

# The dynamics of the relativistic Kepler problem

Elbaz I. Abouelmagd<sup>a</sup>, Juan Luis García Guirao<sup>b</sup>, Jaume Llibre<sup>c</sup>

<sup>a</sup>*Celestial Mechanics and Space Dynamics Research Group (CMSDRG), Astronomy Department, National Research Institute of Astronomy and Geophysics (NRIAG), Helwan – 11421, Cairo, Egypt.*

<sup>b</sup>*Departamento de Matemática Aplicada y Estadística. Universidad Politécnica de Cartagena, Hospital de Marina, 30203-Cartagena, Región de Murcia, Spain.*

<sup>c</sup>*Departament de Matemàtiques. Universitat Autònoma de Barcelona, Bellaterra, 08193-Barcelona, Catalonia, Spain.*

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## Abstract

We deal with the Hamiltonian system (HS) associated to the Hamiltonian in polar coordinates

$$H = \frac{1}{2} \left( p_r^2 + \frac{p_\phi^2}{r^2} \right) - \frac{1}{r} - \frac{\epsilon}{2r^2},$$

where  $\epsilon$  is a small parameter. This Hamiltonian comes from the correction given by the special relativity to the motion of the two-body problem, or by the first order correction to the two-body problem coming from the general relativity. This Hamiltonian system is completely integrable with the angular momentum  $C$  and the Hamiltonian  $H$ . We have two objectives.

First we describe the global dynamics of the Hamiltonian system (HS) in the following sense. Let  $S_h$  and  $S_c$  are the subset of the phase space where  $H = h$  and  $C = c$ , respectively. Since  $C$  and  $H$  are first integrals, the sets  $S_c$ ,  $S_h$  and  $S_{hc} = S_h \cap S_c$  are invariant by the action of the flow of the Hamiltonian system (HS). We determine the global dynamics on those sets when the values of  $h$  and  $c$  vary.

Second recently Tudoran in [33] provided a criterion which detects when a non-degenerate equilibrium point of a completely integrable system is Lyapunov stable. Every equilibrium point  $q$  of the completely integrable Hamiltonian system (HS) is degenerate and has zero angular momentum, so the mentioned criterion cannot be applied to it. But we will show that this criterion is also satisfied when it is applied to the Hamiltonian system (HS) restricted to zero angular momentum.

*Keywords:* Kepler problem, Global dynamics, Lyapunov, Stability.

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## 1. Introduction

In celestial mechanics, the Kepler problem is a special dynamical system coming from the two-body problem. In this model, two objects move under their mutual Newtonian gravitational force [1]. This attractive force varies in size as the inverse square of the separation distance between them and it is proportional to the product of the masses of the two bodies. This system is used to find the position and the velocity vectors of the two bodies at specified time. Using the laws of classical mechanics, the solution can be provided as a Kepler orbit through finding the six orbital elements. This problem is called the Kepler problem in honor of the German astronomer Johannes Kepler, after he proposed Kepler's laws of the motion of the planets and illustrated the kinds of forces which can provide orbits obeying those laws [2].

The Kepler problem appears in various fields, some of these are beyond the physics, which have been studied by Kepler himself. This problem is important in celestial mechanics because the Newtonian gravity obeys the law of inverse square distance between two bodies. Thus for example, the motion of two stars around each other, the planets moving surrounding the Sun, a satellite orbiting a planet, and many other examples of orbital motion. The Kepler problem has also a significant relevance in the study of the motion of two charged particles, because Coulomb's law of electrostatics follows the law of inverse square distance too. For Example the hydrogen atom, muonium and positronium, which have played serious roles as models of dynamical systems to test physical theories and measuring constants of nature, see [3–5].

