ON THE PERIODIC STRUCTURE OF THE RABINOVITCH-FABRIKANT SYSTEM

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Abstract. In this work we proof analytically the existence and stability of four families of periodic orbits of the Rabinovitch-Fabrikant system that born from a Zero-Hopf Bifurcation.

1. Introduction and Statements of the main results

The study of the periodic orbits of a differential equation is one of the main objectives of the qualitative theory of differential equations. The periodic orbits are trajectories of the system where the dynamics is completely determined because after a finite period of time the system return to its starting position. The periodic orbits are related with the integrability and chaoticity of the system. In general, the periodic orbits are studied numerically because, usually, their analytical study is very difficult. In this work we shall study analytically the periodic orbits of the three-dimensional Rabinovitch-Fabrikant system defined by

$$\begin{cases} \dot{x} = y\left(z - 1 + x^2\right) + ax, \\ \dot{y} = x\left(3z + 1 - x^2\right) + ay, \\ \dot{z} = -2z\left(b + xy\right). \end{cases} \tag{1}$$

where a, b are two positive real parameters.

This system was introduced in 1979 by Rabinovich and Fabrikant [14] and it was numerically investigated in [5]. The model was initially designed as a physical environment to describe the stochasticity arising from the modulation instability in a non-equilibrium dissipative medium. However, recently in [5] was revealed that beside the physical properties described in [14], the system presents some extremely rich dynamics as "virtual" saddles beside several chaotic attractors with different shapes and hidden chaotic attractors. In fact, the interest in this system is continuously increasing as can be seen in papers [11] or [16].

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Note that the Rabinovitch-Fabrikant (RF) system has some peculiarities that should be noted. It is a biparametric system with third-order polynomial nonlinearities. More significantly, unlike other nonlinear chaotic systems containing only second-order nonlinearities (such as the Lorenz system), the RF system with third-order polynomial nonlinearities, presents some unusual dynamics.

On the other hand, for $a = b = 0$, the system is integrable having a third-order polynomial algebraic integral and another transcendent integral. The flow of the system is obtained as foliation of these two integrals.

We will not discuss in detail the periodic structure of this integrable system with several families of degenerate equilibria. We can point out that there is a wide variety of families of periodic orbits depending on certain regions of the space where we set the initial conditions of the system. The existence of families of heteroclinic orbits associated with the four families of degenerate equilibria of the system will play a fundamental role in a detailed study of the existence of chaotic orbits for certain values of the parameters of the system.

It is natural to study values of the parameters different from $(0, 0)$ for which there are periodic orbits of the system. The first result of the present work is obtained for (a, b) close $(0, 0)$.

We prove, in this case, that the system has two families of periodic orbits. We note that when (a, b) is close to $(0, 0)$ the RF system undergoes a Zero-Hopf Bifurcation at the origin. We just recall that the occurrence of the classical Hopf bifurcation in $\mathbb{R}^3$ takes place in an equilibrium point with eigenvalues of the form $\pm\omega i$ and $\delta \neq 0$, while for a zero-Hopf bifurcation the eigenvalues are $\pm\omega i$ and 0. Here an equilibrium point with eigenvalues $\pm\omega i$ and 0 will be called a zero-Hopf equilibrium. The zero-Hopf bifurcation has been studied by many authors among them Guckenheimer, Holmes, Scheurle, Han, Kuznetsov, Llibre and Zhang in [7], [8], [9], [10], [13].

In this work we study the zero-Hopf bifurcation of the RF system as tool for obtaining periodic orbits of the system. We have inspired ourselves in the works of Llibre et al. in the study of the Zero-Hopf bifurcation of differential systems with quadratic polynomial nonlinearities.

To investigate the periodic structure of a system of differential equations, different analytical, semi-analytical or numerical theories have been developed. Within the analytical theories we can point out, for example, the averaging theory of dynamical systems, normal forms theory or the use of the fixed point theory and variational analysis as standard use in the scientific literature.In our case, we have decided to use the averaging theory because it made it easier for us to obtain asymptotic approximations of the initial conditions for periodic orbits. In addition, these asymptotic approximations of the initial conditions can be used, applying by example the Poincaré-Lindstedt method, to obtain a series expansion of harmonics of the period of the orbit of arbitrarily high order.

Remark 1. *The Rabinovich-Fabrikant system* (1) *has five equilibrium points, that is*

$$\begin{aligned}
O &= (0,0,0)\,, P_+ = \left(+M_-, -\frac{b}{M_-}, 1-\left(1-\frac{a}{b}\right)M_-^2\right),\\
P_- &= \left(-M_-, -\frac{b}{M_-}, 1-\left(1-\frac{a}{b}\right)M_-^2\right), Q_+ = \left(+M_+, -\frac{b}{M_+}, 1-\left(1-\frac{a}{b}\right)M_+^2\right)\\
Q_- &= \left(-M_+, -\frac{b}{M_+}, 1-\left(1-\frac{a}{b}\right)M_+^2\right), \text{ where}\\
M_- &= \sqrt{\frac{1-\sqrt{1-ab\left(1-\frac{3a}{4b}\right)}}{2\left(1-\frac{3a}{4b}\right)}}, M_+ = \sqrt{\frac{1+\sqrt{1-ab\left(1-\frac{3a}{4b}\right)}}{2\left(1-\frac{3a}{4b}\right)}}.
\end{aligned}$$

For more information about the equilibrium points of the Rabinovich-Fabrikant system see [4].

The statement of our main results is the following:

Proposition 2. *There is only a one-parameter family of Rabinovich-Fabrikant system for which the equilibrium point localized at the origin of coordinates is a zero-Hopf equilibrium point, i.e.* $a = b = 0$.

Our goal in the next result is to characterize when the equilibrium points

$$\begin{aligned}
Q_+ &= \left(+M_+, -\frac{b}{M_+}, 1-\left(1-\frac{a}{b}\right)M_+^2\right),\\
Q_- &= \left(-M_+, -\frac{b}{M_+}, 1-\left(1-\frac{a}{b}\right)M_+^2\right).
\end{aligned}$$

where $b > 0$, $a > 0$ and $4 - 4ab + 3a^2 \geq 0$, of the Rabinovich-Fabrikant system, are zero-Hopf equilibrium points.

