ON THE LIMIT CYCLES FOR A CLASS OF GENERALIZED LIÉNARD DIFFERENTIAL SYSTEM

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Abstract. The main aim of the present paper is to study the existence of limit cycles of a class of piecewise generalized Liénard differential system. The tool that we use is the averaging theory of the dynamical systems.

1. Introduction and formulation of the main results

The so called sixteenth Hilbert problem problem is an open problem in mathematics with two parts proposed in 1900 by David Hilbert [6]. The part number two of this problem deals with the evaluation of the maximum number and position of the limit cycles of a polynomial system of degree n of the form

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q\left(x,y\right). \end{cases}$$

This problem was reformulated by Stephen Smale in 1998, see [11]. In such paper, due to the complexness of the problem is suggested the study of a simpler class of differential equations, called Liénard's polynomial differential equations.

At the literature, there are many papers on the existence of limit cycles of the equations of Liénard type, see for instance [13] and references therein.

Definition 1. *The following equation*

$$\ddot{x} + f(x)\dot{x} + x = 0, \tag{1}$$

where $f(x)$ is a polynomial of degree n is called the classical Liénard's polynomial differential equation of degree n.

In 1977, at the paper [9] was conjectured that the equation (1) has at most $\left[\dfrac{n}{2}\right]$ limit cycles, where $n \geqslant 2$. It was proven that the conjecture

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is true for $n = 2, 3$, see [9] and [8]. The conjecture for $n = 4$ is still open. In 2007 [5] Dumortier et al. proved it does not work for $n \geq 6$. In [3] was proven in 2011 that the conjecture is not true for $n \geq 5$ and they added 2 limit cycles for this conjecture. In 2015 [4] De Maesschalck and Huzak proved that it does not work for $n \geq 5$ and they proposed one limit cycle less for this conjecture. By $[\cdot]$ we shall denote the integer part function.

Let consider the following system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon \left(f(x)y + \varphi_w(y) \left(k_1 x + k_2 \right) \right), \end{aligned}$$

where k_1, k_2 are real mumbers, ε is a parameter of small size, $f(x)$ is a polynomial of degree n and $\varphi_w(y)$ is defined in (3). In [10] was proved that the maximum number of limit cycles for such system is $[n/2] + 1$.

Inspired in the previous system, our aim in the present paper is to study the limit cycles of the a generalized system of Liénard type in the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon \left(f(x,y)y + \varphi_w(y) \left(k_1 x + k_2 \right) \right), \end{aligned} \tag{2}$$

where ε is a small parameter, $f(x, y)$ is a two variable polynomial of degree n and the function $\varphi_w : \mathbb{R} \to \mathbb{R}$ is defined as follows

$$\varphi_w(y) = \begin{cases} -1, & \text{if } y < -w \\ \dfrac{y}{w}, & \text{if } -w < y < w \\ 1, & \text{if } y > w \end{cases} \tag{3}$$

Note, that at the present formulation, $\varphi_w(y)$ represents the piecewise linear function.

We underline that if $w \to 0$ in (2), we have the so called discontinuous generalized Liénard polynomial differential system. This system is important for the non-smooth dynamical systems. Moreover

$$\lim_{w \to 0} \varphi_w(y) = sgn(y)$$

Non-smooth systems of differential equations has been used recently in many science areas like mathematics, physics, engineering, see for more details [7],[2] and the references quoted therein. In these problems the detection of limit cycles is of fundamental importance.

The statement of our main results is the following:

Theorem 1. *Let $|\varepsilon|$ be a real number sufficiently small and $n \geq 1$. Then, the piecewise linear generalized Liénard differential system* (2) *has $[n/2] + 1$ as maximum number of limit cycles. Moreover, such limit cycles bifurcate from the periodic orbits of the linear center $\dot{x} =$*

$y, \dot{y} = -x$. Moreover there are systems (2) *having exactly* $[n/2]+1$ *limit cycles.*

2. Averaging theory of dynamical systems: first order approach

The averaging theory of dynamical systems is a theory that allows under certain conditions to provide information on the periodic structure of given system. See the following monograph for more details [1].

