic judgment matrix. Satisfactory consistency reflects the transferability of the relationship between strengths and weaknesses, and further research issue are on the transferability of the degree of strengths and weaknesses and methods for improving inconsistency.

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