Proposition 3. *If $b > 0$, $a > 0$ and $4 - 4ab + 3a^2 \geq 0$, there is only a one-parameter family of Rabinovich-Fabrikant system for which the equilibrium points Q_+ and Q_- are zero-Hopf equilibrium points, i.e. $a = b = 2$.*

Theorem 4. *Let $(a, b) = (\varepsilon\alpha, \varepsilon\beta)$ where $\alpha < 0, \beta < 0$ and ε are parameters being ε small. Then, the Rabinovitch-Fabrikant system (1) has a zero-Hopf bifurcation at the equilibrium point localized at the origin of coordinates, two periodic orbits born at this equilibrium when $\varepsilon = 0$ and it exists for $\varepsilon > 0$ sufficiently small, see Fig. (1).*

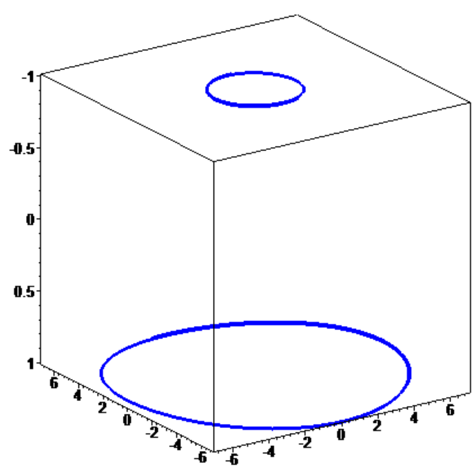

FIGURE 1. The periodic solutions of Theorem 4 for the values of the parameters $\alpha = -1$, $\beta = -0.5$ and $\varepsilon = 0.001$.

Theorem 5. *Let* $(a, b) = (2 + \varepsilon^2\alpha_1, 2 + \varepsilon^2\alpha_1)$ *with* $\alpha_1 < 0$ *and* ε *parameters being* ε *small. Then, the Rabinovitch-Fabrikant system* (1) *has a zero-Hopf bifurcation at the zero-Hopf equilibrium point localized at the equilibrium point* Q_+*, a periodic orbit born at this equilibrium when* $\varepsilon = 0$ *and it exists for* $\varepsilon > 0$ *sufficiently small, see Fig.* (2).

Theorem 6. *Let* $(a, b) = (2 + \varepsilon^2\alpha_1, 2 + \varepsilon^2\alpha_1)$ *with* $\alpha_1 < 0$ *and* ε *parameters being* ε *small. Then, the Rabinovitch-Fabrikant system* (1) *has a zero-Hopf bifurcation at the zero-Hopf equilibrium point localized at the equilibrium point* Q_-*, a periodic orbit born at this equilibrium when* $\varepsilon = 0$ *and it exists for* $\varepsilon > 0$ *sufficiently small, see Fig.* (2).

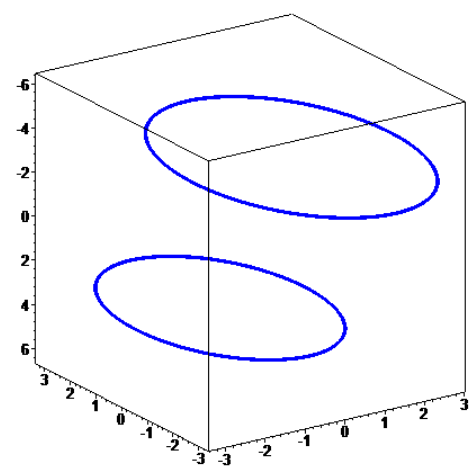

FIGURE 2. The periodic solutions of Theorem 5 and 6 for the values of the parameters $\alpha_1 = -200$ and $\varepsilon = 0.01$.

2. The averaging theory of first order and second order

In this section, we recall the main results of the averaging theory of dynamical systems of first and second order to find periodic orbits, see for more details [12] and [3] and the references therein. The averaging theory is an effective method for studying nonlinear differential equations, especially the study of their periodic orbits in terms of their number and stability. In general the method of averaging has a long history going back to the classic works of Lagrange and Laplace, who provided an intuitive justification for the process. The first formalization of this procedure is due to Fatou [6] in 1928. Important practical and theoretical contributions in this theory were made by Krylov and Bogoliubov [1] in the 1930 and Bogoliubov [2] in 1945.

Theorem 7. *Consider the differential equation*

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{2}$$

with $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n, R : \mathbb{R} \times D \times (0, \varepsilon_f) \to \mathbb{R}^n$ *are continuous functions,* T *-periodic in the first variable, and* D *is an open subset of* $\mathbb{R}^n$*. We assume that*
(i) $F_1(t,\cdot), F_2(t,\cdot) \in C^1(D)$ *for all* $t \in \mathbb{R}, F_1, F_2, R, D_x F_1$ *and* $D_x F_2$ *are locally lipschitz with respect to* x*, and* R *is differentiable with respect to* ε*.*
We define $f_1, f_2 : D \to \mathbb{R}^n$ *as*

$$f_1(z) = \frac{1}{T}\int_0^T F_1(s,z)ds,$$
$$f_2(z) = \frac{1}{T}\int_0^T \left[D_z F_1(s,z)\int_0^s F_1(t,z)dt + F_2(s,z)\right] ds.$$

(ii) For $V \subset D$*, an open and bounded set and for each* $\varepsilon \in (0,\varepsilon_f)\setminus\{0\}$*, there exists* $p \in V$ *such that* $f_1(p) + \varepsilon f_2(p) = 0$ *and*

$$det\left(\left.\frac{\partial(f_1+\varepsilon f_2)}{\partial z}\right|_{z=p}\right) \neq 0. \tag{3}$$

Then, for $\varepsilon > 0$ *sufficiently small, there exists a* $T-$ *periodic solution* $\varphi(t,\varepsilon)$ *of system (2) such that* $\varphi(0,\varepsilon) \to p$ *when* $\varepsilon \to 0$*. If the function* f_1 *is not identically zero, then the zeros of* $f_1 + \varepsilon f_2$ *are mainly the zeros of* f_1 *for* ε *sufficiently small.*

In this case, Theorem 7 provides the so-called averaging theory of first order.

If the function f_1 is identically zero and f_2 is not identically zero, then the zeros of $f_1 + \varepsilon f_2$ are the zeros of f_2. In this case, Theorem 7 provides the so-called averaging theory of second order. In the frame of the averaging theory of first order, we shall consider in D the averaged differential equation

$$\dot{y} = \varepsilon f_1(y), \quad y(0) = x_0, \tag{4}$$

where

$$f_1(y) = \frac{1}{T}\int_0^T F_1(t, y)dt. \tag{5}$$

Then, Theorem 7 provides us information about the stability or instability of the limit cycle $\varphi(t, \varepsilon)$. In fact, it is given by the stability or instability of the equilibrium point p of the averaged system (4). In fact, the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\varphi(t, \varepsilon)$. In the case of the averaging theory of second order, that is, $f_1 \equiv 0$ and f_2 non-identically zero, we have that the stability and instability of the limit cycle $\varphi(t, \varepsilon)$ coincide with the type of stability or instability of the equilibrium point p of the averaged system

$$\dot{y} = \varepsilon^2 f_2(y), \quad y(0) = x_0,$$

that is, it is the same that the singular point p associated the Poincar é map of the limit cycle $\varphi(t, \varepsilon)$. For additional information on averaging theory, see the book [15].

