ON THE PERIODIC STRUCTURE OF THE RABINOVITCH-FABRIKANT SYSTEM

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ABSTRACT. In this work we proof analytically the existence and stability of four families of periodic orbits of the Rabinovitch-Fabrikant system that born of a Zero-Hopf Bifurcation.

1. Introduction and Statements of the main results

The study of the periodic orbits of a differential equation is one of the main objectives of the qualitative theory of differential equations. In general, the periodic orbits are studied numerically because, usually, their analytical study is very difficult. In this work we shall study analytically the periodic orbits of the three-dimensional Rabinovitch-Fabrikant system defined by

$$\begin{cases} \dot{x} = y\left(z - 1 + x^2\right) + ax, \\ \dot{y} = x\left(3z + 1 - x^2\right) + ay, \\ \dot{z} = -2z\left(b + xy\right). \end{cases} \tag{1}$$

where a, b are two positive real parameters.

This system was introduced in 1979 by Rabinovich and Fabrikant [14] and numerically investigated in [5], was initially designed as a physical model describing the stochasticity arising from the modulation instability in a non-equilibrium dissipative medium. However, recently in [5] we revealed that beside the physical properties described in [14], the system presents some extremely rich dynamics as "virtual" saddles beside several chaotic attractors with different shapes and hidden chaotic attractors. In fact, the interest in this system is continuously increasing (see [11], [16]).

Note that the Rabinovitch-Fabrikant system has some peculiarities that should be noted. It is a biparametric system with third-order polynomial nonlinearities. More significantly, unlike other nonlinear chaotic systems containing only second-order nonlinearities (such as

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the Lorenz system), the RF system with third-order polynomial nonlinearities, presents some unusual dynamics.

On the other hand, for $a = b = 0$, the system is integrable having a third-order polynomial algebraic integral and another transcendent integral. The flow of the system is obtained as foliation of these two integrals.

We will not go discussing in detail the periodic structure of this integrable system with several families of degenerate equilibria. We can point out that there is a wide variety of families of periodic orbits depending on certain regions of the space where we set the initial conditions of the system. The existence of families of heteroclinic orbits associated with the four families of degenerate equilibria of the system will play a fundamental role in a detailed study of the existence of chaotic orbits for certain values of the parameters of the system.

It is natural to study values of the parameters other than $(0, 0)$ for which there are periodic orbits of the system. The first result of the work is obtained for (a, b) close $(0, 0)$.

We probe, in this case, the system has two families of periodic orbits. Whe note that when (a, b) is close to $(0, 0)$ the RF system undergoes a Zero-Hopf Bifurcation in the origin. We just recall that the occurrence of the classical Hopf bifurcation in $\mathbb{R}^3$ takes place in an equilibrium point with eigenvalues of the form $\pm\omega i$ and $\delta \neq 0$, while for a zero-Hopf bifurcation the eigenvalues are $\pm\omega i$ and 0. Here an equilibrium point with eigenvalues $\pm\omega i$ and 0 will be called a zero-Hopf equilibrium. The zero-Hopf bifurcation it has been studied by many authors among them Guckenheimer, Holmes, Scheurle, Han, Kuznetsov, Llibre and Zhang in [7], [8], [9], [10], [13].

In this work we study the zero-Hopf bifurcation of the RF system as tool for obtaining periodic orbits of the system in line with the numerous works published by Llibre et al in the study of the Zero-Hopf bifurcation of differential systems with quadratic polynomial nonlinearities with own names (Chua, Rössler,....). The main results are the following.

Remark 1 (Remark about Equilibrium points)
The Rabinovich-Fabrikant system (1) has five equilibrium points, that

is

$$
\begin{aligned}
O &= (0,0,0)\,, P_+ = \left(+M_-, -\frac{b}{M_-}, 1-\left(1-\frac{a}{b}\right)M_-^2\right),\\
P_- &= \left(-M_-, -\frac{b}{M_-}, 1-\left(1-\frac{a}{b}\right)M_-^2\right), Q_+ = \left(+M_+, -\frac{b}{M_+}, 1-\left(1-\frac{a}{b}\right)M_+^2\right)\\
Q_- &= \left(-M_+, -\frac{b}{M_+}, 1-\left(1-\frac{a}{b}\right)M_+^2\right), \text{ where}\\
M_- &= \sqrt{\frac{1-\sqrt{1-ab\left(1-\frac{3a}{4b}\right)}}{2\left(1-\frac{3a}{4b}\right)}}, M_+ = \sqrt{\frac{1+\sqrt{1-ab\left(1-\frac{3a}{4b}\right)}}{2\left(1-\frac{3a}{4b}\right)}}.
\end{aligned}
$$

For more information about the equilibrium points of the Rabinovich-Fabrikant system see [4].

Proposition 1 There is only a one-parameter family of Rabinovich-Fabrikant system for which the equilibrium point localized at the origin of coordinates is a zero-Hopf equilibrium point, that is

$$a = b = 0.$$

Our goal in the next proposition, we characterize when the equilibrium points

$$
\begin{aligned}
Q_+ &= \left(+M_+, -\frac{b}{M_+}, 1-\left(1-\frac{a}{b}\right)M_+^2\right),\\
Q_- &= \left(-M_+, -\frac{b}{M_+}, 1-\left(1-\frac{a}{b}\right)M_+^2\right).
\end{aligned}
$$

if $b > 0$, $a > 0$ and $4 - 4ab + 3a^2 \geq 0$ of the Rabinovich-Fabrikant system is a zero-Hopf equilibrium point.

Proposition 2

If $b > 0$, $a > 0$ and $4 - 4ab + 3a^2 \geq 0$, There is only a one-parameter family of Rabinovich-Fabrikant system for which the equilibrium points Q_+ and Q_- is a zero-Hopf equilibrium points, that is

$$a = b = 2$$

Theorem 3

Let $(a, b) = (\varepsilon\alpha, \varepsilon\beta)$ with $\alpha < 0, \beta < 0$ are two parameters and ε is a small parameter. Then, the Rabinovitch-Fabrikant system (1) has a zero-Hopf bifurcation at the equilibrium point localized at the origin of coordinates, and a two periodic orbits borns at this equilibrium when

$\varepsilon = 0$ and it exists for $\varepsilon > 0$ sufficiently small. See **Fig. 1**.