Let consider here the following dynamical systems

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{4}$$

where ε is a small parameter, the map F_1 is $C^0(\mathbb{R}\times D, \mathbb{R}^n)$, $R \in C^0(\mathbb{R}\times D \times (-\varepsilon_f, \varepsilon_f), \mathbb{R}^n)$ are maps periodic with period T in t and $D \subset \mathbb{R}^n$ is an open subset.
In addition, we define the averaging function $F_{10}: D \to \mathbb{R}^n$ by

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s,z)ds.$$

We also add the following assumptions
(i) The maps F_1 and R are Lipschitz in a local way regarding the second variable.
(ii) If $V \subset D$ is a bounded open region, there exists $a \in V$ holding that $F_{10}(a) = 0$, $F_{10}(z) \neq 0$ for every $z \in \bar{V}\backslash\{a\}$ and $d_B(F_{10}, V, 0) \neq 0$.
In this setting, given $|\varepsilon| > 0$ sufficiently small the averaging theory of dynamical systems at first order guarantee the existence of a T -periodic orbit $\psi(t,\varepsilon)$ of system (4) holding $\psi(0,\varepsilon) \to a$ when $\varepsilon \to 0$.
Here $d_B(F_{10}, V, 0)$ represents the Brouwer degree of map F_{10}.

3. Proof of Theorem 1

In order to be able to apply the averaging theory of dynamical systems we shall perform a change of variable in polar coordinates of the form $(x,y) = (r\cos(\Theta), r\sin(\Theta))$ with $r > 0$. We want to obtain the standard form of system (4). If $f(x,y) = \sum\limits_{i+j=0}^{n} a_{ij}x^i y^j$ then the system

(2) writes as follows

$$\begin{cases} \dot{r} = -\varepsilon \left(\sum_{i+j=0}^{n} a_{ij} r^{i+j+1} \cos^i \Theta \sin^{j+2} \Theta + \varphi_w(r \sin \Theta) \left(k_1 r \cos \Theta + k_2 \right) \sin \Theta \right), \\ \dot{\Theta} = -1 - \dfrac{\varepsilon}{r} \left(\sum_{i+j=0}^{n} a_{ij} r^{i+j+1} \cos^{i+1} \Theta \sin^{j+1} \Theta + \varphi_w(r \sin \Theta) \left(k_1 r \cos \Theta + k_2 \right) \cos \Theta \right). \end{cases} \tag{5}$$

In (5) we use the new independent variable Θ, we obtaint the equation

$$\frac{dr}{d\Theta} = \varepsilon \left(\left(\sum_{i+j=0}^{n} a_{ij} r^{i+j+1} \cos^i \Theta \sin^{j+2} \Theta \right) + \varphi_w(r \sin \Theta) \left(k_1 r \cos \Theta \sin \Theta + k_2 \sin \Theta \right) \right) + O\left(\varepsilon^2\right),$$

where

$$\varphi_w(r \sin \Theta) = \begin{cases} -1, & \text{if } \sin \Theta < -\dfrac{w}{r} \\ \dfrac{r \sin \Theta}{w}, & \text{if } -\dfrac{w}{r} < \sin \Theta < \dfrac{w}{r} \\ 1, & \text{if } \sin \Theta > \dfrac{w}{r} \end{cases}$$

and

$$\begin{aligned} F_{10}(r) \quad = \quad & \frac{1}{2\pi} \int_0^{2\pi} \left(\left(\sum_{i+j=0}^{n} a_{ij} r^{i+j+1} \cos^i \Theta \sin^{j+2} \Theta \right) \right. \\ & \left. + \varphi_w(r \sin \Theta) \left(k_1 r \cos \Theta \sin \Theta + k_2 \sin \Theta \right) \right) d\Theta, \end{aligned}$$

denoting

$$F_{10}(r) = \frac{1}{2\pi} \left(F_{10a}(r) + F_{10b}(r) \right),$$

with

$$F_{10a}(r) = \int_0^{2\pi} \left(\sum_{i+j=0}^{n} a_{i,j} r^{i+j+1} \cos^i \Theta \sin^{j+2} \Theta \right) d\Theta,$$

and

$$F_{10b}(r) = \int_0^{2\pi} \left(\varphi_w(r \sin \Theta) \left(k_1 r \cos \Theta \sin \Theta + k_2 \sin \Theta \right) \right) d\Theta,$$

we use the following formulas

$$\int_0^{2\pi} \cos^i \Theta \sin^{j+2} \Theta d\Theta = \begin{cases} 0 & \text{if } i \text{ odd, or } j \text{ odd,} \\ \pi \alpha_{ij} & \text{if } i, j \text{ even,} \end{cases}$$

$$\int_0^{2\pi} \cos^i \Theta \sin \Theta d\Theta \quad = 0, \quad \text{for} \quad i = 0, 1, \ldots$$

for calculate the expressions of F_{10a}. Hence

$$F_{10a}(r) = \sum_{\substack{i+j=0 \\ i,\, j \text{ even}}}^{n} \pi \alpha_{ij} a_{ij} r^{i+j+1}.$$