3. Proofs of the main results

3.1. **Proofs of Propositions 2 and 3.** Note that the characteristic polynomial of the linear part of the Rabinovich-Fabrikant system at the origin of coordinates is

$$p(\lambda) = (\lambda + 2b)\left(\lambda^2 - 2a\lambda + a^2 + 1\right).$$

Since we must have one null eigenvalue, we consider $b = 0$. Now, we must choose the other two eigenvalues in the form $\pm i\omega$, then

$$p(\lambda) = \lambda\left(\lambda^2 + \omega^2\right),$$

and we must have

$$a = 0, \omega^2 = 1,$$

Now, we observe the characteristic polynomial of the linear part of the Rabinovich-Fabrikant system at the points Q_+ and Q_-

$$\begin{aligned} p(\lambda) &= \lambda^3 - \frac{1}{M_+^2 b^2}\left(-2\,M_+^2 b^3 + 2\,b^2 M_+^2 a\right)\lambda^2 \\ &-\frac{1}{M_+^2 b^2}(-2\,b^4 + 6\,M_+^6 b^2 + 3\,M_+^6 a^2 - 6\,M_+^4 b^2 \\ &+2\,M_+^2 b^4 - 12\,M_+^6 ba - M_+^2 a^2 b^2 + 4\,bM_+^4 a)\lambda \\ &-\frac{1}{M_+^2 b^2}(18\,M_+^6 ba^2 - 32\,M_+^4 b^3 + 24\,M_+^6 b^3 - 2\,M_+^2 b^4 a \\ &+2\,b^4 a + 2\,M_+^2 a^2 b^3 + 8\,M_+^2 b^3 + 26\,M_+^4 b^2 a - 42\,M_+^6 b^2 a) \end{aligned}$$

Again, we impose that the roots of Q_+ and $Q_-$are 0 and the other two eigenvalues have the form $\pm i\omega$, so the following conditions must hold

$$\begin{array}{l} 18\,M_+^6 ba^2 - 32\,M_+^4 b^3 + 24\,M_+^6 b^3 - 2\,M_+^2 b^4 a + 2\,b^4 a \\ +2\,M_+^2 a^2 b^3 + 8\,M_+^2 b^3 + 26\,M_+^4 b^2 a - 42\,M_+^6 b^2 a = 0, \\ -2\,M_+^2 b^3 + 2\,b^2 M_+^2 a = 0 \end{array}$$

and

$$\begin{aligned} \omega^2 &= -\frac{1}{M_+^2 b^2}(-2\,b^4 + 6\,M_+^6 b^2 + 3\,M_+^6 a^2 - 6\,M_+^4 b^2 \\ &\quad +2\,M_+^2 b^4 - 12\,M_+^6 ba - M_+^2 a^2 b^2 + 4\,bM_+^4 a). \end{aligned}$$

From these conditions

$$\left\{\begin{array}{l} 18\,M_+^6 ba^2 - 32\,M_+^4 b^3 + 24\,M_+^6 b^3 - 2\,M_+^2 b^4 a + 2\,b^4 a \\ +2\,M_+^2 a^2 b^3 + 8\,M_+^2 b^3 + 26\,M_+^4 b^2 a - 42\,M_+^6 b^2 a = 0, \\ -2\,M_+^2 b^3 + 2\,b^2 M_+^2 a = 0 \\ -(-2\,b^4 + 6\,M_+^6 b^2 + 3\,M_+^6 a^2 - 6\,M_+^4 b^2 \\ +2\,M_+^2 b^4 - 12\,M_+^6 ba - M_+^2 a^2 b^2 + 4\,bM_+^4 a) < 0, \\ \text{ and } b > 0, a > 0, 4 - 4ab + 3a^2 \geq 0, \end{array}\right.$$

we conclude that $a = b = 2$, ending the proofs of Propositions 2 and 3, respectively. ∎

3.2. **Proof of Theorem 4.** If we consider $(a, b) = (\varepsilon\alpha, \varepsilon\beta)$ with $\varepsilon \geq 0$ a parameter sufficiently small, then Rabinovich-Fabrikant system becomes

$$\left\{\begin{array}{l} \dot{x} = -y + y\,(z + x^2) + \varepsilon\alpha x, \\ \dot{y} = x + x\,(3z - x^2) + \varepsilon\alpha y, \\ \dot{z} = -2z\,(\varepsilon\beta + xy)\,. \end{array}\right. \tag{6}$$

The eigenvalues at the origin of the linear part of system (6) when $\varepsilon = 0$ are 0 and $\pm i$.

By the rescaling of variables $(x,y,z)=(\sqrt{\varepsilon}X,\sqrt{\varepsilon}Y,\varepsilon Z)$, the system (6) becomes

$$\begin{cases} \dot{X}=-Y+\varepsilon\left(YZ+YX^2+\alpha X\right), \\ \dot{Y}=X+\varepsilon\left(3XZ-X^3+\alpha Y\right), \\ \dot{Z}=-2\varepsilon\left(\beta Z\right)-2\varepsilon\left(XYZ\right). \end{cases} \tag{7}$$

Now, we write this differential system in cylindrical coordinates (r,θ,w) defined by $X=r\cos\theta$, $Y=r\sin\theta$, $Z=Z$. Considering θ as the new independent variable, we arrive to the following system

$$\begin{aligned} \frac{dr}{d\theta} &= (r\left(4\,\cos(\theta)\sin(\theta)\,Z+\alpha\right)\varepsilon - r(16\,\left(\cos(\theta)\right)^3\sin(\theta)\,Z^2 \\ &\quad -4\,\cos(\theta)\sin(\theta)\,Z^2-4\,\left(\cos(\theta)\right)^3\sin(\theta)\,Zr^2 \\ &\quad +4\,\alpha\,\left(\cos(\theta)\right)^2 Z-\alpha\,Z-\alpha\,r^2\left(\cos(\theta)\right)^2)\varepsilon^2+O\left(\varepsilon^3\right), \\ &= \varepsilon F_{11}\left(\theta,r,Z\right)+\varepsilon^2F_{21}\left(\theta,r,Z\right)+O\left(\varepsilon^3\right), \\ \frac{dZ}{d\theta} &= (-2\,Z\left(\beta+r^2\cos(\theta)\sin(\theta)\right)\varepsilon+2\,Z(4\,\beta\,\left(\cos(\theta)\right)^2 Z-\beta\,Z \\ &\quad -\beta\,r^2\left(\cos(\theta)\right)^2+4\,\left(\cos(\theta)\right)^3\sin(\theta)\,Zr^2 \\ &\quad -r^2\cos(\theta)\sin(\theta)\,Z-r^4\left(\cos(\theta)\right)^3\sin(\theta))\varepsilon^2+O\left(\varepsilon^3\right), \\ &= \varepsilon F_{12}\left(\theta,r,Z\right)+\varepsilon^2F_{22}\left(\theta,r,Z\right)+O\left(\varepsilon^3\right). \end{aligned} \tag{8}$$

