LIMIT CYCLES OF A GENERALIZED MATHIEU DIFFERENTIAL SYSTEM

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Abstract. We study the maximum number of limit cycles which bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, when it is perturbed in the form

$$\dot{x} = y - \varepsilon(1 + \cos^l \theta)P(x, y), \quad \dot{y} = -x - \varepsilon\,(1 + \cos^m \theta)\,Q(x, y), \tag{1}$$

where $\varepsilon > 0$ is a small parameter, l and m are positive integers, $P(x, y)$ and $Q(x, y)$ are arbitrary polynomials of degree n, and $\theta = \arctan(y/x)$. As we shall see the differential system (1) is a generalization of the Mathieu differential equation. The tool for studying such limit cycles is the averaging theory.

1. Introduction and Statement of the Main Results

A *limit cycle* of a differential system is a periodic orbit having a neighborhood where it is the unique periodic orbit of the differential system. The notion of limit cycle was introduced in 1881 by Poincaré [11].

The study of the existence and number of limit cycles that a differential system in $\mathbb{R}^2$ can exhibit is one of the more difficult problems in the qualitative theory of the differential system in the plane. Thus in 1900 Hilbert [6] presented a list of 23 problems to the International Conference of Mathematicians in Paris, most of these problems were solved partially or completely, but the second part of the 16th problem remains unsolved up today. This problem ask about the existence of an upper bound for the maximal number of limit cycles that polynomial differential systems in $\mathbb{R}^2$ of a given degree can exhibit.

A source of producing limit cycles is by perturbing the periodic orbits of a center, see for instance the papers [3, 12] and the book of Christhoper and Li [5], and the hundreds of references quoted there.

The classical Mathieu's differential equation [10]) is

$$\ddot{x} + b\,(1 + \cos\theta)\,x = 0,$$

where b is real parameter, and the dots denote second derivative with respect to the time t. This equation was first discussed in 1868 by Mathieu while studying the problem of vibrations on an elliptical drumhead. Matthieu's equation has many applications in engineering [13, 15] and also in theoretical and experimental physics [2, 7, 16]. information on its periodic orbits can be found in [18].

Mathieu's equation can be written as the differential system

$$\dot{x} = y, \qquad \dot{y} = -b\,(1 + \cos\theta)\,x,$$

In [4] Chen and Llibre studied the limit cycles of the differential system

$$\dot{x} = y, \qquad \dot{y} = -x - \varepsilon\,(1 + \cos^m \theta)\,Q(x, y), \tag{2}$$

where $\varepsilon > 0$ is a small parameter and $Q\,(x, y)$ is an arbitrary polynomial of degree n.

In the present work we study the limit cycles of the following generalization of the differential system (2)

$$\dot{x} = y - \varepsilon\left(1 + \cos^l \theta\right) P(x, y), \qquad \dot{y} = -x - \varepsilon\,(1 + \cos^m \theta)\,Q(x, y), \tag{3}$$

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where $\varepsilon > 0$ is a small parameter, l and m are positive integers, and $P(x, y)$ and $Q(x, y)$ are arbitrary polynomials of degree n. More precisely, we study the maximum number of limit cycles which bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, when it is perturbed in the form (3).

Our main result is the following one.

Theorem 1. *Using the averaging theory of first order the maximum number of limit cycles of the differential system* (3) *bifurcating from the periodic solutions of the linear center $\dot{x} = y, \dot{y} = -x$ is at most:*

- *n if n is even, and l and m are not both even;*
- *$n/2$ if n, l and m are even;*
- *n if n is odd and l and m are one odd and the other even;*
- *$(n-1)/2$ if n is odd, and l and m are even;*
- *n if if n, l and m are odd.*

Theorem 1 is proved in section 3.

In section 2 we present a summary of the averaging theory of first-order and of the Descartes theorem that we shall need for proving Theorem 1.

2. The averaging theory of first-order and the Descartes theorem

2.1. **Averaging theory of first-order.** In these subsection we summarize the result stated in Theorems 11.5 of the book of Verhulst [17] on the averaging theory. For a general introduction to the computation of periodic orbits using the averaging theory see the book [8].

