On the Implementation of Fractional Homotopy Perturbation Transform Method to the Emden-Fowler Equations

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Abstract

In this paper, we used the homotopy perturbation transform method (HPTM) to offer an efficient semi-analytical technique for solving fractional Emden-Fowler equations. A mixture of Laplace transform, Caputo-Fabrizio derivative, and homotopy perturbation transformation process has the projected technique. To assess the efficacy of the suggested technique, test examples have been provided. The series have been used to represent semi-analytical solutions. Also, covered have the convergence position, estimation, and semi-analytical simulation results. The HPTM efficiently managed and controlled a series solution that quickly converges to a precise result in a narrow admissible region. The new findings essentially improve and simplify some of the previously published findings (see ref [46]). By assigning appropriate values to free parameters, dynamical wave structures of some semi-analytical solutions are graphically demonstrated using 2-dimensional and 3-dimensional figure. Furthermore,various simulations are used to demonstrate the physical behaviors of the acquired solution with respect to fractional integer order.

Keywords: Time-fractional Emden-Fowler equations (EFEs); Homotopy Perturbation Transform Method; CF-derivative; Laplace Transform;

1. Introduction

The fractional calculus (FC), is the generic generalization of integer-order calculus to licentious order integration and differentiation with non integer order. Fractional calculus (FC) dates back to 1695, when l'Hopital addressed Leibnitz a letter regarding the probable meaning of $\frac{d^{1/2}x(t)}{dt^{1/2}}$, which represents the semi derivative of x(t) with respect to t. Due to its advantageous qualities such as analyticity,linearity, and non locality, fractional calculus has recently become a powerful tool. Furthermore, there are numerous pioneering references accessible for various definitions of FC,which lay the foundation for FC [1-4], With the rapid advancement of digital computer technology, many researchers are turning their attention to the theory and applications of fractional calculus. The Jacob Robert Emden (1862-1940), a swiss

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astrophysicist, and Sir Ralph Howard Fowler (1889-1944), an English astronomer, are the namesakes of the famous Emden-Fowler equation [5,6], The generalized derivative operator [7]. The HAM [8,9]. The caputo-fabrizio fractional derivative [10]. asymptotic behavior [11]. The adomian decomposition method [12]. The conformable derivative [13]. The q-homotopy analysis transform method [14]. The homotopy perturbation transform method [15]. The modified (G'/G)-expansion method [16,17]. The modified invariant subspace method [18]. The finite difference method [19]. The multi variable aleph -function [20]. The aboodh adomian decomposition method, a powerful research tool, is successfully used to extract the solution of ZKEs [21,22]. Bernoulli sub scheme is also used to observe Josephson effect [23]. The nonlinear complex Kundu–Eckhaus and Zakharov–Kuznetsov–Benjamin–Bona–Mahony equations have been investigated in conformable [24]. The banach's fixed point speculation is investigated for the controlling fractional-order model in order to determine the existence and uniqueness of the achieved solution [25], Some new voltage behavior such as dark-bright soliton solution, trigonometric, and complex function solutions [26-30]. Magnetohydrodynamic [31]. The uniform haar wavelet resolution technique [32]. The singular boundary value problems [33]. The analytical solutions [34]. $exp(-k(p))$-expansion technique [35]. The caputo fractional derivatives [36]. The generalized adams-bash forth-moulton method [37, 38]. Some standard fixed point theorems and fractional calculus theories [39]. The pseudo-spectral collocation method [40]. The mittag-leffler rule with fractal derivative generalized [42]. and so on.

To investigate these equations, mathematicians developed some of the most extensively used statistics. Emden-differential Fowler's equation is one of these equations. It has numerous applications in a variety of scientific fields. This equation is expressed in its general form as:

$$x^{-\kappa}\frac{d^m}{dx^m}(x^{\kappa}\frac{d^n}{dx^n})y + h(x)j(x) = 0,\ \kappa > 0. \tag{1}$$

The Wazwaz [12] this equation was proposed, and it explains a lot of remarkable facts. The Emden-Fowler (EF) Equations were first proposed by Fowler as a solution to an astronomical problem [5]. Berkovich [6], followed up with a consideration of its particular situations and change into simpler forms. Other properties of the above equation, such as fastness, asymptotic evolution, continuity, boundary value problem, oscillations, and boundedness, were discussed by Wong [5] in 1975.

The homotopy perturbation transform method (HPTM) is described, in which continuous mapping is produced from the initial obligation to the exact solutions. The subsidiary parameter confirms solutions convergence. HPTM is recognized even if a given non-linear problem doesn't restrain any small/large parameters. The convergence zone and rate of approximation category can be adjusted and controlled. It can also be used to approximate a nonlinear issue by varying the base functions. The connection of semi-analytical approaches with the Laplace transform is well-known for avoiding time-consuming repercussions and requiring less CPU time to examine numerical solutions to nonlinear problems describe real-life applications. By selecting a suitable value for the auxiliary parameter alpha, we may easily alter and regulate the convergence zone of solution series in a vast allowed realm. Also, with the same grade point and order of solution range, it can yield many more acceptable solutions than all other analytical techniques. The development in HPTM is the creation of a novel correction function using homotopy polynomials. Five test issues confirm the accuracy of this strategy. This method can be used to solve multi-dimensional fractional physical problems with ease. The motion of a drop with memory in time is described by time-fractional differential equations. When variations are heavy-tailed, space fractional derivatives emerge to depict drop motion that accounts for a transform in the flow field over the exhaustive system. In addition, the fractional derivatives shows that the system memory is modulated or weighted. Electrical signal publicity in a transmission line, wave propagation, signal dissection, and other applications use the Emden-Fowler equation. Because of this, fractional modelling is appropriate for such systems. As a result, understanding the multi-dimensional fractional order Emden-Fowler equation is crucial. It appears to be intriguing to discover a numerical solutions of the fractional order Emden-Fowler equation using HPTM because of its ability to provide a parameter that allows us to regulate and change the series solution's convergence zone. HPTM also eliminates the need for linearisation, discretised, small dislocation, or any restricted assumptions, significantly reduces mathematical computational, provides nonlocal effect, promises a big convergence zone, and eliminates the need to calculate complicated polynomials, integrations, or small/large physical parameters. Conformable derivatives yield Caputo type fractional operators [43,44]. The mittag-leffler power law [45]. The application of the improved q-HAM and the optimal perturbation iteration process yield semi-analytical solutions to the Emden-Fowler problem [46], modified iterative method [47], cylindrical coordinate system [48], For temporal and spatial discretization, a modified Leap-Frog finite difference scheme with stabilised term and a central finite difference scheme are used [49], On the basis of strength and stiffness theory and calculation, applied materials were determined, and applied physics calculations were carried out [50], criteria for oscillation in second-order Emden-Fowler delay differential equations with a sub-linear neutral term [51], the extended sinh-Gordon equation expansion method [52], the Incomplete Global GMERR algorithm and the Global GMERR algorithm [53], various simulations are used to demonstrate the physical behaviors of the acquired solution with respect to fractional integer order. [54-62]. Laplace transform [63,64]. second-order Emden-Fowler neutral delay DEs as an application of oscillation criteria [65,66]. The EFEs under the dirichlet boundary value problem are the application of the variational method [67], q-homotopy analysis transform method [68], statistical analysis [69] The development, analysis,and application of a free coefficient algorithm can also reveal a desirable or undesirable property/behavior [70-72], to the best of our knowledge, this is the first time the caputo-fabrizio derivative has been applied to a singular differential equation problem. The present article (i) the brief introduction and some basic definition of fractional calculus is presented in section 2, (ii) the (HPTM) is discussed in section 3, (iii) The solution of Emden-Fowler equation using the Caputo Fabrizio type fractional operator by HPTM is given in section 4, (iv) The results and discussion in section 5. Finally, we have presented the detailed conclusion in section 6.