The importance of the Kepler problem is not only related to its varied applications in different fields of science, but rather to its use in developing some mathematical methods. Thus, this problem has been used to develop new methods in classical mechanics, like Hamiltonian mechanics, the Hamilton–Jacobi equation, Lagrangian me-

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*Email addresses:* eabouelmagd@gmail.com or elbaz.abouelmagd@nriag.sci.eg (Elbaz I. Abouelmagd), juan.garcia@upct.es (Juan Luis García Guirao), jllibre@mat.uab.cat (Jaume Llibre)

chanics and action-angle coordinates. Furthermore, the Kepler and simple harmonic oscillator problems are two of the most fundamental problems in classical mechanics. They are also the only integrable dynamical systems which have closed orbits for open sets of possible initial conditions

The Kepler problem is an unperturbed version form of the two-body problem, which must be extended or generalized to include to additional forces as perturbations, that represent realistic natural phenomena or to obtain precise and accuracy results. Within the frame of anisotropic perturbation in [6], the authors have showed that the anisotropic Kepler problem is equivalent to two massless particles which move in a plane or on perpendicular lines and which interact according to the Newton's law of gravitation. They have also proved that the generalized issue of anisotropic Kepler problem and anisotropic two centres problem are non-integrable systems. For an analysis of the Kepler problem, specific to radial periodic perturbation of a central force field, it has been proved the existence of rotating periodic solutions nearly circular orbits [7]. Implicit Function Theorem is used to find such solutions.

The solution of the Kepler problem allowed researchers to investigate the unperturbed and the perturbed Lagrange's planetary motion. Thus, these motions were clarified entirely by the classical mechanics and the Newton's law of universal gravitation [8, 9]. More recently, the perturbed Kepler problem have been investigated by [10, 11]. The perturbed models are not limited to Kepler models but also include the restricted three-body problem, which can be reduced to the perturbed Kepler problem in some cases [12–14].

The Manev Hamiltonian is

$$\mathcal{H} = \frac{1}{2} \left( p_r^2 + \frac{p_\phi^2}{r^2} \right) + \frac{a}{r} + \frac{b}{r^2},$$

where  $a$  and  $b$  are arbitrary constants. This Hamiltonian describes the motion of a two-body problem, which is governed by the potential  $a/r + b/r^2$ , where  $r$  is the separation distance between the two bodies and  $(p_r, p_\phi)$  are the momenta in polar coordinates.

The analysis of the motion in the two-body problems has a long history. Just after the Newton's work on the two-body gravitational problem, some discrepancies appeared between the theoretical motions of the pericenters of the planets and the observed ones. Consequently, some doubts on the accuracy of the inverse square law of Newton for gravitation. Motivated scientists to construct alternative gravitational models and corrections trying to reconcile these discrepancies. In fact, Newton was the first to consider what we now call the Manev systems, see the book I, section IX, proposition XLIV, theorem XIV and corollary 2 of the Principia.

Several authors tried to find or construct an appropriate model with the features of the Newtonian one but with the convenient corrections, to make closer theory and

observation.

There were several pre- and post-relativistic attempts to get such modified models. This is the case of the Manev Hamiltonian introduced by Manev in [15–18]. See for more information on the two-body problem, for instance, [19–23].

The correction given by the special relativity to the motion of the two-body problem, or by the first order correction to this problem coming from the general relativity is

$$H = \frac{1}{2} \left( p_r^2 + \frac{p_\phi^2}{r^2} \right) - \frac{1}{r} - \frac{\epsilon}{2r^2}, \quad (1)$$

where  $|\epsilon| \ll 1$  is a small parameter, for details see [24–28]. Note that, when  $\epsilon = 0$ , we have the rotating Kepler problem, for more information on this last problem see [29].

Note that the Hamiltonian  $H$  is a particular case of the Manev Hamiltonian  $\mathcal{H}$  when  $a = -1$  and  $b = -\epsilon/2$ .

The Hamiltonian system defined by the Hamiltonian  $H$  is

$$\begin{aligned} \dot{r} &= p_r, \\ \dot{\phi} &= \frac{p_\phi}{r^2}, \\ \dot{p}_r &= \frac{p_\phi^2 - \epsilon}{r^3} - \frac{1}{r^2}, \\ \dot{p}_\phi &= 0. \end{aligned} \quad (2)$$

For a fixed  $\epsilon < 0$  small, this Hamiltonian system has the circle of equilibria

$$\{q(\phi) = (r, \phi, p_\phi, p_r) = (-\epsilon, \phi, 0, 0) : \phi \in \mathbb{S}^1\}.$$

The Hamiltonian dynamical system (2) is perfectly integrable within frame the of Liouville–Arnold because it has two independent first integrals the Hamiltonian  $H$  and the angular momentum  $p_\phi$  in involution. For more details on perfectly integrable Hamiltonian systems see [30, 31].