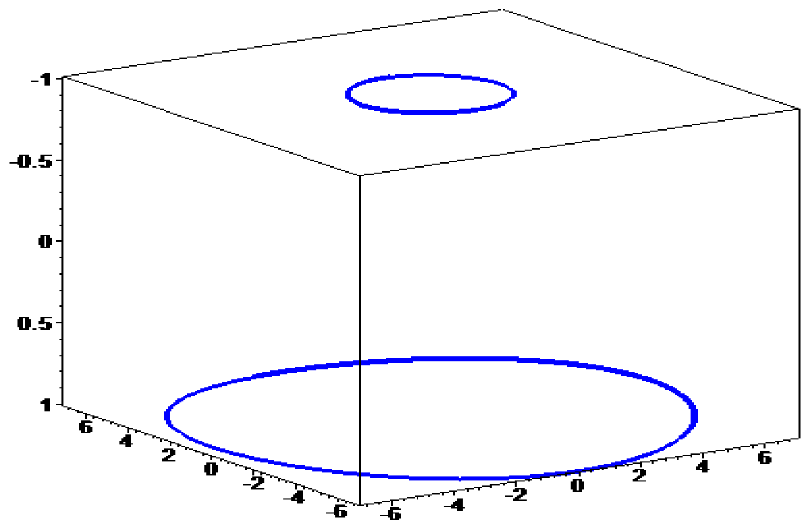

Fig. 1. The periodic solutions of Theorem 3 for the values of the parameters $\alpha = -1$, $\beta = -0.5$ and $\varepsilon = 0.001$.

Theorem 4
Let $(a, b) = (2 + \varepsilon^2\alpha_1, 2 + \varepsilon^2\alpha_1)$ with $\alpha_1 < 0$ is parameter, ε is a small parameter. Then, the Rabinovitch-Fabrikant system (1) has a zero-Hopf bifurcation at the zero-Hopf equilibrium point localized at the equilibrium point Q_+, and a periodic orbit born at this equilibrium when $\varepsilon = 0$ and it exists for $\varepsilon > 0$ sufficiently small. See **Fig. 2**.
Theorem 5
Let $(a, b) = (2 + \varepsilon^2\alpha_1, 2 + \varepsilon^2\alpha_1)$ with $\alpha_1 < 0$ is parameter, ε is a small parameter. Then, the Rabinovitch-Fabrikant system (1) has a zero-Hopf bifurcation at the zero-Hopf equilibrium point localized at the equilibrium point Q_-, and a periodic orbit born at this equilibrium when $\varepsilon = 0$ and it exists for $\varepsilon > 0$ sufficiently small. See **Fig. 2**.

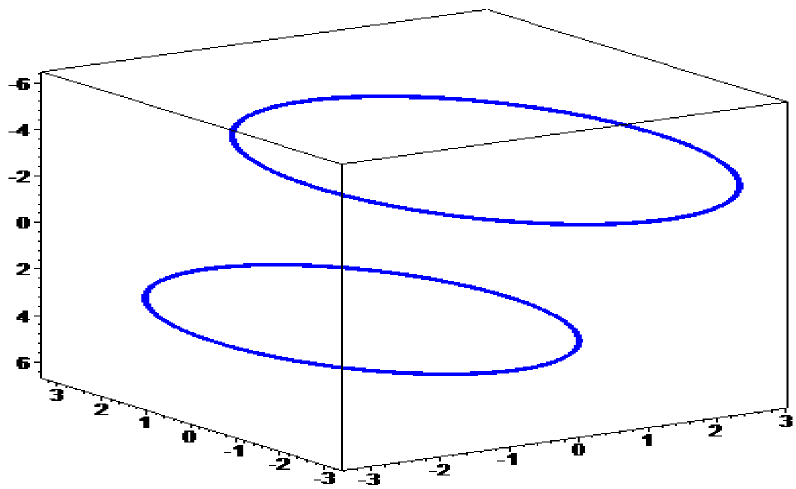

Fig. 2. The periodic solutions of Theorem 4 and 5 for the values of the parameters $\alpha_1 = -200$ and $\varepsilon = 0.01$.

2. The averaging theory of first order and second order

In this section, we recall the averaging theory of first and second order to find periodic orbits, see for more details [12] and [3]. The averaging theory is an effective method for studying nonlinear differential equations, especially the study of their periodic orbits in terms of their number and stability. In general the method of averaging has a long history going back to the classic works of Lagrange and Laplace, who provided an intuitive justification for the process. The first formalization of this procedure is due to Fatou [6] in 1928. Important practical and theoretical contributions in this theory were made by Krylov and Bogoliubov [1] in the 1930 and Bogoliubov [2] in 1945.

Theorem 6

Consider the differential equation

$$\dot{x}(t) = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon), \tag{2}$$

with $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n, R : \mathbb{R} \times D \times (0, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T -periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. We assume that

(i) $F_1(t, \cdot), F_2(t, \cdot) \in C^1(D)$ for all $t \in \mathbb{R}, F_1, F_2, R, D_x F_1$ and $D_x F_2$ are

locally lipschitz with respect to x, and R is differentiable with respect to ε.
We define $f_1, f_2 : D \to \mathbb{R}^n$ as

$$f_1(z) = \frac{1}{T}\int_0^T F_1(s,z)ds,$$
$$f_2(z) = \frac{1}{T}\int_0^T \left[D_z F_1(s,z)\int_0^s F_1(t,z)dt + F_2(s,z)\right] ds.$$

(ii) For $V \subset D$, an open and bounded set and for each $\varepsilon \in (0,\varepsilon_f)\setminus\{0\}$, there exists $p \in V$ such that $f_1(p) + \varepsilon f_2(p) = 0$ and

$$\det\left(\left.\frac{\partial(f_1+\varepsilon f_2)}{\partial z}\right|_{z=p}\right) \neq 0. \tag{3}$$

Then, for $\varepsilon > 0$ sufficiently small, there exists a $T-$ periodic solution $\varphi(t,\varepsilon)$ of system (2) such that $\varphi(0,\varepsilon) \to p$ when $\varepsilon \to 0$ If the function f_1 is not identically zero, then the zeros of $f_1 + \varepsilon f_2$ are mainly the zeros of f_1 for ε sufficiently small. In this case, Theorem 6 provides the so-called averaging theory of first order.