2. Preliminaries

Some fundamental definitions of the Riemann-Liouville (R-L) fractional differentiation, Laplace transform (LT) and FCD are presented [15, 35].

Definition 2.1 The Caputo derivative is define for $\alpha \geq 0$ & $n \in N \cup 0$ is defined as below [15]

$$ {}_{0}^{CF}D_{t}^{\alpha}\, u(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-\xi) \frac{d^{n}}{dt^{n}} u(\xi) d\xi, \tag{2} $$

where ${}_{0}^{CF}D_{t}^{\alpha}$ is a Caputo Fabrizio derivative.

Definition 2.2 Assume u be a function $u \in H^1(a_1, b_1),\ b_1 > 0,\ 0 < \alpha < 1$. Then,the fractional caputo-fabrizio factional operator is define as [15]:

$$ {}_0^{CF} D_t^{\alpha}\ u(t) = \frac{M(\alpha)}{1-\alpha)} \int_0^t exp[-\frac{\alpha(1-\xi)}{1-\alpha}]u'(\xi)d\xi, t \geq 0,\ 0 < \alpha < 1, \tag{3} $$

with a normalize functions $M(\alpha)$ which is depend on $\alpha \in M(0) = M(1) = 1$.
Definition 2.3 The CFD of order $0 < \alpha < 1$. is given by [15]

$$ {}_0^{CF} D_t^{\alpha}\ u(t) = \frac{2(1-\alpha)}{M(\alpha)(2-\alpha)} u(t) + \frac{2\alpha}{M(\alpha)(2-\alpha)} \int_0^t u(\xi)d\xi,\ t \geq 0, \tag{4} $$

where ${}_0^{CF} D_t^{\alpha}\ u(t) = 0$, if u is a constant function.
Definition 2.4 The Laplace transform (LT) for the (CFD) of order $0 < \alpha < 1$. and $m \in N$ is gives by [15]

$$ \begin{aligned} \mathscr{L}\left[{}_0^{CF} D_t^{(m+\alpha)}\ u(t)\right](s) =& \frac{1}{1-\alpha}\mathscr{L}[u^{m+1}(t)]\mathscr{L}\left[exp\left(\frac{-\alpha}{(1-\alpha)}t\right)\right] \\ =& \frac{s^{m+1}\mathscr{L}[u(t)] - s^m u(0) - s^{m-1}u'(0) \ldots - u^m(0)}{s+\alpha(1-s)} \end{aligned} \tag{5} $$

In particular, we have

$$ \mathscr{L}\left[{}_0^{CF} D_t^{(m+\alpha)}\ u(t)\right](s) = \frac{s\mathscr{L}(u(t))}{s+\alpha(1-s)}, \quad m = 0, $$

$$ \mathscr{L}\left[{}_0^{CF} D_t^{(m+\alpha)}\ u(t)\right](s) = \frac{s^2\mathscr{L}(u(t)) - su(o) - u'(0)}{s+\alpha(1-s)}, \quad m = 1. $$

3. General description of homotopy perturbation transform method via Caputo-Fabrizio type operator:

In this section presents a powerful scheme called the homotopy perturbation transform method. It is as follows [15]. We look at the following equation of Nonlinear partial differential equation along with caputo-Fabrizio derivative:

$$ {}_0^{CF} D_t^{m+\alpha} u(x,t) + \beta u(x,t) + \varphi u(x,t) = k(x,t), \quad n-1 < \alpha + m \leq n, \tag{6} $$

such that

$$ \frac{\partial^l u(x,0)}{\partial t^l} = f_l(x), \quad l = 0, 1, 2, \ldots n-1. \tag{7} $$

Now, be applying the (LT) on both Eq.(6) and Eq.(7), we receive:

$$ \mathscr{L}[u(x,t)] = \Theta(x,s) - \left(\frac{s+\alpha(1-s)}{s^{n+1}}\right)\mathscr{L}[\beta u(x,t) + \varphi u(x,t)]. \tag{8} $$

where

$$ \Theta(x,s) = \frac{1}{s^{m+1}}[s^m f_0(x) + s^{m-1} f_1(x) + \ldots + f_m(x)] + \frac{s+\alpha(1-s)}{s^{n+1}}\tilde{k}(x,s). \tag{9} $$

Taking the inverse Laplace transformation the Eq.(8) we have

$$u(x,t) = \Theta(x,s) - \mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{n+1}}\right)\mathscr{L}[\beta u(x,t) + \varphi u(x,t)]\right], \tag{10}$$

where $\Theta(x,s)$ is the term that arises from the source term, and it specifies the initial conditions. The solution $u(x,t)$ can be extended into an infinite sequence using the regular homotopy perturbation method as follows:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t). \tag{11}$$

where $u_m(x,t)$ are known functions is given by

$$\varphi u(x,t) = \sum_{n=0}^{\infty} p^n H_n(x,t). \tag{12}$$

The poly. $H_n(x,t)$ are define as [8-9]

$$H_m(u_0, u_1, u_2, \ldots u_n) = \frac{1}{n!}\frac{\partial^m}{\partial p^m}\left[\left(\sum_{m=0}^{\infty} p^i u_i\right)\right]_{p=0}, \qquad m = 0, 1, 2, \ldots; \tag{13}$$

substitute Eq.(11) and Eq.(12) into Eq.(10) we are getting

$$\sum_{m=0}^{\infty} u_m(x,t) = \Theta(x,s) - p\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}\left[\beta\sum_{m=0}^{\infty} p^m u_m(x,t) + \sum_{n=0}^{\infty} p^m H_m\right]\right]. \tag{14}$$