The objective of this paper is double. Our first objective is to describe the global dynamics of the Hamiltonian system (2) in the following sense. Let  $S_h, S_c$  are the subset of the phase space,  $H = h$  and  $p_\phi = c$  where  $h$  and  $c$  are the constants of integration. Due to the fact that  $H$  and  $p_\phi$  are first integrals, the sets  $S_h, S_c$  and  $S_{hc} = S_h \cap S_c$  are invariant under the flow of the Hamiltonian system (2). We determine the global dynamics on those sets when  $h$  and  $c$  vary. Moreover, we describe the foliation of the phase space by the invariant sets  $S_h$ , and the foliation of  $S_h$  by the invariant sets  $S_{hc}$ . See section 2.

In fact, this first objective is a particular case of the general cases studied inside the Manev Hamiltonian  $\mathcal{H}$  with arbitrary values of  $a$  and  $b$ , see [32]. However, from that work, it is not easy to obtain the global dynamics for the particular case  $a = -1$  and  $b = -\epsilon/2$ , corresponding to our Hamiltonian (1). We cover this case in this work.

Second, we shall show that every equilibrium point  $q(\phi)$  of the Hamiltonian system (2) restricted to zero angular

momentum is Lyapunov stable. However, our interest in this Lyapunov stability rises in showing that the criterion provided recently by Tudoran in [33] which detects when a non-degenerate equilibrium point of a completely integrable system is Lyapunov stable, also can be extended to the degenerate equilibrium points of the completely integrable system (2) restricted to zero angular momentum, where the equilibrium points of system (2) live, see section 3.

## 2. On the global dynamics

### 2.1. Critical points and critical values

We will consider the notation and terminology of [32], as consequence, we suppose that  $S_h$  and  $S_c$  are the subset of the phase space where  $H = h$  and  $p_\phi$ . Since  $H$  and  $p_\phi$  are first integrals, the sets  $S_h$ ,  $S_c$  and  $S_{hc} = S_h \cap S_c$  are invariant under the flow of the Hamiltonian system (2), i.e. if an orbit of the Hamiltonian system has a point on the set  $S_{hc}$  then the whole orbit is contained in this set.

Let  $r \in \mathbb{R}^+ = (0, \infty)$ ,  $\phi \in \mathbb{S}^1$ ,  $(p_r, p_\phi) \in \mathbb{R}^2$  and  $\mathfrak{N} = \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R}^2$ , where

$$S_h = \{(r, \phi, p_r, p_\phi) \in \mathfrak{N} : H(r, \phi, p_r, p_\phi) = h\},$$

$$S_c = \{(r, \phi, p_r, p_\phi) \in \mathfrak{N} : p_\phi = c\}.$$

In this case the set of critical points of  $H$  is

$$\mathcal{C} = \{(r, \phi, p_r, p_\phi) \in \mathfrak{N} : r + \epsilon = 0, \phi \in \mathbb{S}^1\}$$

since  $r > 0$ , then

$$\mathcal{C} = \begin{cases} \emptyset & \text{if } \epsilon \geq 0, \\ -\epsilon & \text{if } \epsilon < 0. \end{cases}$$

hence the critical value of  $H$  is  $1/2\epsilon$  if  $\epsilon < 0$

### 2.2. Hill Regions

Let  $\mathfrak{N}$  and  $\mathcal{R} = \mathbb{R}^+ \times \mathbb{S}^1$  be the phase space and the configuration space of the Hamiltonian system (2), and let  $\Gamma : \mathfrak{N} \rightarrow \mathcal{R}$  be the projection from  $\mathfrak{N}$  to  $\mathcal{R}$ . Then for each  $h$  belongs to real set  $\mathbb{R}$ , the regions of motion  $R_h$  (Hill regions) of  $S_h$  are defined by  $\Gamma(S_h) = R_h$ , for more details see [32, 34], hence