Our previous system has the form of the differential equation (8) with $t=\theta$, $x=(r,Z)\in D=(0,+\infty)\times\mathbb{R}$, $T=2\pi$, and $F_1\left(\theta,r,Z\right)=\left(F_{11}\left(\theta,r,Z\right),F_{12}\left(\theta,r,Z\right)\right)$, $F_2\left(\theta,r,Z\right)=\left(F_{21}\left(\theta,r,Z\right),F_{22}\left(\theta,r,Z\right)\right)$. An easy computation shows that

$$f_1\left(r,Z\right)=\left(f_{11}\left(r,Z\right),f_{12}\left(r,Z\right)\right)$$

is given by

$$\begin{aligned} f_{11}\left(r,Z\right) &= \frac{1}{2\pi}\int_0^{2\pi}F_{11}\left(\theta,r,Z\right)d\theta=\alpha r, \\ f_{12}\left(r,Z\right) &= \frac{1}{2\pi}\int_0^{2\pi}F_{12}(\theta,r,Z)d\theta=-2\beta Z, \end{aligned}$$

Note here that the averaging theory of first order, cannot be used.

For applying the averaging theory of second order, we must compute the expression

$$\begin{pmatrix} \frac{\partial F_{11}}{\partial r} & \frac{\partial F_{11}}{\partial Z} \\ \frac{\partial F_{12}}{\partial r} & \frac{\partial F_{12}}{\partial Z} \end{pmatrix}\begin{pmatrix} \int_0^\theta F_{11}(s,r,Z)ds \\ \int_0^\theta F_{12}(s,r,Z)d\theta \end{pmatrix}+\begin{pmatrix} F_{21}(\theta,r,Z) \\ F_{22}(\theta,r,Z \end{pmatrix}.$$

By computing the integral of this expression between 0 and 2π and dividing by 2π, we obtain

$$f_2(r,Z)=\left(\begin{array}{c} f_{21}(r,Z)\\ f_{22}(r,Z)\end{array}\right)=\left(\begin{array}{c}\pi\, r\alpha^2-r\alpha\, Z+r^3\alpha/2\\ Zr^2\alpha+2\,\beta\, Z^2-\beta\, Zr^2\end{array}\right).$$

Now we aim to solve the system of equations

$$\begin{array}{rcl} f_{21}(r,Z) &=& 0,\\ f_{22}(r,Z) &=& 0.\end{array}\tag{9}$$

By solving the first equation with respect to Z, we obtain the solution

$$Z=r^2/2+\alpha\pi,$$

By substituting it in second equation, we find that

$$r^2\alpha^2\pi+1/2\,r^4\alpha+2\,\beta\,\alpha^2\pi^2+\beta\,\alpha\, r^2\pi=0.$$

Solving this equation with respect to r, we obtain the solutions

$$r_1=\sqrt{-2\beta\pi}, r_2=-\sqrt{-2\beta\pi}, r_3=\sqrt{-2\alpha\pi}, r_4=-\sqrt{-2\alpha\pi}.$$

Since $\alpha<0$ and $\beta<0$, by choosing completely positive roots we find

$$r_1=\sqrt{-2\beta\pi}, r_3=\sqrt{-2\alpha\pi}.$$

The system $f_{21}(r,Z)=f_{22}(r,Z)=0$ has two solutions $(\sqrt{-2\beta\pi},(\alpha-\beta)\,\pi)$, $\left(\sqrt{-2\alpha\pi},0\right)$.

The Jacobian at $(\sqrt{-2\beta\pi},(\alpha-\beta)\,\pi)$, $\left(\sqrt{-2\alpha\pi},0\right)$ takes the value

$$\begin{array}{rcl}\det\left.\dfrac{\partial(f_{21},f_{22})}{(r,Z)}\right|_{(r,Z)=\left(\sqrt{-2\beta\pi},(\alpha-\beta)\pi\right)} &=& 4\pi^2\alpha^2\beta\,(\beta-\alpha)\,,\\ \det\left.\dfrac{\partial(f_{21},f_{22})}{(r,Z)}\right|_{(r,Z)=\left(\sqrt{-2\alpha\pi},0\right)} &=& 4\pi^2\alpha^3\,(\alpha-\beta)\,,\end{array}\tag{10}$$

In short the solutions $(\sqrt{-2\beta\pi},(\alpha-\beta)\,\pi)$, $\left(\sqrt{-2\alpha\pi},0\right)$ of system (9) which verify condition (10) satisfy the assumptions (i) and (ii) of Theorem 7. So, we conclude that applying the averaging theory of second order system (8) has two periodic orbits. From it we conclude the Rabinovitch-Fabrikant differential system (1) has a zero-Hopf bifurcation at the zero-Hopf equilibrium point localized at the origin of coordinates, two periodic orbits born at this equilibrium when $\varepsilon=0$ and it exists $\varepsilon>0$ sufficiently small and the proof is over.