Comparing the coefficients of p^0, p^1, p^2, p^3 and p^4 we get

$$\begin{aligned}
p^0 : u_0(x,t) &= \Theta(x,s),\\
p^1 : u_1(x,t) &= -\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}[\beta u_0(x,t) + H_0(u)]\right]\\
p^2 : u_2(x,t) &= -\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}[\beta u_1(x,t) + H_1(u)]\right]\\
p^3 : u_3(x,t) &= -\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}[\beta u_2(x,t) + H_2(u)]\right]\\
&\vdots\\
p^{m+1} : u_{m+1}(x,t) &= -\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)\mathscr{L}[\beta u_{m+1}(x,t) + H_{m+1}(u)]\right].
\end{aligned}$$

By the help of HPTM, the series solutions is

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t) \tag{15}$$

This approach avoids linearization and weak nonlinearity assumptions, and the solution is created in the form of a general solution, making it more practical than the method of simplifying physical problems.

4. Semi-analytical experiments

In this section, we will solve various types of Emden-Fowler equations using the homotopy perturbation transform method [46].

Example 4.1. Contemplate the Emden-Fowler equation

$$ {}_{0}^{CF}D_{t}^{\alpha}\, u(x,t) = \frac{\partial^2 u}{\partial x^2} + \frac{5}{x}\frac{\partial u}{\partial x} - (12t^2 - 2tx^2 + 4t^4x^2)u(x,t), \tag{16} $$

with the IC

$$ u(x,0) = 1. \tag{17} $$

The Laplace transformation on both sides Eq.(16), we get

$$ L[u(x,t)] = \frac{1}{s} + \left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\frac{\partial^2 u}{\partial x^2} + \frac{5}{x}\frac{\partial u}{\partial x} - (12t^2 - 2tx^2 + 4t^4x^2)u(x,t)\right]. \tag{18} $$

Applying the inverse of the (LT) to Eq.(18)

$$ u(x,t) = 1 + \mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\frac{\partial^2 u}{\partial x^2} + \frac{5}{x}\frac{\partial u}{\partial x} - (12t^2 - 2tx^2 + 4t^4x^2)u(x,t)\right]\right]. \tag{19} $$

Now, we apply the HPTM

$$ \sum_{m=0}^{\infty} u_m = + p\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\sum_{m=0}^{\infty} p^m u_m + \frac{5}{x}\sum_{m=0}^{\infty} p^m u_m - (12t^2 - 2tx^2 + 4t^4x^2)\sum_{m=0}^{\infty} p^m u_m\right]\right], \tag{20} $$

Firmly facing the above conditions, we get

$$ \begin{aligned} p^0 : u_0(x,t) =& 1 \\ p^1 : u_1(x,t) =& -\frac{4}{3}t^3\left(3\alpha + x^2\alpha\right) \\ & - 2t\left(-x^4 + x^4\alpha\right) \\ & + t^2\left(-12 - 4x^2 + 12\alpha + 4x^2\alpha + x^4\alpha\right), \\ p^2 : u_2(x,t) =& \frac{8}{9}t^6\left(9\alpha^2 + 6x^2\alpha^2 + x^4\alpha^2\right) \\ & + 2t\left(32x^2 + x^4 - 64x^2\alpha - 2x^4\alpha + 32x^2\alpha^2 + x^4\alpha^2\right) \\ & - 2t^2\left(30 + 2x^2 - 60\alpha - 36x^2\alpha - x^4\alpha + 30\alpha^2 + 34x^2\alpha^2 + x^4\alpha^2\right) \\ & + \frac{1}{3}t^3\left(-108\alpha - 4x^2\alpha + 108\alpha^2 + 36x^2\alpha^2 + x^4\alpha^2\right) \\ & - \frac{8}{15}t^5\left(-90\alpha - 60x^2\alpha - 10x^4\alpha + 90\alpha^2 + 60x^2\alpha^2 + 13x^4\alpha^2 + x^6\alpha^2\right) \\ & + \frac{4}{3}t^4\left(-6x^4\alpha - 2x^6\alpha - 3\alpha^2 + 6x^4\alpha^2 + 2x^6\alpha^2\right), \end{aligned} $$

$$
\begin{aligned}
p^3 : u_3(x,t) =& -2t\left(-384 - 64x^2 - x^4 + 1152\alpha + 192x^2\alpha + 3x^4\alpha - 1152\alpha^2 - 192x^2\alpha^2 - 3x^4\alpha^2 + 384\alpha^3 + \ldots\right)\\
&+ t^2\left(-108 - 4x^2 + 1476\alpha + 204x^2\alpha + 3x^4\alpha - 2628\alpha^2 - 396x^2\alpha^2 - 6x^4\alpha^2 + 1260\alpha^3 + 196x^2\alpha^3 \ldots\right)\\
&- \frac{1}{3}t^3\left(264\alpha + 8x^2\alpha - 1680\alpha^2 - 208x^2\alpha^2 - 3x^4\alpha^2 + 1416\alpha^3 + 200x^2\alpha^3 + 3x^4\alpha^3\right)\\
&+ \frac{32}{81}t^9\left(27\alpha^3 + 27x^2\alpha^3 + 9x^4\alpha^3 + x^6\alpha^3\right)\\
&- \frac{4}{45}t^6\left(-1530\alpha^2 - 700x^2\alpha^2 - 10x^4\alpha^2 + 1530\alpha^3 + 796x^2\alpha^3 + 73x^4\alpha^3 + x^6\alpha^3\right)\\
&+ \frac{4}{15}t^5\left(1620\alpha + 760x^2\alpha + 20x^4\alpha - 3240\alpha^2 - 1904x^2\alpha^2 - 292x^4\alpha^2 - 4x^6\alpha^2 + 1617\alpha^3 + \ldots\right)\\
&- \frac{1}{12}t^4\left(3072x^2\alpha + 2016x^4\alpha + 32x^6\alpha + 204\alpha^2 - 6140x^2\alpha^2 - 4032x^4\alpha^2 - 64x^6\alpha^2 - \ldots\right)\\
&- \frac{2}{9}t^8\left(-432\alpha^2 - 432x^2\alpha^2 - 144x^4\alpha^2 - 16x^6\alpha^2 + 432\alpha^3 + 432x^2\alpha^3 + 153x^4\alpha^3 + \ldots\right)\\
&+ \frac{16}{63}t^7\left(-63x^4\alpha^2 - 42x^6\alpha^2 - 7x^8\alpha^2 + 36\alpha^3 + 16x^2\alpha^3 + 63x^4\alpha^3 + 42x^6\alpha^3 + 7x^8\alpha^3\right).\\
&\vdots
\end{aligned}
$$