$$R_h = \{(r, \phi) \in \mathcal{R} : -\frac{1}{r} - \frac{\epsilon}{2r^2} \leq h\}$$

$$= \{r \in \mathbb{R}^+ : 2hr^2 + 2r + \epsilon \geq 0\} \times \mathbb{S}^1.$$

Therefore, if  $h < 0$  then  $R_h$  is homeomorphic to

$$\begin{aligned} \emptyset & \text{if } \epsilon < \frac{1}{2h}, \\ \left\{-\frac{1}{h}\right\} \times \mathbb{S}^1 & \text{if } \epsilon = \frac{1}{2h}, \\ \left[-\frac{1 - \sqrt{1 - 2h\epsilon}}{h}, -\frac{1 + \sqrt{1 - 2h\epsilon}}{h}\right] \times \mathbb{S}^1 & \text{if } \frac{1}{2h} < \epsilon < 0, \\ \left(0, -\frac{1 + \sqrt{1 - 2h\epsilon}}{h}\right] \times \mathbb{S}^1 & \text{if } \epsilon \geq 0. \end{aligned} \quad (3)$$

If  $h = 0$  then  $R_h$  is homeomorphic to

$$\begin{cases} \left[-\frac{\epsilon}{2}, \infty\right) \times \mathbb{S}^1 & \text{if } \epsilon < 0, \\ \mathbb{R}^+ \times \mathbb{S}^1 & \text{if } \epsilon \geq 0. \end{cases} \quad (4)$$

If  $h > 0$  then  $R_h$  is homeomorphic to

$$\begin{cases} \left[\frac{\sqrt{1 - 2h\epsilon} - 1}{h}, \infty\right) \times \mathbb{S}^1 & \text{if } \epsilon < 0, \\ \mathbb{R}^+ \times \mathbb{S}^1 & \text{if } \epsilon \geq 0. \end{cases} \quad (5)$$

### 2.3. The sets $S_h$

Now we determine the topology of the invariant energy levels  $S_h$  by using the fact:

$$S_h = \{(r, \phi, p_r, p_\phi) \in \mathfrak{N} : g(r, p_r, p_\phi) = h\}$$

$$\approx \{g^{-1}(h)\} \times \mathbb{S}^1,$$

where

$$g(r, p_r, p_\phi) = \frac{1}{2} \left( p_r^2 + \frac{p_\phi^2}{r^2} \right) - \frac{1}{r} - \frac{\epsilon}{2r^2},$$

If  $h$  is a regular momentum value of the map  $g : \mathbb{R}^+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$  and the set of points  $\{g^{-1}(h)\} \neq \emptyset$ , then the set  $\{g^{-1}(h)\}$  is a surface in  $\mathbb{R}^+ \times \mathbb{R}^2$ . Hence the intersection of  $\{g^{-1}(h)\}$  with  $\{r = r_0 = \text{constant}\}$ , is

either an ellipse if  $\frac{1}{r_0} + \frac{\epsilon}{2r_0^2} + h > 0$ ,

or a point if  $\frac{1}{r_0} + \frac{\epsilon}{2r_0^2} + h = 0$ ,

or the empty set if  $\frac{1}{r_0} + \frac{\epsilon}{2r_0^2} + h < 0$ .

From the definition of the energy levels  $S_h$  we obtain

$$S_h = \bigcup_{(r, \phi) \in R_h} \mathfrak{N}_{(r, \phi)} \quad (6)$$

where

$$\mathfrak{N}_{(r, \phi)} = \{(r, \phi, p_r, p_\phi) \in \mathfrak{N} : p_r^2 + \frac{p_\phi^2}{r^2} = \frac{1}{4r^2} (2hr^2 + 2r + \epsilon)\}.$$