Now for every $r_1 > 0$ we note that

$$\begin{aligned} G_1\left(r_1, w\right) &= \int_0^{2\pi} \left(\varphi_w\left(r_1 \sin\Theta\right)\left(k_1 r \cos\Theta \sin\Theta + k_2 \sin\Theta\right)\right) d\Theta \\ &= \begin{cases} \pi k_2 \dfrac{r_1}{w} & 0 < r_1 \leq w \\ 2k_2 \left(\dfrac{r_1}{w} arcsec\left(\dfrac{r_1}{w}\right) + \dfrac{\sqrt{r_1^2 - w^2}}{r_1}\right) & r_1 \geq w \end{cases} \end{aligned}$$

for calculate the expressions of F_{10b}.
So the averaging function F_{10} is given by

$$F_{10}\left(r_1\right) = \left(\sum_{\substack{i+j=0 \\ i,\, j \text{ even}}}^{n} \pi \alpha_{ij} a_{ij} r_1^{i+j+1}\right) + G_1\left(r_1, w\right),$$

If $F_{10}\left(r_1\right) = 0$. We distinguish two cases: $0 < r_1 \leq w$ and $r_1 > w$. we write $F_{10}\left(r_1\right)$ as follows

$$F_{10}\left(r_1\right) = \begin{cases} F_{10}^{I}\left(r_1\right) & , \quad r_1 < w \\ F_{10}^{II}\left(r_1\right) & , \quad r_1 \geq w \end{cases}$$

with

$$\begin{aligned} F_{10}^{I}\left(r_1\right) &= \frac{1}{2\pi}\left(\left(\sum_{\substack{i+j=0 \\ i,\, j \text{ even}}}^{n} \pi \alpha_{ij} a_{ij} r_1^{i+j+1}\right) + \pi k_2 \frac{r_1}{w}\right), \\ &= \frac{r_1}{2}\left(\left(\sum_{\substack{i+j=0 \\ i,\, j \text{ even}}}^{n} \alpha_{ij} a_{ij} r_1^{i+j}\right) + \frac{k_2}{w}\right), \\ &= \frac{r_1}{2}\left(\left(\sum_{\substack{i+j=2 \\ i,\, j \text{ even}}}^{n} \alpha_{ij} a_{ij} r_1^{i+j}\right) + \left(\alpha_{00} a_{00} + \frac{k_2}{w}\right)\right), \end{aligned}$$

and

$$F_{10}^{II}\left(r_1\right)=\frac{1}{2\pi}\left(\sum_{\substack{i+j=0\\ i,\ j \text{ even}}}^{n}\pi\alpha_{ij}a_{ij}r_1^{i+j+1}+2k_2\left(\frac{r_1}{w}arcsec\left(\frac{r_1}{w}\right)+\frac{\sqrt{r_1^2-w^2}}{r_1}\right)\right),$$

from now on we can choose w small so that the function $F_{10}^{I}\left(r_1\right)=0$ does not accept zeros in the interval $(0,w)$.
For $r_1>w$. We suppose that

$$G\left(r_1,w\right)=2\left(\frac{r_1}{w}arcsec\left(\frac{r_1}{w}\right)+\frac{\sqrt{r_1^2-w^2}}{r_1}\right),$$