Hence series solution is given by

$$
u(x,t) = \sum_{m=0}^{\infty} u_m(x,t) = u_0 + u_1 + u_2 + u_3 + \ldots. \tag{21}
$$

Therefore, converges to exact solution $u(x,t) = e^{x^2t^2}$ of the integer-order EFEs as

Example 4.2. Contemplate the Emden-Fowler equation

$$
{}_{0}^{CF}D_t^{\alpha}\, u(x,t) = \frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x} - (5+4x^2)u(x,t) - (6-5x^2-4x^4), \tag{22}
$$

with the IC

$$
u(x,0) = x^2 + e^{x^2}. \tag{23}
$$

The Laplace transformation on both sides Eqs.(22-23) , we get

$$
\mathscr{L}[u(x,t)] = \frac{1}{s}(x^2+e^{x^2}) - \left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}(6+5x^2-4x^2) + \left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x} - (5+4x^2)u\right]. \tag{24}
$$

Applying the inverse of the (LT) to Eq.(24)

$$
u = x^2 + e^{x^2} + (-6+5x^2-4x^2)(1-\alpha+\alpha t) + \mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x} - (5+4x^2)\right]\right]. \tag{25}
$$

Now, we apply the HPTM

$$\sum_{m=0}^{\infty} u_m = x^2 + e^{x^2} - (6 + 5x^2 - 4x^2)(1 - \alpha + \alpha t) + p\mathscr{L}^{-1}\left[\left(\frac{s + \alpha(1-s)}{s}\right)\mathscr{L}\left[\sum_{m=0}^{\infty} p^m u_m + \frac{2}{x}\sum_{m=0}^{\infty} p^m u_m - (5 + 4x^2)\sum_{m=0}^{\infty} p^m u_m\right]\right], \tag{26}$$

Firmly facing the above conditions, we get

$$\begin{aligned}
p^0 : u_0(x,t) &= x^2 + e^{x^2} + \left(-6 + 5x^2 - 4x^4\right)(1 - \alpha + t\alpha),\\
p^1 : u_1(x,t) &= 30 + 7e^{x^2} - 80x^2 + 4e^{x^2}x^2 - 8x^4 - 54\alpha - 7e^{x^2}\alpha + 54t\alpha + 7e^{x^2}t\alpha + 155x^2\alpha - 4e^{x^2}x^2\alpha\\
&\quad - 155tx^2\alpha + 4e^{x^2}tx^2\alpha + 12x^4\alpha - 12tx^4\alpha + 24\alpha^2 - 48t\alpha^2 + 12t^2\alpha^2\\
&\quad - 75x^2\alpha^2 + 150tx^2\alpha^2 - \frac{75}{2}t^2x^2\alpha^2 - 4x^4\alpha^2 + 8tx^4\alpha^2 - 2t^2x^4\alpha^2\\
p^2 : u_2(x,t) &= \frac{1}{2}t^2\alpha^2\left(-1302 + 73e^{x^2} - 550x^2 + 88e^{x^2}x^2 - 16x^4 + 16e^{x^2}x^4 + 1278\alpha + 465x^2\alpha + 12x^4\alpha\right)\\
&\quad + (-1 + \alpha)\left(630 - 73e^{x^2} + 360x^2 - 88e^{x^2}x^2 + 8x^4 - 16e^{x^2}x^4 - 876\alpha + 73e^{x^2}\alpha - 395x^2\alpha \ldots\right)\\
&\quad - t\alpha\left(1506 - 146e^{x^2} + 755x^2 - 176e^{x^2}x^2 + 20x^4 - 32e^{x^2}x^4 - 2604\alpha + 146e^{x^2}\alpha - 1100x^2\alpha \ldots\right.\\
&\quad + \frac{1}{6}t^3\left(-426\alpha^3 - 155x^2\alpha^3 - 4x^4\alpha^3\right),
\end{aligned}$$

$$\begin{aligned}
p^3 : u_3(x,t) = &- 2790 + 1039e^{x^2} - 520x^2 + 1932e^{x^2}x^2 - 8x^4 + 720e^{x^2}x^4 + 64e^{x^2}x^6 + 8826\alpha - 3117e^{x^2}\alpha + \ldots\\
&- \frac{1}{6}t^3\alpha^3\left(-1854 - 601e^{x^2} - 7403x^2 - 1112e^{x^2}x^2 - 2276x^4 - 272e^{x^2}x^4 - 64x^6 + 2244\alpha + \ldots\right)\\
&+ t(-1 + \alpha)\alpha\left(8826 - 3117e^{x^2} + 1675x^2 - 5796e^{x^2}x^2 + 28x^4 - 2160e^{x^2}x^4 - 192e^{x^2}x^6 - \ldots\right)\\
&+ \frac{1}{2}t^2\alpha^2\left(-2826 + 2679e^{x^2} + 6483x^2 + 4976e^{x^2}x^2 + 2260x^4 + 1712e^{x^2}x^4 + 64x^6 + 128e^{x^2}x^6 + \ldots\right.\\
&+ \frac{1}{24}t^4\left(-1356\alpha^4 - 235x^2\alpha^4 - 4x^4\alpha^4\right).\\
&\vdots
\end{aligned}$$

Hence series solution is given by

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots, \tag{27}$$

therefore, converges to exact solution $u(x,t) = e^{t+x^2} + x^2$ of the integer-order EFEs as

Example 4.3. Contemplate the Emden-Fowler equation

$${}_{0}^{CF}D_t^{\alpha}\, u(x,t) = \frac{\partial^2 u}{\partial x^2} + \frac{4}{x}\frac{\partial u}{\partial x} - (18x + 9x^4)u(x,t) + 2 + (18x + 9x^4)t^2, \tag{28}$$

with the IC

$$u(x,0) = e^{x^3}. \tag{29}$$

The Laplace transformation on both sides Eqs. (28-29) , we get

$$\mathscr{L}[u(x,t)] = \frac{1}{s}e^{x^3}+\left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}(18x+9x^4)t^2+\left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\frac{\partial^2 u}{\partial x^2}+\frac{4}{x}\frac{\partial u}{\partial x}-(18x+9x^4)u\right]. \tag{30}$$