For every point  $(r, \phi) \in R_h$  the set  $\mathfrak{N}_{(r, \phi)}$  is an ellipse, a point or the empty set if the point  $(r, \phi)$  belongs to the interior of the Hill region  $R_h$ , to the boundary of  $R_h$ , or does not belong to  $R_h$ , respectively. Hence every energy level  $S_h$  of the planar relativistic Kepler problem is homeomorphic, from (3) and (6), to

$$\begin{aligned} \emptyset & \text{if } \epsilon < \frac{1}{2h}, \\ \mathbb{S}^1 & \text{if } \epsilon = \frac{1}{2h}, \\ \mathbb{S}^2 \times \mathbb{S}^1 & \text{if } \frac{1}{2h} < \epsilon < 0, \\ (\mathbb{S}^2 \times \mathbb{S}^1) \setminus \mathbb{S}^1 & \text{if } 0 \leq \epsilon, \end{aligned}$$

if  $h < 0$ ; and from (4), (5) and (6) to

$$\begin{aligned} (\mathbb{S}^2 \times \mathbb{S}^1) \setminus \mathbb{S}^1 & \text{ if } \epsilon < 0, \\ \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{S}^1 & \text{ if } \epsilon \geq 0, \end{aligned}$$

if  $h \geq 0$ .

#### 2.4. The sets $S_c$

Since  $S_c = \{(r, \phi, p_r, p_\phi) \in \mathfrak{N} : p_\phi = c\}$ , we get that  $S_c$  is homeomorphic to  $\mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R}$  for all  $c \in \mathbb{R}$ .

#### 2.5. The foliation of $S_h$ by $S_{hc}$

We can evaluate the invariant set  $S_{hc}$  from knowing the set  $\{g^{-1}(h)\}$  and

$$\begin{aligned} S_{hc} &= S_h \cap S_c \\ &= S_h \cap \{p_\phi = c\} \\ &= (\{g^{-1}(h)\} \cap \{p_\phi = c\}) \times \mathbb{S}^1 \end{aligned}$$

Hence the foliation of  $S_h$  by  $S_{hc}$  can be described when  $h$  varies through the following cases:

*Case 1:*  $h < 0$ . Then the surface  $g^{-1}(h)$  is the topological plane  $\mathbb{R}^2$  of Fig. 1(a). The curves  $\gamma_{hc} = \{g^{-1}(h)\} \cap \{p_\phi = c\}$  for each  $|c| \leq c_1 = \sqrt{(2\epsilon h - 1)/(2h)}$  are homeomorphic to:

- one component homeomorphic to  $\mathbb{R}$  if  $0 \leq c \leq c_2$ ,
- one component homeomorphic to  $\mathbb{S}^1$  if  $c_2 = \sqrt{\epsilon} < |c| < c_1$ , and
- one component homeomorphic to a point if  $|c| = c_1$ .

The manifold  $S_h$  is homeomorphic to a solid torus without its boundary. Hence we can find the dynamics on  $S_h$  by rotating Fig. 1(b) around the  $e$ -axis. From this figure we have

- one periodic orbit (topologically a circle)  $S_{hc}$  for  $|c| = c_1$ ,
- one two-dimensional torus  $S_{hc}$  for  $c_2 < |c| < c_1$ , and
- one cylinder  $S_{hc}$  for  $0 < |c| < c_2$ ,

which foliate  $S_h$ .

*Case 2:*  $h \geq 0$  and  $\epsilon \geq 0$ . Then  $g^{-1}(h)$  is homeomorphic to a cylinder  $\mathbb{R} \times \mathbb{S}^1$  see Fig. 2(a), and the curves  $\gamma_{hc}$  are formed by

- two components each of them homeomorphic to  $\mathbb{R}$  if  $c = 0$ , and
- one component homeomorphic to  $\mathbb{R}$  if  $c \geq 0$ .

The manifold  $S_h$  is homeomorphic to a solid torus without its boundary and without its central circular axis. Hence we can find the dynamics on  $S_h$  by rotating Fig. 2(b) around the  $e$  axis. From this figure we obtain one cylinder  $S_{hc}$  for every  $c \in \mathbb{R} \setminus \{0\}$ , and two cylinders  $S_{hc}$  for every  $c = 0$ , which foliate  $S_h$ .