If the function f_1 is identically zero and f_2 is not identically zero, then the zeros of $f_1 + \varepsilon f_2$ are the zeros of f_2. In this case, Theorem 6 provides the so-called averaging theory of second order. In the case of the averaging theory of first order, we consider in D the averaged differential equation

$$\dot{y} = \varepsilon f_1(y), \quad y(0) = x_0, \tag{4}$$

where

$$f_1(y) = \frac{1}{T}\int_0^T F_1(t,y)dt. \tag{5}$$

Then Theorem 6 gives us information about the stability or instability of the limit cycle $\varphi(t,\varepsilon)$. In fact, it is given by the stability or instability of the equilibrium point p of the averaged system (4). In fact, the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\varphi(t,\varepsilon)$. In the case of the averaging theory of second order, that is, $f_1 \equiv 0$ and f_2 non-identically zero, we have that the stability and instability of the limit cycle $\varphi(t,\varepsilon)$ coincide with the type of stability or instability of the equilibrium point p of the averaged system

$$\dot{y} = \varepsilon^2 f_2(y), \quad y(0) = x_0,$$

that is, it is the same that the singular point p associated the Poincaré map of the limit cycle $\varphi(t,\varepsilon)$. For additional information on averaging theory, see the book [15].

3. Proofs and Some Remarks

3.1. **Proof of Propositions 1 and 2.** Note that the characteristic polynomial of the linear part of the Rabinovich-Fabrikant system at the origin of coordinates is

$$p(\lambda) = (\lambda + 2b)\left(\lambda^2 - 2a\lambda + a^2 + 1\right),$$

because we must have one null eigenvalue, for this we put

$$b = 0,$$

now, we must choose the other two eigenvalues in the form $\pm i\omega$, then

$$p(\lambda) = \lambda\left(\lambda^2 + \omega^2\right),$$

then, we must have

$$a = 0, \omega^2 = 1,$$

Now, we observe the characteristic polynomial of the linear part of the Rabinovich-Fabrikant system at the points Q_+ and Q_-

$$\begin{aligned} p(\lambda) &= \lambda^3 - \frac{1}{M_+^2 b^2}\left(-2\,M_+^2 b^3 + 2\,b^2 M_+^2 a\right)\lambda^2 \\ &\quad - \frac{1}{M_+^2 b^2}(-2\,b^4 + 6\,M_+^6 b^2 + 3\,M_+^6 a^2 - 6\,M_+^4 b^2 \\ &\quad + 2\,M_+^2 b^4 - 12\,M_+^6 ba - M_+^2 a^2 b^2 + 4\,bM_+^4 a)\lambda \\ &\quad - \frac{1}{M_+^2 b^2}(18\,M_+^6 ba^2 - 32\,M_+^4 b^3 + 24\,M_+^6 b^3 - 2\,M_+^2 b^4 a \\ &\quad + 2\,b^4 a + 2\,M_+^2 a^2 b^3 + 8\,M_+^2 b^3 + 26\,M_+^4 b^2 a - 42\,M_+^6 b^2 a) \end{aligned}$$

Again, we impose that the roots of Q_+ and $Q_-$are 0 and the other two eigenvalues in the form $\pm i\omega$, so the following conditions must hold

$$\begin{aligned} &18\,M_+^6 ba^2 - 32\,M_+^4 b^3 + 24\,M_+^6 b^3 - 2\,M_+^2 b^4 a + 2\,b^4 a \\ &+2\,M_+^2 a^2 b^3 + 8\,M_+^2 b^3 + 26\,M_+^4 b^2 a - 42\,M_+^6 b^2 a = 0, \\ &-2\,M_+^2 b^3 + 2\,b^2 M_+^2 a = 0 \end{aligned}$$

and

$$\begin{aligned} \omega^2 &= -\frac{1}{M_+^2 b^2}(-2\,b^4 + 6\,M_+^6 b^2 + 3\,M_+^6 a^2 - 6\,M_+^4 b^2 \\ &\quad + 2\,M_+^2 b^4 - 12\,M_+^6 ba - M_+^2 a^2 b^2 + 4\,bM_+^4 a). \end{aligned}$$

Of the following conditions

$$\left\{\begin{array}{l} 18\,M_+^6ba^2-32\,M_+^4b^3+24\,M_+^6b^3-2\,M_+^2b^4a+2\,b^4a\\ +2\,M_+^2a^2b^3+8\,M_+^2b^3+26\,M_+^4b^2a-42\,M_+^6b^2a=0,\\ -2\,M_+^2b^3+2\,b^2M_+^2a=0\\ -(-2\,b^4+6\,M_+^6b^2+3\,M_+^6a^2-6\,M_+^4b^2\\ +2\,M_+^2b^4-12\,M_+^6ba-M_+^2a^2b^2+4\,bM_+^4a)<0,\\ \text{ and } b>0, a>0, 4-4ab+3a^2\geq 0.\end{array}\right.$$

We conclude that $a=b=2$.

3.2. **Proof of Theorem 3.** If we consider $(a,b)=(\varepsilon\alpha,\varepsilon\beta)$ with $\varepsilon\geq 0$ a sufficiently small parameter, then Rabinovich-Fabrikant system becomes

$$\left\{\begin{array}{l} \dot{x}=-y+y\left(z+x^2\right)+\varepsilon\alpha x,\\ \dot{y}=x+x\left(3z-x^2\right)+\varepsilon\alpha y,\\ \dot{z}=-2z\left(\varepsilon\beta+xy\right).\end{array}\right. \tag{6}$$

The eigenvalues at the origin of the linear part of system (6) when $\varepsilon=0$ are 0 and $\pm i$

By the rescaling of variables $(x,y,z)=(\sqrt{\varepsilon}X,\sqrt{\varepsilon}Y,\varepsilon Z)$, the system (6) becomes

$$\left\{\begin{array}{l} \dot{X}=-Y+\varepsilon\left(YZ+YX^2+\alpha X\right),\\ \dot{Y}=X+\varepsilon\left(3XZ-X^3+\alpha Y\right),\\ \dot{Z}=-2\varepsilon\left(\beta Z\right)-2\varepsilon\left(XYZ\right).\end{array}\right. \tag{7}$$