Applying the inverse of the (LT) to Eq.(30)

$$\begin{aligned} u(x,t) =& e^{x^3} + \; 2+18t^2x+9t^2x^4-2\alpha+2t\alpha-18t^2x\alpha+6t^3x\alpha-9t^2x^4\alpha+3t^3x^4 \\ &+\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\frac{\partial^2 u}{\partial x^2}+\frac{4}{x}\frac{\partial u}{\partial x}-(18x+9x^4)\right]\right]. \end{aligned} \tag{31}$$

Now, we apply the HPTM

$$\begin{aligned} \sum_{m=0}^{\infty} u_m =& e^{x^3} + \; 2+18t^2x+9t^2x^4-2\alpha+2t\alpha-18t^2x\alpha+6t^3x\alpha-9t^2x^4\alpha+3t^3x^4 \\ &+p\mathscr{L}^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\mathscr{L}\left[\sum_{m=0}^{\infty}p^m u_m+\frac{4}{x}\sum_{m=0}^{\infty}p^m u_m-(18x+9x^4)\sum_{m=0}^{\infty}p^m u_m\right]\right]. \end{aligned} \tag{32}$$

Firmly facing the above conditions, we get

$$\begin{aligned} p^0: u_0(x,t) &= e^{x^3} + \; 2+18t^2x+9t^2x^4-2\alpha+2t\alpha-18t^2x\alpha+6t^3x\alpha-9t^2x^4\alpha+3t^3x^4 \\ p^1: u_1(x,t) &= 2+\frac{72t^2}{x}+18t^2x+252t^2x^2+9t^2x^4-4\alpha+4t\alpha-\frac{144t^2\alpha}{x}+\frac{48t^3\alpha}{x}-36t^2x\alpha+12t^3x\alpha \\ &-504t^2x^2\alpha+168t^3x^2\alpha-18t^2x^4\alpha+6t^3x^4\alpha+2\alpha^2-4t\alpha^2+t^2\alpha^2+\frac{72t^2\alpha^2}{x}-\frac{48t^3\alpha^2}{x} \\ &+\frac{6t^4\alpha^2}{x}+18t^2x\alpha^2-12t^3x\alpha^2+\frac{3}{2}t^4x\alpha^2+252t^2x^2\alpha^2-168t^3x^2\alpha^2 \\ &+21t^4x^2\alpha^2+9t^2x^4\alpha^2-6t^3x^4\alpha^2+\alpha^2 \\ \ldots p^2: u_2(x,t) &= -2t\left(-2\alpha+18x\alpha+9x^4\alpha+6\alpha^2-3\alpha^3\right)+2\left(-18x-9x^4-2\alpha+18x\alpha+9x^4\alpha+3\alpha^2-\ldots\right. \\ &+\frac{3t^5\left(-16\alpha^3+16x^2\alpha^3+280x^3\alpha^3+2x^4\alpha^3+56x^5\alpha^3+x^7\alpha^3\right)}{20x^3} \\ &-\frac{9t^4\left(16\alpha^2-16x^2\alpha^2-280x^3\alpha^2-2x^4\alpha^2-56x^5\alpha^2-x^7\alpha^2-16\alpha^3+16x^2\alpha^3\ldots\right)}{4x^3} \\ &-\frac{1}{x^3}3t^2\left(48-48x^2-840x^3-6x^4-168x^5-3x^7-144\alpha+144x^2\alpha+2520x^3\alpha+18x^4\alpha+\ldots\right. \\ &\frac{1}{3x^3}t^3\left(-432\alpha+432x^2\alpha+7560x^3\alpha+54x^4\alpha+1512x^5\alpha+27x^7\alpha+864\alpha^2-864x^2\alpha^2-\ldots\right. \\ &\vdots \end{aligned}$$

Hence series solution is given by

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t) = u_0 + u_1 + u_2 + u_3 + ... \tag{33}$$

therefore, converges to exact solution $u(x,t) = e^{x^3} + t^2$ of the integer-order EFEs as

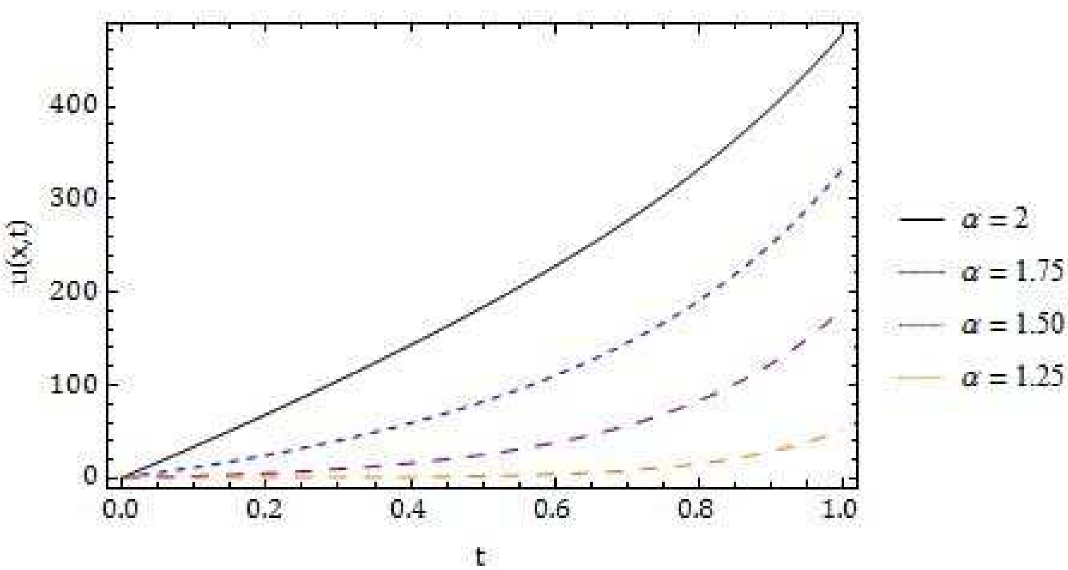

Figure 1: Comparison of our approximate solution $u(x,t)$ for different values of $\alpha = 1.25,\ \alpha = 1.50, \alpha = 1.75$ and $\alpha = 2$ for Ex.4.1.