*Case 3:*  $h \geq 0$  and  $\epsilon < 0$ . Now  $g^{-1}(h)$  is homeomorphic to a plane  $\mathbb{R}^2$  see Fig. 3(a), and the curves  $\gamma_{hc}$  are homeomorphic to  $\mathbb{R}$ .

The manifold  $S_h$  is homeomorphic to a solid torus without its boundary. The dynamics of  $S_h$  can be obtained by rotating Fig. 3(b) around the  $e$  axis. From this figure the foliation of  $S_h$  is done by the cylinders  $S_{hc}$  for every  $c \in \mathbb{R} \setminus \{0\}$ . Figures (1, 2, 3) already have been appeared in [32] in a more general context.

### 3. On the Lyapunov stability of the equilibria

Let  $\Upsilon_t(p)$  be the solution of a differential system (S) such that  $\Upsilon_t(p) = p$ , i.e.  $\Upsilon_t(p)$  is the flow defined by the system (S). An equilibrium point  $q$  of the differential system (S) is called *Lyapunov stable* if for each open neighborhood  $U$  of  $q$ , there exists an open neighborhood  $V \subseteq U$  of  $q$  where  $\Upsilon_t(p) \in U$  for any  $p \in V$  and any  $t \geq 0$ . An equilibrium state which is not Lyapunov stable is called *unstable*.

For  $\epsilon < 0$  sufficiently small we restrict the dynamics of the Hamiltonian system (2) to the space  $p_\phi = 0$ , where we have the circle of equilibria  $q(\phi)$  for  $\phi \in \mathbb{S}^1$ . Then we have the differential system

$$\begin{aligned} \dot{r} &= p_r, \\ \dot{\phi} &= 0, \\ \dot{p}_r &= -\frac{1}{r^2} - \frac{\epsilon}{r^3}. \end{aligned} \tag{7}$$

This differential system is completely integrable because it has the two functionally independent first integrals

$$C_1 = \phi \quad \text{and} \quad C_2 = \frac{1}{2}p_r^2 - \frac{1}{r} - \frac{\epsilon}{2r^2}.$$

According with [33] for our system (7) a non-degenerate equilibrium point is a point which satisfies that the determinant of the Hessian of the function  $C_2$  is non-zero at that equilibrium, see Definition 3.3 of [33]. Then every equilibrium point  $(-\epsilon, \phi, 0)$  of system (7) is degenerate, because the determinant of the Hessian of the function  $C_2$  at this equilibrium is zero. Indeed, the mentioned Hessian is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \frac{1}{\epsilon^3} & 0 & 0 \end{pmatrix} \bigg|_{(r, \phi, p_r) = (-\epsilon, \phi, 0)}, \tag{8}$$

and clearly its determinant is zero.

In [33] to a non-degenerate equilibria  $q$  is associated a real number  $I(q)$  as follows. If  $P(x)$  is the characteristic