∎

Remark 8. *We underline the following facts that can be obtained as consequences of the application of Theorem 7:*

(1) *The Jacobian matrix at* $\left(r_1=\sqrt{-2\pi\alpha},\ Z_1=0\right)$ *of the averaged equations* $\left(f_{21}\left(r,Z\right),f_{22}\left(r,Z\right)\right)$ *are*

$$\left(\begin{array}{cc} -2\pi\alpha^2 & \sqrt{2\pi}(-\alpha)^{3/2} \\ 0 & 2\pi\alpha\left(\beta-\alpha\right) \end{array}\right),$$

with eigenvalues $\lambda_1=-2\pi\alpha^2$ *and* $\lambda_2=2\pi\alpha\left(\beta-\alpha\right)$,

(2) *The Jacobian matrix at* $\left(r_2=\sqrt{-2\pi\beta},\ Z_2=2\pi\alpha\left(\alpha-\beta\right)\right)$ *of the averaged equations* $\left(f_{21}\left(r,Z\right),f_{22}\left(r,Z\right)\right)$ *are*

$$\left(\begin{array}{cc} -2\pi\alpha\beta & -\sqrt{2\pi}\alpha\sqrt{-\beta} \\ 2\sqrt{2}\pi^{3/2}\sqrt{-\beta}\left(\alpha-\beta\right)^2 & 2\pi\beta\left(\alpha-\beta\right) \end{array}\right),$$

with eigenvalues

$$\lambda_1=-\pi\left(\beta^2+\sqrt{\beta\left(4\alpha^3-4\alpha^2\beta+\beta^3\right)}\right),$$

and

$$\lambda_2=\pi\left(\beta^2-\sqrt{\beta\left(4\alpha^3-4\alpha^2\beta+\beta^3\right)}\right).$$

(3) *The periodic orbit with initial condition* $r_1=\sqrt{-2\pi\alpha}$, $Z_1=0$ *is linearly stable in the parametric region*

$$R_1=\{(\alpha,\beta)/\alpha<0,\beta<0,\beta-\alpha>0\},$$

and unstable in the parametric region

$$R_2=\{(\alpha,\beta)/\alpha<0,\beta<0,\beta-\alpha<0\}.$$

(4) *The periodic orbit with initial condition* $r_2=\sqrt{-2\pi\beta}$, $Z_2=\pi(\alpha-\beta)$ *is linearly stable in the parametric region*

$$R_1=\{(\alpha,\beta)/\alpha<0,\beta<0,\beta-\alpha>0\},$$

and unstable in the parametric region

$$R_2=\{(\alpha,\beta)/\alpha<0,\beta<0,\beta-\alpha<0\}.$$

3.3. **Proofs of Theorems 5 and 6.** Consider (a,b) of the form $(2+\varepsilon^2\alpha_1,2+\varepsilon^2\alpha_1)$ with $\alpha_1<0$ and $\varepsilon>0$ sufficiently small.

We translate the equilibrium point Q_+ to the origin of coordinates and, maintaining the notation (x,y,z) for the new coordinates, then

Rabinovitch Fabrikant system (1) becomes

$$
\left\{
\begin{aligned}
&\dot{x} = -\left(y - \frac{2+\varepsilon^2\alpha_1}{\sqrt{2+\sqrt{4-(2+\varepsilon\alpha_1)^2}}}\right) + \left(y - \frac{2+\varepsilon^2\alpha_1}{\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}}\right)((z+1\\
&+\left(x+\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}\right)^2\right) + (2+\varepsilon^2\alpha_1)\left(x+\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}\right),\\
&\dot{y} = x+\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}\\
&+\left(x+\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}\right)\left(3\,(z+1)-\left(x+\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}\right)^2\right)\\
&+(2+\varepsilon\alpha_1)\left(y - \frac{2+\varepsilon^2\alpha_1}{\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}}\right),\\
&\dot{z} = -2\,(z+1)\left(2+\varepsilon^2\alpha_1+\left(x+\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}\right)(y\right.\\
&\left.-\frac{2+\varepsilon^2\alpha_1}{\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}}\right)\Bigg).
\end{aligned}
\right.
\tag{11}
$$

The eigenvalues at the origin of the linear part of system (11) when $\varepsilon = 0$ are 0 and $\pm 4i$.

By the rescaling of variables $(x, y, z) = (\varepsilon X, \varepsilon Y, \varepsilon Z)$, the system (11) becomes

$$
\left\{
\begin{aligned}
&\dot{X} = \frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(2\,Y\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right.\\
&+Y\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-2\,Z\\
&\left.-2\,X\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right)+\frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(4\,YX\right.\\
&\left.+YZ\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-2\,X^2+2\,YX\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\right)\varepsilon\\
&+\frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(YX^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right.\\
&\left.-\alpha_1Z-\alpha_1X\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right)\varepsilon^2-\left(\frac{\alpha_1X^2}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\right)\varepsilon^3,\\
&\dot{Y} = -\frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(2\,X\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right.\\
&+3\,X\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&\left.-6\,Z-3\,Z\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}-2\,Y\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right)\\
&-\frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(-3\,XZ\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+6\,X^2\right.\\
&\left.+3\,X^2\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\right)\varepsilon-\frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(X^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right.\\
&\left.-\alpha_1Y\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right)\varepsilon^2,\\
&\dot{Z} = -\frac{2}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(2\,Y+Y\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}-2\,X\right)\\
&-\frac{2}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(2\,ZY+ZY\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\right.\\
&\left.-2\,XZ+XY\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right)\varepsilon-\frac{2}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(-X\alpha_1\right.\\
&\left.+ZXY\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right)\varepsilon^2+\left(\frac{2ZX\alpha_1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\right)\varepsilon^3.
\end{aligned}
\right.
\tag{12}
$$

We need to write the linear part at the origin of the system (12) when $\varepsilon = 0$ into its real Jordan normal form, i.e. into the form

$$J = \begin{pmatrix} 0 & -4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the new variables (u, v, w) defined by

$$\begin{cases} X = \left(\dfrac{1-2\sqrt{2}}{12}\right) u - \left(\dfrac{2+\sqrt{2}}{12}\right) v + \dfrac{\sqrt{2}}{8} w, \\ Y = -\left(\dfrac{2\sqrt{2}+3}{12}\right) u + \left(\dfrac{3\sqrt{2}-2}{12}\right) v + \dfrac{\sqrt{2}}{8} w, \\ Z = \dfrac{1}{3} u + \dfrac{\sqrt{2}}{6} v, \end{cases}$$

the system (12) becomes

$$\begin{aligned}
\dot{u} = &-\frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\Big(-\tfrac{2}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}+4\sqrt{2}v \\
&+\tfrac{3}{2}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}v-\tfrac{2}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v \\
&-\tfrac{1}{2}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
&+\tfrac{2}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
&+\tfrac{2}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
&+\tfrac{1}{2}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}w\Big)-\frac{\varepsilon}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\Big(-\tfrac{85}{144}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u^2 \\
&-\tfrac{5}{32}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}w^2+\tfrac{17}{72}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v^2 \\
&+\tfrac{1}{8}w^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{11}{36}u^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
&-\tfrac{2}{9}v^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{6}\sqrt{2}v^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
&-\tfrac{1}{3}uv\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{4}vw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
&-\tfrac{1}{3}uw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{5}{24}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}vw
\end{aligned}$$