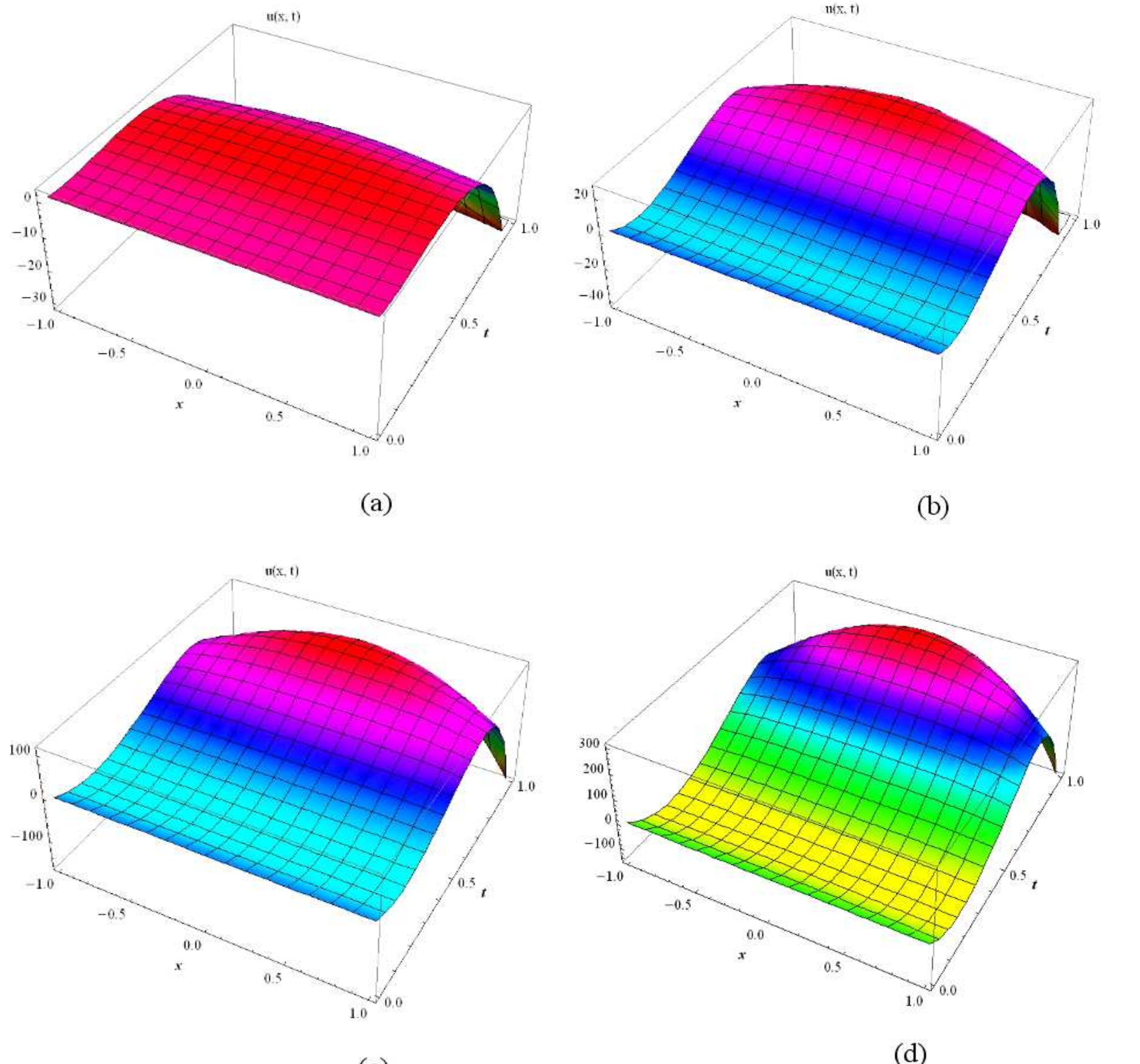

Figure 2: Surface show the 3D wave function $u(x,t)$ at (a) $\alpha = 1.25,$ (b) $\alpha = 1.50,$ (c) $\alpha = 1.75$ and (d) $\alpha = 2.$

5. Results and discussion

In this section, we show the 2-dimensional and 3-dimensional graphs for some of the reported solutions with a suitable parameter choice. Fig.1 shows the comparison of approximate solution for Eq.(16) attained by HPTM versus t for different values of α. Fig.2 (a)-(d) shows the profile of the third order approximation solution for 3D wave function for second order fractional nonlinear EFEs for $-1 \leq x \leq 1$ and $0 \leq t \leq 1$ at$\alpha = 1.25,\ 1.50,\ 1.75$ and $\alpha = 2$,for Eq.(16) by the application of initial condition represented by the Eq.(17) of $u(x,t)$. Fig.2 depicts the solitary wave nature of the approximate solution produced by HPTM for the second order fractional nonlinear EFEs. Fig.3

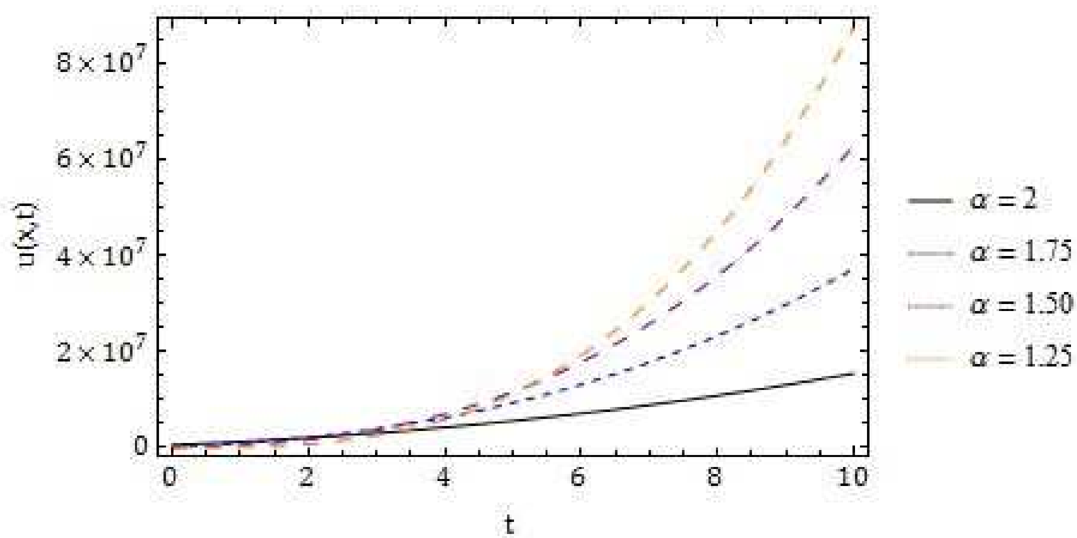

Figure 3: Comparison of approximate solution $u(x,t)$ for different values of $\alpha = 1.25,\ \alpha = 1.50, \alpha = 1.75$ and $\alpha = 2$ for Ex.4.2.