Now, we write this differential system in cylindrical coordinates (r,θ,w) defined by $X=r\cos\theta$, $Y=r\sin\theta$, $Z=Z$ and after we introduce θ the new independent variable, and so we arrive to the system

$$\begin{aligned} \frac{dr}{d\theta} &= (r\left(4\,\cos\left(\theta\right)\sin\left(\theta\right)Z+\alpha\right)\varepsilon-r(16\,\left(\cos\left(\theta\right)\right)^3\sin\left(\theta\right)Z^2\\ &\quad -4\,\cos\left(\theta\right)\sin\left(\theta\right)Z^2-4\,\left(\cos\left(\theta\right)\right)^3\sin\left(\theta\right)Zr^2\\ &\quad +4\,\alpha\,\left(\cos\left(\theta\right)\right)^2Z-\alpha\,Z-\alpha\,r^2\left(\cos\left(\theta\right)\right)^2)\varepsilon^2+O\left(\varepsilon^3\right),\\ &= \varepsilon F_{11}\left(\theta,r,Z\right)+\varepsilon^2F_{21}\left(\theta,r,Z\right)+O\left(\varepsilon^3\right), \qquad (8)\\ \frac{dZ}{d\theta} &= (-2\,Z\left(\beta+r^2\cos\left(\theta\right)\sin\left(\theta\right)\right)\varepsilon+2\,Z(4\,\beta\,\left(\cos\left(\theta\right)\right)^2Z-\beta\,Z\\ &\quad -\beta\,r^2\left(\cos\left(\theta\right)\right)^2+4\,\left(\cos\left(\theta\right)\right)^3\sin\left(\theta\right)Zr^2\\ &\quad -r^2\cos\left(\theta\right)\sin\left(\theta\right)Z-r^4\left(\cos\left(\theta\right)\right)^3\sin\left(\theta\right))\varepsilon^2+O\left(\varepsilon^3\right),\\ &= \varepsilon F_{12}\left(\theta,r,Z\right)+\varepsilon^2F_{22}\left(\theta,r,Z\right)+O\left(\varepsilon^3\right).\end{aligned}$$

Our previous system has the form of the differential equation (8) with $t=\theta$, $x=(r,Z)\in D=(0,+\infty)\times\mathbb{R}$, $T=2\pi$, and $F_1\left(\theta,r,Z\right)=$

$(F_{11}(\theta,r,Z), F_{12}(\theta,r,Z))$, $F_2(\theta,r,Z) = (F_{21}(\theta,r,Z), F_{22}(\theta,r,Z))$ and an easy computation shows that

$$f_1(r,Z) = (f_{11}(r,Z), f_{12}(r,Z))$$

is given by

$$\begin{aligned} f_{11}(r,Z) &= \frac{1}{2\pi}\int_0^{2\pi} F_{11}(\theta,r,Z)\,d\theta = \alpha r, \\ f_{12}(r,Z) &= \frac{1}{2\pi}\int_0^{2\pi} F_{12}(\theta,r,Z)d\theta = -2\beta Z, \end{aligned}$$

Note here that the averaging theory of first order, cannot be used.
For applying the averaging theory of second order, we must compute the expression

$$\begin{pmatrix} \frac{\partial F_{11}}{\partial r} & \frac{\partial F_{11}}{\partial Z} \\ \frac{\partial F_{12}}{\partial r} & \frac{\partial F_{12}}{\partial Z} \end{pmatrix} \begin{pmatrix} \int_0^\theta F_{11}(s,r,Z)ds \\ \int_0^\theta F_{12}(s,r,Z)d\theta \end{pmatrix} + \begin{pmatrix} F_{21}(\theta,r,Z) \\ F_{22}(\theta,r,Z \end{pmatrix}.$$

By computing the integral of this expression between 0 and 2π and dividing by 2π, we obtain

$$f_2(r,Z) = \begin{pmatrix} f_{21}(r,Z) \\ f_{22}(r,Z) \end{pmatrix} = \begin{pmatrix} \pi\, r\alpha^2 - r\alpha\, Z + r^3\alpha/2 \\ Zr^2\alpha + 2\,\beta\, Z^2 - \beta\, Zr^2 \end{pmatrix},$$

we solve the system of equations

$$\begin{aligned} f_{21}(r,Z) &= 0, \\ f_{22}(r,Z) &= 0, \end{aligned} \tag{9}$$

By solving the first equation with respect to Z, we obtain the one solution

$$Z = r^2/2 + \alpha\pi,$$

By substituting in the second equation, we find that

$$r^2\alpha^2\pi + 1/2\, r^4\alpha + 2\,\beta\,\alpha^2\pi^2 + \beta\,\alpha\, r^2\pi = 0$$

Solving this equation with respect to r, we obtain the solutions

$$r_1 = \sqrt{-2\beta\pi}, r_2 = -\sqrt{-2\beta\pi}, r_3 = \sqrt{-2\alpha\pi}, r_4 = -\sqrt{-2\alpha\pi}.$$

Where $\alpha < 0$ and $\beta < 0$, by choosing completely positive roots we find

$$r_1 = \sqrt{-2\beta\pi}, r_3 = \sqrt{-2\alpha\pi}.$$

The system $f_{21}(r,Z) = f_{22}(r,Z) = 0$ has a two solutions $(\sqrt{-2\beta\pi}, (\alpha-\beta)\pi)$, $(\sqrt{-2\alpha\pi}, 0)$

The Jacobian at $(\sqrt{-2\beta\pi}, (\alpha-\beta)\pi)$, $\left(\sqrt{-2\alpha\pi}, 0\right)$ takes the value

$$\det \left.\frac{\partial(f_{21}, f_{22})}{(r,Z)}\right|_{(r,Z)=\left(\sqrt{-2\beta\pi},(\alpha-\beta)\pi\right)} = 4\pi^2\alpha^2\beta(\beta-\alpha), \quad (10)$$

$$\det \left.\frac{\partial(f_{21}, f_{22})}{(r,Z)}\right|_{(r,Z)=\left(\sqrt{-2\alpha\pi},0\right)} = 4\pi^2\alpha^3(\alpha-\beta),$$

In short the solutions $(\sqrt{-2\beta\pi}, (\alpha-\beta)\pi)$, $\left(\sqrt{-2\alpha\pi}, 0\right)$ of system (9) which verify condition (10) satisfy the assumptions (i) and (ii) of Theorem 6. So we conclude that applying the averaging theory of second order system (8) has two periodic orbits. From it we conclude the Rabinovitch-Fabrikant differential system (1) has a zero-Hopf bifurcation at the zero-Hopf equilibrium point localized at the origin of coordinates, and a two periodic orbits borns at this equilibrium when $\varepsilon = 0$ and it exists $\varepsilon > 0$ sufficiently small.