$$
\begin{aligned}
&-\tfrac{17}{36}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}uv+\tfrac{5}{12}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}uw\\
&-\tfrac{7}{36}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u^2\sqrt{2}-\tfrac{5}{36}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}v^2\\
&-\tfrac{11}{72}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}v-\tfrac{1}{6}v\sqrt{2}w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{11}{36}u\sqrt{2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{7}{48}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}w\\
&+\tfrac{5}{24}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v\sqrt{2}w-\tfrac{23}{18}u^2+\tfrac{1}{3}v\sqrt{2}w\\
&-\tfrac{1}{9}u\sqrt{2}v+\tfrac{2}{3}uw+\tfrac{7}{9}v^2-\tfrac{1}{4}w^2\Big)\\
&-\frac{\varepsilon^2}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\Bigg(\tfrac{19}{144}uv^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{23}{432}u^3\sqrt{2}\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{\sqrt{2}}{128}w^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{1}{32}u\sqrt{2}w^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{7}{216}v^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{11}{288}u^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{\sqrt{2}}{144}v^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{3}\alpha_1v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{1}{6}\alpha_1u\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{32}vw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{5}{72}v^2w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{7}{192}uw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{13}{192}u^2\sqrt{2}w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{6}\alpha_1\sqrt{2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{1}{16}uvw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{17\sqrt{2}}{144}uvw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{1}{96}v^2\sqrt{2}w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{43\sqrt{2}}{288}u^2v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{1}{36}u\sqrt{2}v^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{\sqrt{2}}{4}\alpha_1w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{5}{192}\sqrt{2}vw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{\sqrt{2}}{3}\alpha_1u\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{7}{72}u^2w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{16}u^2v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{1}{2}\alpha_1\sqrt{2}v+2/3\,\alpha_1u\sqrt{2}-\tfrac{1}{2}\alpha_1\sqrt{2}w+\tfrac{2}{3}\alpha_1v\\
&-\frac{\varepsilon^3}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\Bigg(\frac{7}{72}\alpha_1u\sqrt{2}v+\frac{11}{72}\alpha_1v^2+1/32\,\alpha_1w^2\\
&+\frac{17}{36}\alpha_1uv-1/12\,\alpha_1uw-\frac{5}{24}\alpha_1vw-\frac{7}{48}\alpha_1u\sqrt{2}w\\
&+\frac{5}{36}\alpha_1\sqrt{2}v^2+\frac{7}{36}\alpha_1u^2\sqrt{2}-\frac{7}{144}\alpha_1u^2-1/24\,\alpha_1v\sqrt{2}w\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\dot{v} = {} & \frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\Big(\tfrac{3}{2}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2} \\
& +\tfrac{2}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u-\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{4}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+4\,u\sqrt{2} \\
& -\tfrac{1}{2}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}w+\tfrac{2\sqrt{2}}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{1}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}v\Big)+\frac{\varepsilon}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\Big(-\tfrac{5\sqrt{2}}{32}\sqrt{-\alpha_1\left(4+\alpha_1\epsilon^2\right)\epsilon^2}w^2 \\
& +\tfrac{7}{72}\,u^2\sqrt{2}\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{16}\sqrt{2}w^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{2\sqrt{2}}{3}uw+\tfrac{1}{18}u^2\sqrt{2}-\tfrac{5}{9}\sqrt{2}v^2-\tfrac{\sqrt{2}}{4}w^2+\tfrac{5}{18}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u^2 \\
& +\tfrac{5}{12}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}w-\tfrac{\sqrt{2}}{6}\,uv\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{7}{36}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}v-\tfrac{1}{24}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v\sqrt{2}w \\
& -\tfrac{1}{3}u^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{2}{3}vw-\tfrac{14}{9}\,uv \\
& -\tfrac{5}{24}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}uw-\tfrac{5}{9}uv\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{1}{6}vw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{4}\,uw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& -\tfrac{29}{36}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}uv-\tfrac{13}{144}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u^2\sqrt{2} \\
& +\tfrac{5}{12}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}vw-\tfrac{5\sqrt{2}}{36}\,v^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& -\tfrac{19}{72}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}v^2+\tfrac{\sqrt{2}}{6}\,uw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{1}{18}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v^2+\frac{\varepsilon^2}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\Big(\tfrac{1}{2}\,\alpha_1 w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{\sqrt{2}}{32}vw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{\sqrt{2}}{3}\,\alpha_1 v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{7}{72}\,uvw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{\sqrt{2}}{6}\alpha_1 u\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& -\tfrac{5\sqrt{2}}{192}\,uw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{5\sqrt{2}}{144}\,uv^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{5}{72}\,u^2\sqrt{2}w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{16}u^2\sqrt{2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{1}{72}\,v^2\sqrt{2}w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{216}\,u^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{1}{72}\,v^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{64}\,w^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{1}{2}\,\alpha_1 w-\tfrac{2}{3}\alpha_1 u-\tfrac{\sqrt{2}}{3}\,\alpha_1 v+\tfrac{\sqrt{2}}{2}\alpha_1 u \\
& -\tfrac{23}{144}\,u^2v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{9}uv^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& -\tfrac{5}{96}\,u^2w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{16}\,uw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& -\tfrac{1}{16}v^2w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{2}{3}\,\alpha_1 u\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}
\end{aligned}
$$