shows the comparison of approximate solution for Eq. (22) attained by HPTM versus t for different values of α, fig.3 (a)-(d) shows the profile of the third order approximation solution for 3D wave function for second order fractional nonlinear EFEs for $-1 \leq x \leq 1$ and $0 \leq t \leq 1$ at $\alpha = 1.25,\ 1.50,\ 1.75$ and $\alpha = 2$. for Eq.(22) by the application of initial condition represented by the Eq.(23) of $u(x,t)$, fig.3 depicts the solitary wave nature of the approximate solution produced by HPTM for the second order fractional nonlinear EFEs. Fig.4 shows the comparison of approximate solution for Eq.(28) attained by HPTM versus t for different values of α, fig.4 (a)-(d) shows the profile of the third order approximation solution for 3D wave function for second order fractional nonlinear EFEs for $-1 \leq x \leq 1$ and $0 \leq t \leq 1$ at $\alpha = 1.25,\ 1.50,\ 1.75$ and $\alpha = 2$, for Eq.(28) by the application of initial condition represented by the Eq.(29) of $u(x,t)$, fig.4 depicts the solitary wave nature of the approximate solutions produced by HPTM for the second order fractional nonlinear EFEs.

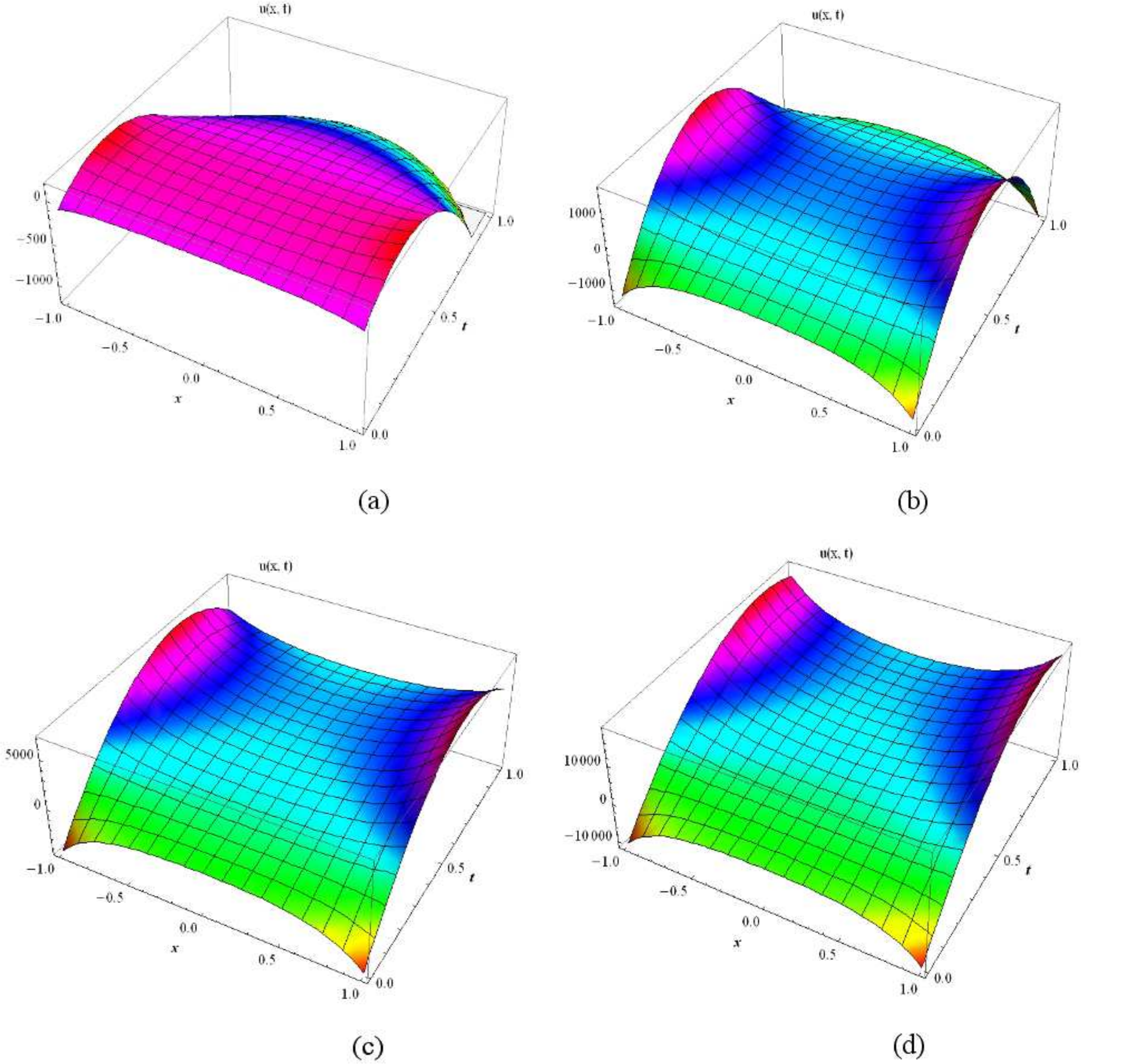

Figure 4: Surface show the 3D wave function $u(x,t)$ at (a) $\alpha = 1.25$, (b) $\alpha = 1.50$, (c) $\alpha = 1.75$ and (d) $\alpha = 2$.