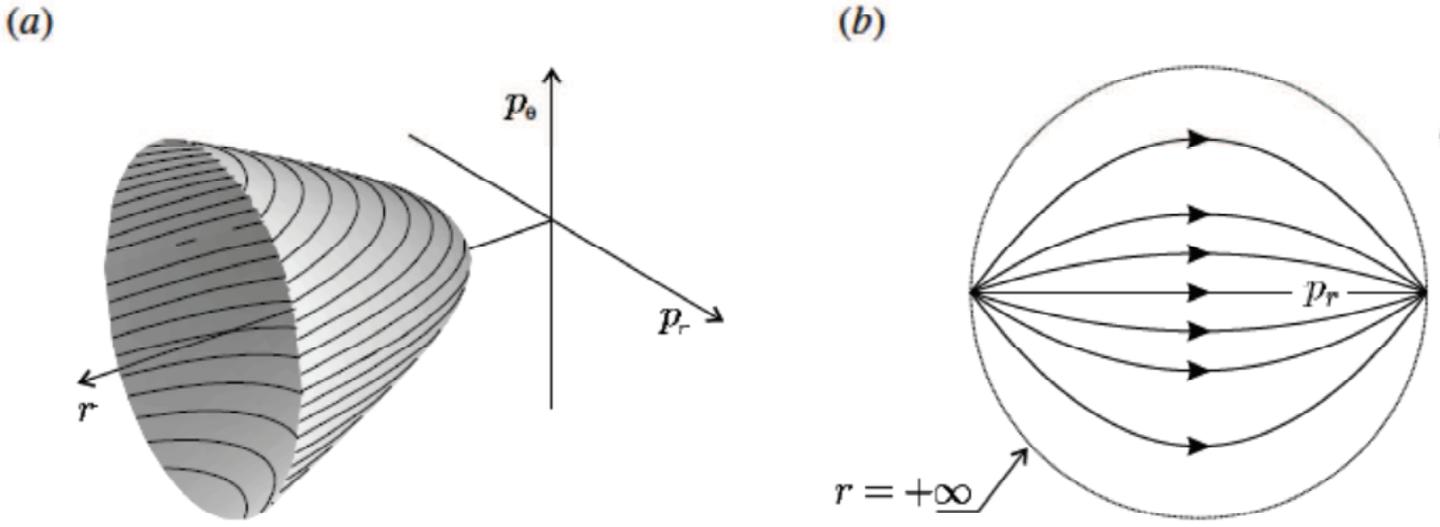


Figure 1: When  $h < 0$ : (a) The surface  $g^{-1}(h)$ . (b) Manifold  $S_h/\mathbb{S}^1$ .

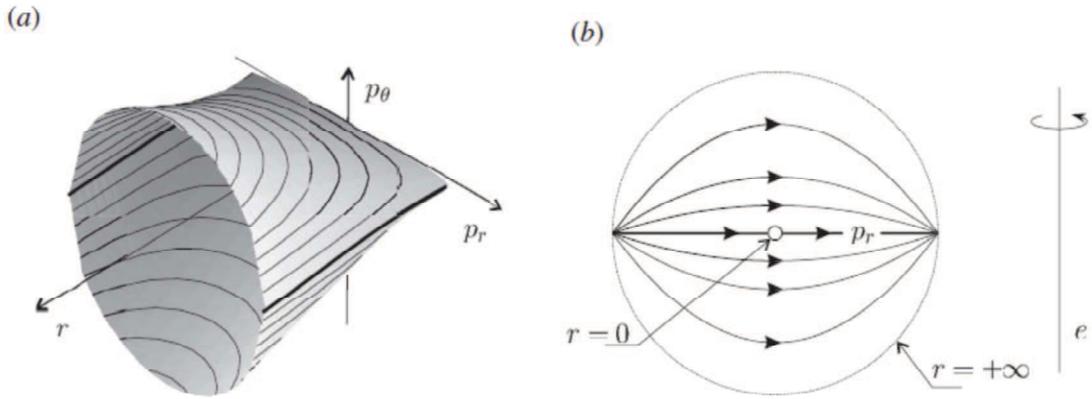


Figure 2: When  $h \geq 0$  and  $\epsilon \geq 0$ : (a) The surface  $g^{-1}(h)$ . (b) Manifold  $S_h/\mathbb{S}^1$ .

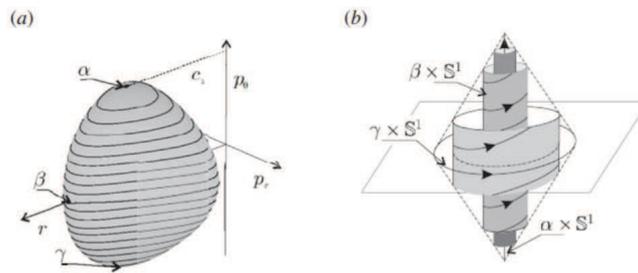


Figure 3: When  $h \geq 0$  and  $\epsilon < 0$ : (a) The surface  $g^{-1}(h)$ . (b) Manifold  $S_h/\mathbb{S}^1$ .

polynomial defined by the linear part of the complete integrable differential system at  $q$ , then  $P(x) = (-x)^{n-2}(x^2 + I(q))$  where  $n$  is the dimension of the differential system, see Theorem 5.4 of [33].