$$-\tfrac{1}{16}v^2w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{2}{3}\alpha_1u\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$+\tfrac{1}{3}\alpha_1v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{96}\,vw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$+\tfrac{5\sqrt{2}}{216}v^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{\sqrt{2}}{32}u^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$\left.-\tfrac{5\sqrt{2}}{48}\,uvw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right)$$
$$+\frac{\varepsilon^3}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(\tfrac{17\sqrt{2}}{144}\,\alpha_1u^2+\tfrac{1}{36}\alpha_1uv+\tfrac{\sqrt{2}}{24}\,\alpha_1vw\right.$$
$$-\tfrac{7\sqrt{2}}{36}\,\alpha_1uv-\tfrac{\sqrt{2}}{12}\alpha_1uw-\tfrac{\sqrt{2}}{72}\,\alpha_1v^2-\tfrac{1}{12}\alpha_1vw+\tfrac{\sqrt{2}}{32}\alpha_1w^2$$
$$\left.-\tfrac{1}{18}\,\alpha_1v^2-\tfrac{5}{18}\,\alpha_1u^2+\tfrac{5}{24}\,\alpha_1uw\right),$$
$$\dot{w}=-\frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(-\tfrac{7}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v\right.$$
$$-\tfrac{7}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}-2\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$-2\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u-\tfrac{16}{3}u+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}w$$
$$+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{16\sqrt{2}}{3}v$$
$$+2\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}v-\tfrac{16}{3}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$\left.+8/3\,u\sqrt{2}\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right)$$
$$-\frac{\varepsilon}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(-\tfrac{7\sqrt{2}}{12}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}uv-\tfrac{3\sqrt{2}}{8}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}vw\right.$$
$$+\tfrac{13}{9}\,u\sqrt{2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{3}vw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$-\tfrac{7}{18}\,uv\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{7\sqrt{2}}{18}\,v^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$+\tfrac{7}{12}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}vw-\tfrac{65}{36}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}uv$$
$$-\tfrac{1}{72}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}v^2-\tfrac{67}{144}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u^2\sqrt{2}$$
$$+\tfrac{3}{8}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}uw-\tfrac{7}{6}uw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$+uw-\sqrt{2}vw+\tfrac{2}{9}v^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{4}w^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$+\tfrac{7}{12}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}w-\tfrac{7\sqrt{2}}{12}\,vw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$+\tfrac{2}{3}\sqrt{2}v^2+\tfrac{1}{3}u^2\sqrt{2}-\tfrac{4}{3}uv+\tfrac{17}{18}u^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$-\tfrac{7}{6}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u^2+\tfrac{7}{6}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v^2+\tfrac{14\sqrt{2}}{9}\,uv$$
$$+\tfrac{28}{9}\,v^2-\tfrac{28}{9}\,u^2-\tfrac{3\sqrt{2}}{32}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}w^2$$
$$\left.+\tfrac{7\sqrt{2}}{18}\,u^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{\sqrt{2}}{6}uw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right)$$

$$
\begin{aligned}
&-\frac{\varepsilon^2}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(-\alpha_1 w\sqrt{2}-\tfrac{53\sqrt{2}}{144}\,uvw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right.\\
&-\tfrac{13}{144}\,v^2w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{5}{18}\,uv^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{2}{3}\,\alpha_1 u\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\alpha_1 v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{11\sqrt{2}}{216}\,v^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{32}vw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{\sqrt{2}}{2}\alpha_1 u\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{3}\,\alpha_1\sqrt{2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{7\sqrt{2}}{72}\,u^2w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{64}\,u\sqrt{2}w^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{17}{144}\,u\sqrt{2}v^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{7}{72}\,uvw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{61}{144}\,u^2\sqrt{2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{7}{96}\,\sqrt{2}vw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{7}{72}\,v^2\sqrt{2}w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{64}\,w^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{2}{3}\alpha_1 u+\tfrac{7}{48}\,uw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{31\sqrt{2}}{288}\,u^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{5}{144}\,u^2v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{79}{288}\,u^2w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{8}v^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{19}{216}\,u^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{2}{3}\alpha_1\sqrt{2}v+7/3\,\alpha_1 u\sqrt{2}+7/3\,\alpha_1 v\\
&\left.+\tfrac{1}{2}\alpha_1 w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right)-\frac{\varepsilon^3}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(\tfrac{3\sqrt{2}}{32}\,\alpha_1 w^2\right.\\
&+\tfrac{25\sqrt{2}}{72}\,\alpha_1 v^2+\tfrac{7\sqrt{2}}{36}\,\alpha_1 uv+\tfrac{7}{18}\,\alpha_1 v^2-\tfrac{7}{18}\,\alpha_1 u^2+\tfrac{41}{36}\,\alpha_1 uv-\tfrac{7}{12}\,\alpha_1 vw\\
&\left.-\tfrac{7\sqrt{2}}{12}\,\alpha_1 uw+\tfrac{1}{8}\alpha_1 uw-\tfrac{\sqrt{2}}{8}\alpha_1 vw+\tfrac{91}{144}\,\alpha_1 u^2\sqrt{2}\right).
\end{aligned}
$$

Now, we write this differential system in cylindrical coordinates (r,θ,w) defined by $u=r\cos\theta$, $v=r\sin\theta$, $w=w$ and after considering θ as the new independent variable, we arrive to the system

$$
\begin{aligned}
\frac{dr}{d\theta} \;=\;& \frac{1}{576}\Big(48\,\sqrt{2}\sqrt{-\alpha_1}r+12\,\sqrt{2}rw+2\,\sqrt{2}r^2\left(\cos\left(\theta\right)\right)^3-144\,\sin\left(\theta\right)\sqrt{2}\sqrt{-\alpha_1}w\\
&-10\,r^2\left(\cos\left(\theta\right)\right)^2\sin\left(\theta\right)-9\,\cos\left(\theta\right)\sqrt{2}w^2-384\,\sqrt{-\alpha_1}r\left(\cos\left(\theta\right)\right)^2\\
&+144\,\cos\left(\theta\right)\sqrt{-\alpha_1}w-72\,\cos\left(\theta\right)\sqrt{2}\sqrt{-\alpha_1}w+24\,\sqrt{2}\sin\left(\theta\right)r\cos\left(\theta\right)w\\
&-72\,\sin\left(\theta\right)\sqrt{-\alpha_1}w+192\,\cos\left(\theta\right)\sqrt{-\alpha_1}r\sin\left(\theta\right)+12\,\sqrt{2}r\left(\cos\left(\theta\right)\right)^2w\\
&+48\,\sqrt{2}\sqrt{-\alpha_1}r\left(\cos\left(\theta\right)\right)^2+48\,r\cos\left(\theta\right)\sin\left(\theta\right)w \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
& +52\sqrt{2}\sin(\theta)\,r^2(\cos(\theta))^2+96\,\sin(\theta)\sqrt{2}\sqrt{-\alpha_1}r\cos(\theta)\\
& -168\,\cos(\theta)\,r^2-9\,\sin(\theta)\,w^2+18\,\cos(\theta)\,w^2+260\,r^2(\cos(\theta))^3\\
& -40\,\sin(\theta)\sqrt{2}r^2-96\,r(\cos(\theta))^2\,w-18\sqrt{2}\sin(\theta)\,w^2-24\,\cos(\theta)\sqrt{2}r^2\\
& -20\,\sin(\theta)\,r^2+192\sqrt{-\alpha_1}r+48\,rw\Big)\sqrt{2}\varepsilon+O\left(\varepsilon^2\right)\\
= \; & \varepsilon F_{11}(\theta,r,w)+O\left(\varepsilon^2\right),\\
\frac{dw}{d\theta} = \; & \Bigg(-\frac{1}{2}\sqrt{-\alpha_1}w-\frac{1}{16}w^2-\frac{7\sqrt{2}}{24}\,r^2-\frac{7}{72}\,r^2(\cos(\theta))^2\\
& +\frac{7\sqrt{2}}{48}\,r\sin(\theta)\,w-\frac{7\sqrt{2}}{36}\,\cos(\theta)\,r^2\sin(\theta)\\
& -\frac{\sqrt{2}}{3}\sqrt{-\alpha_1}r\cos(\theta)-\frac{\sqrt{2}}{12}r\cos(\theta)\,w\\
& +\frac{7\sqrt{2}}{12}\sqrt{-\alpha_1}r\sin(\theta)+\frac{7\sqrt{2}}{12}\,r^2(\cos(\theta))^2\\
& -\frac{7}{24}\,\cos(\theta)\,r^2\sin(\theta)+\frac{7}{6}\sqrt{-\alpha_1}r\cos(\theta)+\frac{7}{24}\,r\cos(\theta)\,w\\
& +\frac{1}{6}r\sin(\theta)\,w+\frac{2}{3}\sqrt{-\alpha_1}r\sin(\theta)-\frac{2}{9}r^2\Bigg)\varepsilon+O\left(\varepsilon^2\right)\\
= \; & \varepsilon F_{12}(\theta,r,w)+O\left(\varepsilon^2\right).
\end{aligned}
$$