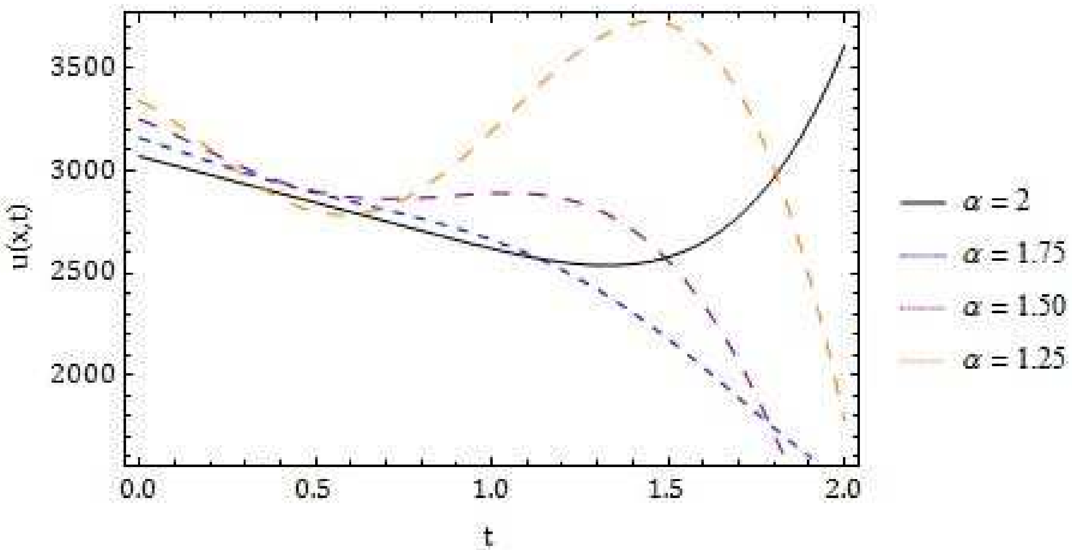

Figure 5: Comparison of approximate solution $u(x,t)$ for different values of $\alpha = 1.25$, $\alpha = 1.50, \alpha = 1.75$ and $\alpha = 2$ for Ex.4.3.

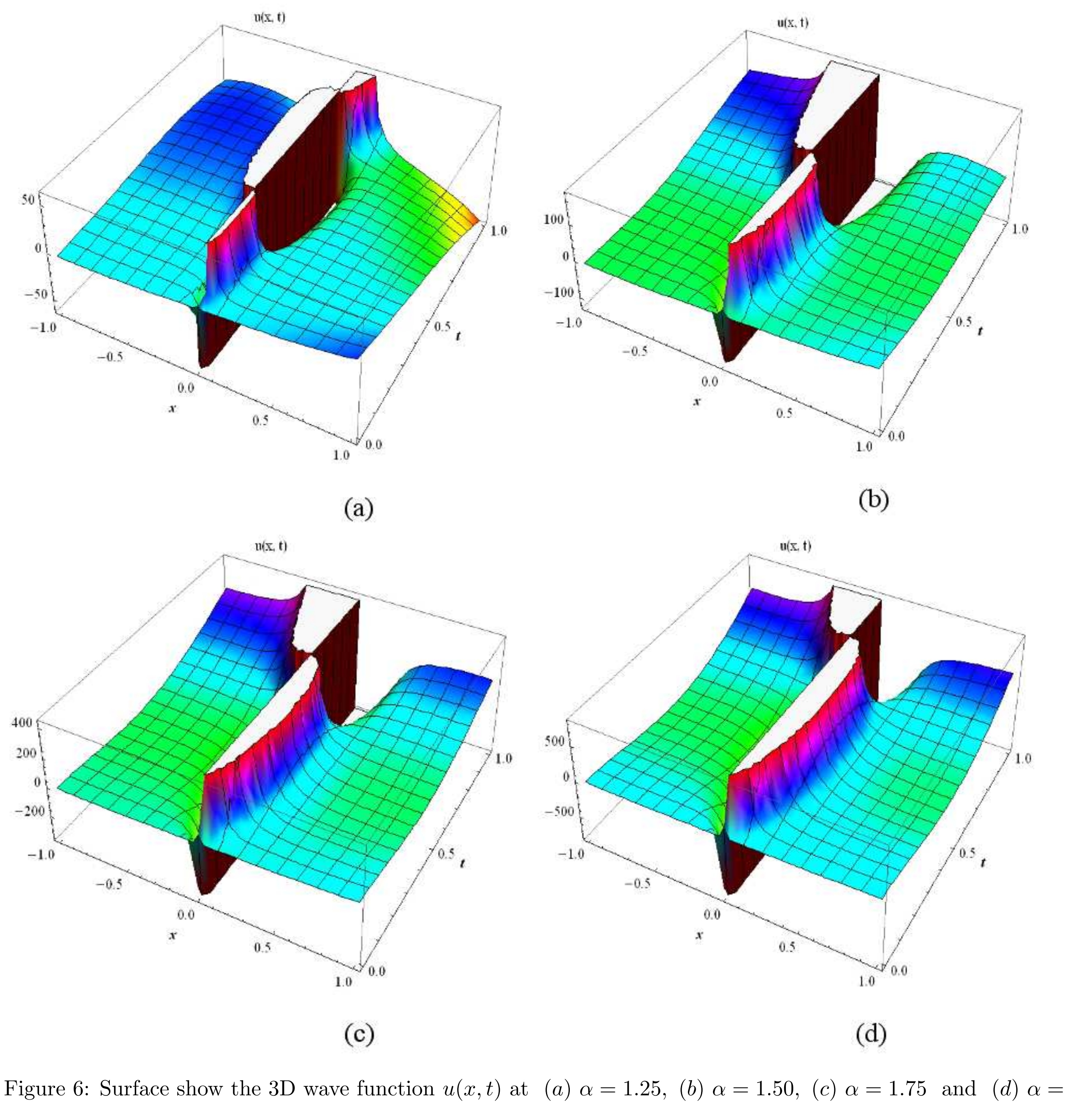

Figure 6: Surface show the 3D wave function $u(x,t)$ at $(a)\ \alpha = 1.25,$ $(b)\ \alpha = 1.50,$ $(c)\ \alpha = 1.75$ and $(d)\ \alpha = 2.$

6. Conclusion

In this present work, the homotopy perturbation transform method has been used to obtain semi-analytical solutions to the nonlinear time fractional nonlinear EFEs with great precision and accuracy. The collected findings reveal that up to third order approximation. The accuracy is very high. By setting $1 < \alpha \leq 2$, we can discover the classical solution to these model. The results show that the HPTM is a very effective and powerful approach for studying various quantum nonlinear model. This approach can also be used to investigate more complex phenomena in science and engineering. This technique is also ideal for studying higher order nonlinear model, which can be found in a wide range of physical sciences fields. To demonstrate the relevance and efficacy of the considered strategy, we looked at three different examples of the projected model. The secure outputs show that a basic HPTM algorithm was used to generate a standardized semi-analytical solutions. The suggested approach is unique in that it provides a simple solution, a critical convergence zone, and a non-local influence. Finally, the proposed scheme can be used to examine the behavior of nonlinear systems that exist in quantum mechanics as a novel tool over previous available analytical techniques.

7. Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

8. Availability of data and material

All data are included in the paper.

9. Funding

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10. Institutional Review Board Statement

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11. Authors'contributions

Authors contributed equally.

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