we have
(A) For every w fixed $\dfrac{\partial^2 G\left(r_1,w\right)}{\partial r_1^2}=-\dfrac{4w^2}{r^3\sqrt{r^2-w^2}}<0$, from it we conclude that the graph of $G(.,w)$ is concave.
(B) For every w fixed $\dfrac{\partial G\left(r_1,w\right)}{\partial r_1}=\dfrac{2}{w}arcsec\left(\dfrac{r_1}{w}\right)-\dfrac{2\sqrt{r_1^2-w^2}}{r_1^2}$ and,
(B1) $\lim\limits_{r_1\to w}\dfrac{\partial G}{\partial r_1}\left(r_1,w\right)=\dfrac{\pi}{w}$, and
(B2) $\lim\limits_{r_1\to\infty}\dfrac{\partial G}{\partial r_1}\left(r_1,w\right)=0$.
Returning to (A) we conclude that $\dfrac{\partial G}{\partial r_1}$ is decreasing. Then by returning to (A), (B1) and (B2) we conclude that $\dfrac{\partial G}{\partial r_1}\left(r_1,w\right)>0$, $\dfrac{\partial G}{\partial r_1}\left(r_1,w\right)<\dfrac{\pi}{w}$, $G(.,w)$ is increasing and the straight line $\dfrac{\pi}{w}r_1$ is above the graph of $G(.,w)$.
(C) $\lim\limits_{r_1\to\infty}G\left(r_1,w\right)=4, G(.,w):(0,\infty)\to(0,4)$ and G is a diffeomorphism of type C^1.
We conclude that F_{10} is C^1.
We are now studying the presence of zeros for the following equation

$$\sum_{\substack{i+j=0\\ i,\ j \text{ even}}}^{n}\pi\alpha_{ij}a_{ij}r_1^{i+j+1}+k_2G\left(r_1,w\right)=0, \tag{6}$$

For simplification, we take $k_2 = \pi$.
There are two cases, if n is odd, the equation (6) becomes

$$-r_1\left(\sum_{\substack{i+j=0\\ i,\ j \text{ even}}}^{n-1} \alpha_{ij}a_{ij}r_1^{i+j}\right) = G\left(r_1, w\right), \tag{7}$$

while if n is even, the equation (6) becomes

$$-r_1\left(\sum_{\substack{i+j=0\\ i,\ j \text{ even}}}^{n} \alpha_{ij}a_{ij}r_1^{i+j}\right) = G\left(r_1, w\right), \tag{8}$$

From equations (7) and (8), the maximum number of positive solutions is $[n/2] + 1$. So the system (2) has at most $[n/2] + 1$ limit cycles bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$. With this we have completed the proof of Theorem 1.

Example 1. *We consider the piecewise linear generalized Liénard differential system (2), where* $f(x, y) = \sum_{i+j=0}^{4} a_{ij}x^i y^j, k_1 = 1, w = 0,1, \varepsilon = 0,001$ *and*

$a_{00} = -4,923793758$	$a_{10} = 0$	$a_{01} = 0$
$a_{20} = 0,996908332$	$a_{11} = 0$	$a_{02} = 0,996908332$
$a_{30} = 0$	$a_{21} = 0$	$a_{12} = 0$
$a_{03} = 0$	$a_{40} = -0,075928995$	$a_{31} = 0$
$a_{22} = -0,075928995$	$a_{13} = 0$	$a_{04} = -0,075928995$

Now we solving the following equation with $r_1 > 0.1$

$$\begin{aligned} &-0,06643787062r_1^5\pi + 0,0996908332r_1^3\pi \\ &+(-4,923793758\pi + 20\pi arccsc(10r_1))r_1 + \frac{2\pi\sqrt{r_1^2 - 0,01}}{r_1} = 0 \end{aligned}$$

we get $r_1 = 1, r_2 = 2, r_3 = 3$. *So this differential system has* 3 *limit cycles bifurcating from the periodic orbits of the linear center* $\dot{x} = y, \dot{y} = -x$. *see Figure 1.*

Remark 1. *In the previous example, the conditions of Theorem 1 are fulfilled (it can be verified).*

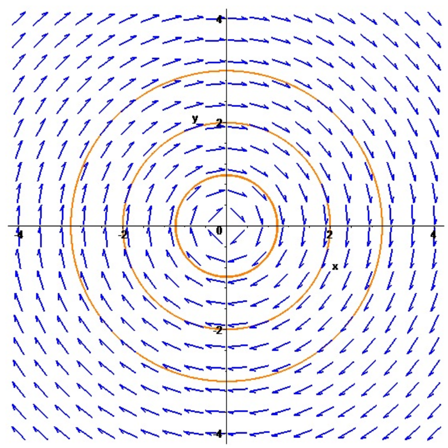

FIGURE 1. Limit cycles for $\varepsilon = 0,001$.

4. Conclusion

We have shown that the piecewise linear generalized Liénard differential system (2) has $[n/2]+1$ as a maximum number of limit cycles using the averaging theory and this result is given in Theorem 1. Moreover, we have given a numerical example to illustrate this.
We recall in this work the perturbation methods (averaging method...) are used the parameter small ε, but if this parameter is large, this methods is not called perturbation methods.

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