Our previous system has the form of the differential equation (13) with $t=\theta$, $x=(r,w)\in D=(0,+\infty)\times\mathbb{R}$, $T=2\pi$, and $F_1(\theta,r,w)=(F_{11}(\theta,r,w),F_{12}(\theta,r,w))$, and an easy computation shows that

$$
f_1(r,w)=(f_{11}(r,w),f_{12}(r,w)),
$$

is given by

$$
\begin{aligned}
f_{11}(r,w) &= \frac{1}{2\pi}\int_0^{2\pi}F_{11}(\theta,r,w)\,d\theta\\
&= \frac{1}{16}r\left(4\sqrt{-\alpha_1}+w\right),\\
f_{12}(r,w) &= \frac{1}{2\pi}\int_0^{2\pi}F_{12}(\theta,r,w)d\theta\\
&= -\frac{1}{2}\sqrt{-\alpha_1}w-\frac{13}{48}\,r^2-\frac{1}{16}\,w^2.
\end{aligned}
$$

We solve the system of equations

$$
\left\{\begin{array}{l}
\frac{1}{16}r\left(4\sqrt{-\alpha_1}+w\right)=0,\\
-\frac{1}{2}\sqrt{-\alpha_1}w-\frac{13}{48}\,r^2-\frac{1}{16}\,w^2=0,
\end{array}\right. \tag{14}
$$

where $r > 0$ and we find that this system accepts only one solution

$$(r, w) = \left(\frac{4\sqrt{-39\alpha_1}}{13}, -4\sqrt{-\alpha_1}\right),$$

where $\alpha_1 < 0$.

The Jacobian at $\left(\frac{4\sqrt{-39\alpha_1}}{13}, -4\sqrt{-\alpha_1}\right)$, take the value

$$\det \left.\frac{\partial\,(f_{11}, f_{12})}{(r, w)}\right|_{(r,w)=\left(\frac{4\sqrt{-39\alpha_1}}{13}, -4\sqrt{-\alpha_1}\right)} = -\frac{1}{8}\alpha_1, \tag{15}$$

In short the solution $\left(\frac{4\sqrt{-39\alpha_1}}{13}, -4\sqrt{-\alpha_1}\right)$ of system (14) which verify condition (15) satisfy the assumptions (i) and (ii) of Theorem 7. So we conclude that applying the averaging theory of first order, system (13) has one periodic orbit. From it we conclude the Rabinovitch-Fabrikant differential system (1) has a zero-Hopf bifurcation at the zero-Hopf equilibrium point localized at equilibrium point Q_+, a periodic orbit born at this equilibrium when $\varepsilon = 0$ and it exists for $\varepsilon > 0$ sufficiently small. This finish the proof of Theorem 5. The proof of Theorem 6 can be developed using similar ideas. ∎

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References

[1] NN. Bogoliubov, N. Krylov, The Application of Methods of NonlinearMechanics in the Theory of Stationary Oscillations, Publ. 8 of the Ukrainian Acad. Sci: Kiev, 1934.

[2] NN. Bogoliubov, On Some Statistical Methods in Mathematical Physics, Izv. vo Akad. Nauk Ukr. SSR: Kiev, 1945.

[3] A. Buică, J. Llibre, Averaging methods for finding periodic orbits via Brouwer degree, Bulletin des Sciences Mathématiques, 128 (2004) 7 – 22.

[4] M. F. Danca, G. Chen, Bifurcation and chaos in a complex model of dissipative medium, International Journal of Bifurcation and Chaos, 14(2004), 3409–3447.

[5] M. F. Danca, M. Feckan, N. Kuznetsov, G. Chen, Looking more closely to the Rabinovich-Fabrikant system, International Journal of Bifurcation and Chaos, 26 (2015) 1650038.

[6] P. Fatou, Sur le mouvement d'un système soumis à des forces à courte période, Bull. Soc. Math, France, 56 (1928), 98 – 139.

[7] J. Guckenheimer, On a codimension two bifurcation, Lecture Notes in Math., 898 (1980), 99 – 142.

[8] J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, Springer, 1983.

[9] M. Han, Existence of periodic orbits and invariant tori in codimension two bifurcations of three dimensional systems, J. Sys. Sci & Math. Scis., 18 (1998), 403 – 409.

[10] Y. A. Kuznetsov, Elements of Applied Bifurcation Theory, Springer-Verlag, 3rd edition, 2004.

[11] Y. Liu, Q. Yang, G. Pang, A hyperchaotic system from the Rabinovich system, J. Comput Appl. Math. 234 (2010) 101.

[12] J. Llibre, Averaging theory and limit cycles for quadratic systems, Rad. Mat, 11 (2002) /03, 215 – 228.

[13] J. Llibre and X. Zhang, Hopf bifurcation in higher dimensional di erential systems via the averaging method, Pacific J. Math., 240 (2009), 321 – 341.

[14] M.I. Rabinovich, A. L. Fabrikant, Stochastic self-modulation of waves in nonequilibrium media, J.E.T.P. (Sov.) 77 (1979) 617.

[15] J. Sanders, F. Verhulst, Murdock J. Averaging method in nonlinear dynamical systems 2nd ed., Vol. 59. Applied Mathematical Sciences, Springer,New York, 2007.

[16] C. X. Zhang, S.-M. Yu, Y. Zhang, Design and realization of multi-wing chaotic attractors via switching control, Int. J. Mod. Phys. B 25 (2011) 2183.

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