3.3. **Some Remarks.**

(1) The Jacobian matrix at $\left(r_1 = \sqrt{-2\pi\alpha},\ Z_1 = 0\right)$ of the averaged equations $(f_{21}(r,Z), f_{22}(r,Z))$ are

$$\begin{pmatrix} -2\pi\alpha^2 & \sqrt{2\pi}(-\alpha)^{3/2} \\ 0 & 2\pi\alpha(\beta-\alpha) \end{pmatrix},$$

with eigenvalues $\lambda_1 = -2\pi\alpha^2$ and $\lambda_2 = 2\pi\alpha(\beta-\alpha)$,

(2) The Jacobian matrix at $\left(r_2 = \sqrt{-2\pi\alpha},\ Z_2 = 2\pi\alpha(\alpha-\beta)\right)$ of the averaged equations $(f_{21}(r,Z), f_{22}(r,Z))$ are

$$\begin{pmatrix} -2\pi\alpha\beta & -\sqrt{2\pi}\alpha\sqrt{-\beta} \\ 2\sqrt{2}\pi^{3/2}\sqrt{-\beta}(\alpha-\beta)^2 & 2\pi\beta(\alpha-\beta) \end{pmatrix},$$

with eigenvalues

$$\lambda_1 = -\pi\left(\beta^2 + \sqrt{\beta\left(4\alpha^3 - 4\alpha^2\beta + \beta^3\right)}\right),$$

and

$$\lambda_2 = \pi\left(\beta^2 - \sqrt{\beta\left(4\alpha^3 - 4\alpha^2\beta + \beta^3\right)}\right).$$

(3) The periodic orbit with initial condition $r_1 = \sqrt{-2\pi\alpha}$, $Z_1 = 0$ is linearly stable in the parametric region

$$R_1 = \{(\alpha,\beta)/\alpha < 0, \beta < 0, \beta-\alpha > 0\},$$

and unstable in the parametric region

$$R_2 = \{(\alpha,\beta)/\alpha < 0, \beta < 0, \beta-\alpha < 0\}.$$

(4) The periodic orbit with initial condition $r_2 = \sqrt{-2\pi\beta}$, $Z_2 = \pi(\beta - \alpha)$ is linearly stable in the parametric region

$$R_1 = \{(\alpha, \beta)/\alpha < 0, \beta < 0, \beta - \alpha > 0\},$$

and unstable in the parametric region

$$R_2 = \{(\alpha, \beta)/\alpha < 0, \beta < 0, \beta - \alpha < 0\}.$$

3.4. **Proof of Theorem 4 and 5.** If we consider $(a, b) = (2 + \varepsilon^2\alpha_1, 2 + \varepsilon^2\alpha_1)$ with $\alpha_1 < 0$ and $\varepsilon > 0$ a sufficiently small. Here we only prove with the Theorem 4 because the proof for the Theorem 5 is similar to it.
We translate the equilibrium point Q_+ to the origin of coordinates and, maintaining the notation (x, y, z) for the new coordinates, then Rabinovitch Fabrikant system (1) becomes

$$\begin{cases} \dot{x} = -\left(y - \dfrac{2+\varepsilon^2\alpha_1}{\sqrt{2+\sqrt{4-(2+\varepsilon\alpha_1)^2}}}\right) + \left(y - \dfrac{2+\varepsilon^2\alpha_1}{\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}}\right)((z+1 \\ +\left(x+\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}\right)^2\Bigg) + (2+\varepsilon^2\alpha_1)\left(x+\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}\right), \\ \dot{y} = x + \sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}} \\ +\left(x+\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}\right)\left(3(z+1) - \left(x+\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}\right)^2\right) \\ +(2+\varepsilon\alpha_1)\left(y - \dfrac{2+\varepsilon^2\alpha_1}{\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}}\right), \\ \dot{z} = -2(z+1)\left(2+\varepsilon^2\alpha_1 + \left(x+\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}\right)(y\right. \\ \left.\left. - \dfrac{2+\varepsilon^2\alpha_1}{\sqrt{2+\sqrt{4-(2+\varepsilon^2\alpha_1)^2}}}\right)\right). \end{cases} \tag{11}$$

The eigenvalues at the origin of the linear part of system (11) when $\varepsilon = 0$ are 0 and $\pm 4i$
By the rescaling of variables $(x, y, z) = (\varepsilon X, \varepsilon Y, \varepsilon Z)$, the system (11) becomes

$$
\left\{
\begin{aligned}
&\dot{X} = \frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(2\,Y\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right.\\
&+Y\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-2\,Z\\
&-2\,X\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(4\,YX\right.\\
&\left.+YZ\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-2\,X^2+2\,YX\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\right)\varepsilon\\
&+\frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(YX^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right.\\
&\left.-\alpha_1Z-\alpha_1X\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right)\varepsilon^2-\left(\frac{\alpha_1X^2}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\right)\varepsilon^3,\\
&\dot{Y} = -\frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(2\,X\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right.\\
&+3\,X\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&\left.-6\,Z-3\,Z\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}-2\,Y\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right)\\
&-\frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(-3\,XZ\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+6\,X^2\right.\\
&\left.+3\,X^2\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\right)\varepsilon-\frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(X^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right.\\
&\left.-\alpha_1Y\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right)\varepsilon^2,\\
&\dot{Z} = -\frac{2}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(2\,Y+Y\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}-2\,X\right)\\
&-\frac{2}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(2\,ZY+ZY\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\right.\\
&\left.-2\,XZ+XY\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right)\varepsilon-\frac{2}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(-X\alpha_1\right.\\
&\left.+ZXY\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right)\varepsilon^2+\left(\frac{2ZX\alpha_1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\right)\varepsilon^3.
\end{aligned}
\right.
\tag{12}
$$