In Theorem 5.4 of [33] it is also shown that a non-

degenerate equilibrium point  $q$  is unstable if  $I(q) < 0$ , and in Theorem 5.6 it is proved that a non-degenerate equilibrium point  $q$  is Lyapunov stable if  $I(q) > 0$ .

From (8) the characteristic polynomial of the linear

part of system (7) at the equilibrium is  $(-\epsilon, \phi, 0)$  is

$$\lambda \left( \lambda^2 - \frac{1}{\epsilon^3} \right).$$

So applying the Tudoran criterium (which in fact we cannot apply because the equilibrium  $(-\epsilon, \phi, 0)$  is degenerate) we obtain that

$$I(-\epsilon, \phi, 0) = -\frac{1}{\epsilon^3} > 0. \quad (9)$$

because  $\epsilon < 0$ , and by the mentioned criterium the equilibrium  $(-\epsilon, \phi, 0)$  would be Lyapunov stable.

Now we shall prove that really the equilibrium point  $(-\epsilon, \phi, 0)$  is Lyapunov stable for all  $\phi \in \mathbb{S}^1$ . Consequently we have proved that the criterion of Tudoran also works for the degenerate equilibrium points of the completely integrable system (7).

We note that we cannot apply the so-called ‘‘Arnol’d stability test’’ to the equilibrium  $(-\epsilon, \phi, 0)$  because the second condition of that test does not hold, see the statement of this test in Theorem 5.5 of [33].

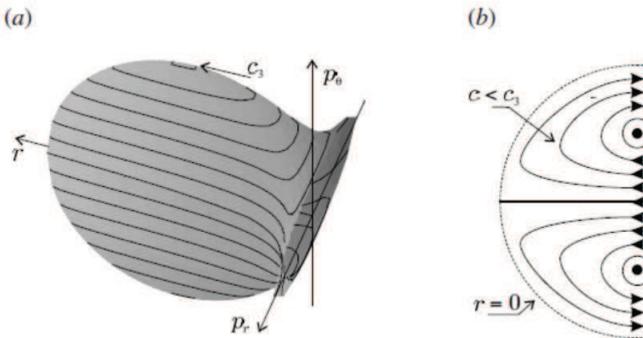


Figure 4: The graphic of the potential  $V(r) = -1/r - \epsilon/r^2$  for  $-\epsilon = 1/10$ .

From the differential system (7) it is clear that the variable  $\phi$  is constant for any solution. So for studying the Lyapunov stability at an equilibrium  $(-\epsilon, \phi, 0)$  of system (7) we can restrict to study it at the equilibrium  $(-\epsilon, 0)$  of the Hamiltonian system with one degree of freedom

$$\begin{aligned} \dot{r} &= p_r, \\ \dot{p}_r &= -\frac{1}{r^2} - \frac{\epsilon}{r^3}. \end{aligned} \quad (10)$$

with Hamiltonian  $C_2$ . Since the potential  $V(r) = -1/r - \epsilon/r^2$  of system (10) has the graphic of the Figure 4, and the minimum of this graphic takes place at  $r = -\epsilon$ , recall that  $\epsilon < 0$  is fixed and small, it follows that the equilibrium point  $(-\epsilon, 0)$  of system (10) is a center, i.e. all the orbits in a convenient neighborhood of it are periodic with the exception of the equilibrium point (see [35] for more information). Hence given any neighborhood  $U$  of  $(-\epsilon, 0)$  in the plane  $(r, p_r)$  there is another neighborhood  $V \subseteq U$

of  $(-\epsilon, 0)$  formed by sufficiently small periodic orbits surrounding the point  $(-\epsilon, 0)$ , and consequently contained in  $U$ . Hence the equilibrium point  $(-\epsilon, 0)$  is Lyapunov stable for system (10), and consequently for the system (7).

In summary, we have proved that the degenerate equilibrium points of the completely integrable system (7) satisfies the Tudoran criterium (see (9)) and are Lyapunov stable.

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