We need to write the linear part at the origin of the system (12) when $\varepsilon = 0$ into its real Jordan normal form, i.e. into the form

$$J = \begin{pmatrix} 0 & -4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the new variables (u, v, w) defined by

$$\begin{cases} X = \left(\dfrac{1-2\sqrt{2}}{12}\right)u - \left(\dfrac{2+\sqrt{2}}{12}\right)v + \dfrac{\sqrt{2}}{8}w, \\ Y = -\left(\dfrac{2\sqrt{2}+3}{12}\right)u + \left(\dfrac{3\sqrt{2}-2}{12}\right)v + \dfrac{\sqrt{2}}{8}w, \\ Z = \dfrac{1}{3}u + \dfrac{\sqrt{2}}{6}v, \end{cases}$$

the system (12) becomes

$$\begin{aligned}
\dot{u} = &-\frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(-\tfrac{2}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}+4\sqrt{2}v\right. \\
&+\tfrac{3}{2}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}v-\tfrac{2}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v \\
&-\tfrac{1}{2}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
&+\tfrac{2}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
&+\tfrac{2}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
&\left.+\tfrac{1}{2}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}w\right)-\frac{\varepsilon}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(-\tfrac{85}{144}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u^2\right. \\
&-\tfrac{5}{32}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}w^2+\tfrac{17}{72}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v^2 \\
&+\tfrac{1}{8}w^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{11}{36}u^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
&-\tfrac{2}{9}v^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{6}\sqrt{2}v^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
&-\tfrac{1}{3}uv\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{4}vw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
&-\tfrac{1}{3}uw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{5}{24}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}vw
\end{aligned}$$

$$
\begin{aligned}
&-\tfrac{17}{36}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}uv+\tfrac{5}{12}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}uw\\
&-\tfrac{7}{36}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u^2\sqrt{2}-\tfrac{5}{36}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}v^2\\
&-\tfrac{11}{72}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}v-\tfrac{1}{6}v\sqrt{2}w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{11}{36}u\sqrt{2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{7}{48}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}w\\
&+\tfrac{5}{24}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v\sqrt{2}w-\tfrac{23}{18}u^2+\tfrac{1}{3}v\sqrt{2}w\\
&-\tfrac{1}{9}u\sqrt{2}v+\tfrac{2}{3}uw+\tfrac{7}{9}v^2-\tfrac{1}{4}w^2\Big)\\
&-\frac{\varepsilon^2}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\Bigg(\tfrac{19}{144}uv^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{23}{432}u^3\sqrt{2}\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{\sqrt{2}}{128}w^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{1}{32}u\sqrt{2}w^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{7}{216}v^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{11}{288}u^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{\sqrt{2}}{144}v^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{3}\alpha_1v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{1}{6}\alpha_1u\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{32}vw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{5}{72}v^2w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{7}{192}uw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{13}{192}u^2\sqrt{2}w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{6}\alpha_1\sqrt{2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{1}{16}uvw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{17\sqrt{2}}{144}uvw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{1}{96}v^2\sqrt{2}w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{43\sqrt{2}}{288}u^2v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{1}{36}u\sqrt{2}v^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{\sqrt{2}}{4}\alpha_1w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{5}{192}\sqrt{2}vw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{\sqrt{2}}{3}\alpha_1u\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{7}{72}u^2w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{16}u^2v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{1}{2}\alpha_1\sqrt{2}v+2/3\,\alpha_1u\sqrt{2}-\tfrac{1}{2}\alpha_1\sqrt{2}w+\tfrac{2}{3}\alpha_1v\\
&-\frac{\varepsilon^3}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(\frac{7}{72}\alpha_1u\sqrt{2}v+\frac{11}{72}\alpha_1v^2+1/32\,\alpha_1w^2\right.\\
&+\frac{17}{36}\alpha_1uv-1/12\,\alpha_1uw-\frac{5}{24}\alpha_1vw-\frac{7}{48}\alpha_1u\sqrt{2}w\\
&\left.+\frac{5}{36}\alpha_1\sqrt{2}v^2+\frac{7}{36}\alpha_1u^2\sqrt{2}-\frac{7}{144}\alpha_1u^2-1/24\,\alpha_1v\sqrt{2}w\right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{v} = {} & \frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\Big(\tfrac{3}{2}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2} \\
& +\tfrac{2}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u-\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{4}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+4\,u\sqrt{2} \\
& -\tfrac{1}{2}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}w+\tfrac{2\sqrt{2}}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{1}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}v\Big)+\frac{\varepsilon}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\Big(-\tfrac{5\sqrt{2}}{32}\sqrt{-\alpha_1\left(4+\alpha_1\epsilon^2\right)\epsilon^2}w^2 \\
& +\tfrac{7}{72}u^2\sqrt{2}\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{16}\sqrt{2}w^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{2\sqrt{2}}{3}uw+\tfrac{1}{18}u^2\sqrt{2}-\tfrac{5}{9}\sqrt{2}v^2-\tfrac{\sqrt{2}}{4}w^2+\tfrac{5}{18}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u^2 \\
& +\tfrac{5}{12}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}w-\tfrac{\sqrt{2}}{6}uv\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{7}{36}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}v-\tfrac{1}{24}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v\sqrt{2}w \\
& -\tfrac{1}{3}u^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{2}{3}vw-\tfrac{14}{9}uv \\
& -\tfrac{5}{24}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}uw-\tfrac{5}{9}uv\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{1}{6}vw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{4}uw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& -\tfrac{29}{36}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}uv-\tfrac{13}{144}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u^2\sqrt{2} \\
& +\tfrac{5}{12}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}vw-\tfrac{5\sqrt{2}}{36}v^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& -\tfrac{19}{72}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}v^2+\tfrac{\sqrt{2}}{6}uw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{1}{18}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v^2+\frac{\varepsilon^2}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\Big(\tfrac{1}{2}\alpha_1 w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{\sqrt{2}}{32}vw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{\sqrt{2}}{3}\alpha_1 v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{7}{72}uvw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{\sqrt{2}}{6}\alpha_1 u\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& -\tfrac{5\sqrt{2}}{192}uw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{5\sqrt{2}}{144}uv^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{5}{72}u^2\sqrt{2}w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{16}u^2\sqrt{2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{1}{72}v^2\sqrt{2}w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{216}u^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{1}{72}v^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{64}w^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& +\tfrac{1}{2}\alpha_1 w-\tfrac{2}{3}\alpha_1 u-\tfrac{\sqrt{2}}{3}\alpha_1 v+\tfrac{\sqrt{2}}{2}\alpha_1 u \\
& -\tfrac{23}{144}u^2v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{9}uv^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& -\tfrac{5}{96}u^2w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{16}uw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}} \\
& -\tfrac{1}{16}v^2w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{2}{3}\alpha_1 u\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}
\end{aligned}
$$

$$
\begin{aligned}
&-\tfrac{1}{16}v^2w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{2}{3}\alpha_1u\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{1}{3}\alpha_1v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{96}\,vw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{5\sqrt{2}}{216}v^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{\sqrt{2}}{32}u^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{5\sqrt{2}}{48}\,uvw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\Bigg)\\
&+\frac{\varepsilon^3}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(\tfrac{17\sqrt{2}}{144}\,\alpha_1u^2+\tfrac{1}{36}\alpha_1uv+\tfrac{\sqrt{2}}{24}\,\alpha_1vw\right.\\
&-\tfrac{7\sqrt{2}}{36}\,\alpha_1uv-\tfrac{\sqrt{2}}{12}\alpha_1uw-\tfrac{\sqrt{2}}{72}\,\alpha_1v^2-\tfrac{1}{12}\alpha_1vw+\tfrac{\sqrt{2}}{32}\alpha_1w^2\\
&\left.-\tfrac{1}{18}\,\alpha_1v^2-\tfrac{5}{18}\,\alpha_1u^2+\tfrac{5}{24}\,\alpha_1uw\right),\\
&\dot{w}=-\frac{1}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\Bigg(-\tfrac{7}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v\\
&-\tfrac{7}{3}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}-2\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-2\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u-\tfrac{16}{3}u+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}w\\
&+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{16\sqrt{2}}{3}v\\
&+2\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}v-\tfrac{16}{3}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+8/3\,u\sqrt{2}\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\Bigg)\\
&-\frac{\varepsilon}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\left(-\tfrac{7\sqrt{2}}{12}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}uv-\tfrac{3\sqrt{2}}{8}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}vw\right.\\
&+\tfrac{13}{9}\,u\sqrt{2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{3}vw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{7}{18}\,uv\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{7\sqrt{2}}{18}\,v^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{7}{12}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}vw-\tfrac{65}{36}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}uv\\
&-\tfrac{1}{72}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}\sqrt{2}v^2-\tfrac{67}{144}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u^2\sqrt{2}\\
&+\tfrac{3}{8}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}uw-\tfrac{7}{6}uw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+uw-\sqrt{2}vw+\tfrac{2}{9}v^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{4}w^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{7}{12}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u\sqrt{2}w-\tfrac{7\sqrt{2}}{12}\,vw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&+\tfrac{2}{3}\sqrt{2}v^2+\tfrac{1}{3}u^2\sqrt{2}-\tfrac{4}{3}uv+\tfrac{17}{18}u^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\\
&-\tfrac{7}{6}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}u^2+\tfrac{7}{6}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}v^2+\tfrac{14\sqrt{2}}{9}\,uv\\
&+\tfrac{28}{9}\,v^2-\tfrac{28}{9}\,u^2-\tfrac{3\sqrt{2}}{32}\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}w^2\\
&\left.+\tfrac{7\sqrt{2}}{18}\,u^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{\sqrt{2}}{6}uw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\right)
\end{aligned}
$$

$$-\frac{\varepsilon^2}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\Bigg(-\alpha_1 w\sqrt{2}-\tfrac{53\sqrt{2}}{144}\,uvw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$-\tfrac{13}{144}\,v^2w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{5}{18}\,uv^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$-\tfrac{2}{3}\,\alpha_1 u\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\alpha_1 v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$-\tfrac{11\sqrt{2}}{216}\,v^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{32}vw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$+\tfrac{\sqrt{2}}{2}\alpha_1 u\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{3}\,\alpha_1\sqrt{2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$-\tfrac{7\sqrt{2}}{72}\,u^2w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{1}{64}\,u\sqrt{2}w^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$-\tfrac{17}{144}\,u\sqrt{2}v^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{7}{72}\,uvw\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$+\tfrac{61}{144}\,u^2\sqrt{2}v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{7}{96}\,\sqrt{2}vw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$+\tfrac{7}{72}\,v^2\sqrt{2}w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{64}\,w^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$-\tfrac{2}{3}\alpha_1 u+\tfrac{7}{48}\,uw^2\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$+\tfrac{31\sqrt{2}}{288}\,u^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{5}{144}\,u^2v\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$-\tfrac{79}{288}\,u^2w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}-\tfrac{1}{8}v^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}$$
$$+\tfrac{19}{216}\,u^3\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}+\tfrac{2}{3}\alpha_1\sqrt{2}v+7/3\,\alpha_1 u\sqrt{2}+7/3\,\alpha_1 v$$
$$+\tfrac{1}{2}\alpha_1 w\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}\Bigg)-\frac{\varepsilon^3}{\sqrt{2+\sqrt{-\alpha_1\left(4+\alpha_1\varepsilon^2\right)\varepsilon^2}}}\Bigg(\tfrac{3\sqrt{2}}{32}\,\alpha_1 w^2$$
$$+\tfrac{25\sqrt{2}}{72}\,\alpha_1 v^2+\tfrac{7\sqrt{2}}{36}\,\alpha_1 uv+\tfrac{7}{18}\,\alpha_1 v^2-\tfrac{7}{18}\,\alpha_1 u^2+\tfrac{41}{36}\,\alpha_1 uv-\tfrac{7}{12}\,\alpha_1 vw$$
$$-\tfrac{7\sqrt{2}}{12}\,\alpha_1 uw+\tfrac{1}{8}\alpha_1 uw-\tfrac{\sqrt{2}}{8}\alpha_1 vw+\tfrac{91}{144}\,\alpha_1 u^2\sqrt{2}\Bigg).$$

Now, we write this differential system in cylindrical coordinates (r,θ,w) defined by $u=r\cos\theta$, $v=r\sin\theta$, $w=w$ and after we introduce θ the new independent variable, and so we arrive to the system

$$\begin{aligned}\frac{dr}{d\theta} \;=\; \frac{1}{576}\Big(&48\,\sqrt{2}\sqrt{-\alpha_1}r+12\,\sqrt{2}rw+2\,\sqrt{2}r^2\left(\cos\left(\theta\right)\right)^3-144\,\sin\left(\theta\right)\sqrt{2}\sqrt{-\alpha_1}w\\ &-10\,r^2\left(\cos\left(\theta\right)\right)^2\sin\left(\theta\right)-9\,\cos\left(\theta\right)\sqrt{2}w^2-384\,\sqrt{-\alpha_1}r\left(\cos\left(\theta\right)\right)^2\\ &+144\,\cos\left(\theta\right)\sqrt{-\alpha_1}w-72\,\cos\left(\theta\right)\sqrt{2}\sqrt{-\alpha_1}w+24\,\sqrt{2}\sin\left(\theta\right)r\cos\left(\theta\right)w\\ &-72\,\sin\left(\theta\right)\sqrt{-\alpha_1}w+192\,\cos\left(\theta\right)\sqrt{-\alpha_1}r\sin\left(\theta\right)+12\,\sqrt{2}r\left(\cos\left(\theta\right)\right)^2w\\ &+48\,\sqrt{2}\sqrt{-\alpha_1}r\left(\cos\left(\theta\right)\right)^2+48\,r\cos\left(\theta\right)\sin\left(\theta\right)w\end{aligned}\qquad(13)$$

$$
\begin{aligned}
& +52\sqrt{2}\sin(\theta)\,r^2(\cos(\theta))^2+96\sin(\theta)\sqrt{2}\sqrt{-\alpha_1}r\cos(\theta)\\
& -168\cos(\theta)\,r^2-9\sin(\theta)\,w^2+18\cos(\theta)\,w^2+260\,r^2(\cos(\theta))^3\\
& -40\sin(\theta)\sqrt{2}r^2-96\,r(\cos(\theta))^2w-18\sqrt{2}\sin(\theta)\,w^2-24\cos(\theta)\sqrt{2}r^2\\
& -20\sin(\theta)\,r^2+192\sqrt{-\alpha_1}r+48\,rw\Big)\sqrt{2}\varepsilon+O\left(\varepsilon^2\right)\\
= {} & \varepsilon F_{11}(\theta,r,w)+O\left(\varepsilon^2\right),\\
\frac{dw}{d\theta} = {} & \Bigg(-\frac{1}{2}\sqrt{-\alpha_1}w-\frac{1}{16}w^2-\frac{7\sqrt{2}}{24}r^2-\frac{7}{72}r^2(\cos(\theta))^2\\
& +\frac{7\sqrt{2}}{48}r\sin(\theta)\,w-\frac{7\sqrt{2}}{36}\cos(\theta)\,r^2\sin(\theta)\\
& -\frac{\sqrt{2}}{3}\sqrt{-\alpha_1}r\cos(\theta)-\frac{\sqrt{2}}{12}r\cos(\theta)\,w\\
& +\frac{7\sqrt{2}}{12}\sqrt{-\alpha_1}r\sin(\theta)+\frac{7\sqrt{2}}{12}r^2(\cos(\theta))^2\\
& -\frac{7}{24}\cos(\theta)\,r^2\sin(\theta)+\frac{7}{6}\sqrt{-\alpha_1}r\cos(\theta)+\frac{7}{24}r\cos(\theta)\,w\\
& +\frac{1}{6}r\sin(\theta)\,w+\frac{2}{3}\sqrt{-\alpha_1}r\sin(\theta)-\frac{2}{9}r^2\Bigg)\varepsilon+O\left(\varepsilon^2\right)\\
= {} & \varepsilon F_{12}(\theta,r,w)+O\left(\varepsilon^2\right).
\end{aligned}
$$

Our previous system has the form of the differential equation (13) with $t=\theta$, $x=(r,w)\in D=(0,+\infty)\times\mathbb{R}$, $T=2\pi$, and $F_1(\theta,r,w)=(F_{11}(\theta,r,w),F_{12}(\theta,r,w))$, and an easy computation shows that

$$
f_1(r,w)=(f_{11}(r,w),f_{12}(r,w)),
$$

is given by

$$
\begin{aligned}
f_{11}(r,w) &= \frac{1}{2\pi}\int_0^{2\pi}F_{11}(\theta,r,w)\,d\theta\\
&= \frac{1}{16}r\left(4\sqrt{-\alpha_1}+w\right),\\
f_{12}(r,w) &= \frac{1}{2\pi}\int_0^{2\pi}F_{12}(\theta,r,w)d\theta\\
&= -\frac{1}{2}\sqrt{-\alpha_1}w-\frac{13}{48}r^2-\frac{1}{16}w^2,
\end{aligned}
$$

we solve the system of equations

$$
\left\{\begin{array}{l}
\frac{1}{16}r\left(4\sqrt{-\alpha_1}+w\right)=0,\\
-\frac{1}{2}\sqrt{-\alpha_1}w-\frac{13}{48}r^2-\frac{1}{16}w^2=0,
\end{array}\right. \tag{14}
$$

where $r > 0$, we find that this system accepts a one solution

$$(r, w) = \left(\frac{4\sqrt{-39\alpha_1}}{13}, -4\sqrt{-\alpha_1}\right),$$

where $\alpha_1 < 0$

The Jacobian at $\left(\frac{4\sqrt{-39\alpha_1}}{13}, -4\sqrt{-\alpha_1}\right)$, take the value

$$\det \left.\frac{\partial (f_{11}, f_{12})}{(r, w)}\right|_{(r,w)=\left(\frac{4\sqrt{-39\alpha_1}}{13}, -4\sqrt{-\alpha_1}\right)} = -\frac{1}{8}\alpha_1, \tag{15}$$

In short the solution $\left(\frac{4\sqrt{-39\alpha_1}}{13}, -4\sqrt{-\alpha_1}\right)$ of system (14) which verify condition (15) satisfy the assumptions (i) and (ii) of Theorem 6. So we conclude that applying the averaging theory of first order system (13) has one periodic orbits. From it we conclude the Rabinovitch-Fabrikant differential system (1) has a zero-Hopf bifurcation at the zero-Hopf equilibrium point localized at equilibrium point Q_+, and a periodic orbit born at this equilibrium when $\varepsilon = 0$ and it exists for $\varepsilon > 0$ sufficiently small.

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