Consider the periodic differential system

$$\frac{dx}{d\theta} = \mathcal{X}(\theta, x) = \varepsilon \mathcal{F}(\theta, x) + \varepsilon^2 \Phi(\theta, x, \varepsilon), \tag{4}$$

where ε is a small parameter, $x \in \mathbb{R}$, $\theta \in \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ and $\mathcal{F} : \mathbb{S}^1 \times D \to \mathbb{R}$, $\Phi : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ are C^2 functions, being D an open interval of $\mathbb{R}$ and $\mathcal{F}$ and Φ are periodic with period 2π in the variable θ.

Now we consider the averaging function $f : D \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{F}(\theta, x)\, d\theta. \tag{5}$$

It is known that if $x(\theta, x_0)$ is the solution of system (4) such that $x(0, x_0) = x_0$, then we have

$$x(2\pi, x_0) - x_0 = \varepsilon f(x_0) + O(\varepsilon^2). \tag{6}$$

So for $\varepsilon > 0$ sufficiently small the simple zeros of the averaged function $f(x)$ provide limit cycles of differential equation (4).

2.2. **The Descartes Theorem.** In order to study the real zeros of the function $f(x)$ we shall use the Descartes Theorem (for a proof see [1]).

Theorem 2 (Descartes Theorem)**.** *Consider the real polynomial $p(\rho) = a_{i_1}\rho^{i_1} + a_{i_2}\rho^{i_2} + \cdots + a_{i_n}\rho^{i_n}$ with $0 \le i_1 \le i_2 \le \cdots \le i_n$ and $a_{i_j} \neq 0$ real constants for $j \in \{1, 2, ..., n\}$. When $a_{i_j} a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs in the polynomial $p(\rho)$ is m, then $p(\rho)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $p(\rho)$ in such a way that $p(\rho)$ has exactly $n-1$ positive real roots.*

3. Proof of Theorem 1

Let the polynomials $P(x,y) = \sum\limits_{i+j=0}^{n} a_{ij}x_iy_j$ and $Q(x,y) = \sum\limits_{i+j=0}^{n} b_{ij}x_iy_j$ be.

We write the differential system (3) in polar coordinates (ρ,θ) defined by $x = \rho\cos\theta$ and $y = \rho\sin\theta$, with $\rho > 0$, we obtain

$$\begin{aligned}
\dot{\rho} &= -\varepsilon\left(\cos\theta(1+\cos^L\theta)P(\rho\cos\theta,\rho\sin\theta) + \sin\theta(1+\cos^m\theta)Q(\rho\cos\theta,\rho\sin\theta)\right),\\
\dot{\theta} &= -1 + \frac{\varepsilon}{\rho}\left(\sin\theta(1+\cos^L\theta)P(\rho\cos\theta,\rho\sin\theta) - \cos\theta(1+\cos^m\theta)Q(\rho\cos\theta,\rho\sin\theta)\right).
\end{aligned} \tag{7}$$

Taking in the differential system (7) the variable θ as the new independent variable, system (7) reduces to the differential equation

$$\begin{aligned}
\frac{d\rho}{d\theta} &= \varepsilon\left(\cos\theta(1+\cos^L\theta)P(\rho\cos\theta,\rho\sin\theta) + \sin\theta(1+\cos^m\theta)Q(\rho\cos\theta,\rho\sin\theta)\right) + O(\varepsilon^2)\\
&= \varepsilon\sum_{i+j=0}^{n}\cos^i\theta\sin^j\theta\left(a_{ij}(\cos\theta+\cos^{L+1}\theta) + b_{ij}(\sin\theta+\sin\theta\cos^m\theta)\right)\rho^{i+j} + O(\varepsilon^2)\\
&= \varepsilon\mathcal{F}(\theta,x) + O(\varepsilon^2).
\end{aligned} \tag{8}$$

Since this differential equation is written in the normal form (4), so we can apply to it the averaging theory of first order.

In our study we shall use the following formulas for computing the averaged function:

$$\begin{aligned}
\int_0^{2\pi}\cos^p\theta\sin^{2q}\theta d\theta &= \frac{(2q-1)!!}{(2q+p)(2q+p-2)\dots(p+2)}\int_0^{2\pi}\cos^p\theta d\theta = 2\pi\alpha_{p,2q},\\
\int_0^{2\pi}\cos^{2s}\theta d\theta &= \frac{(2s-1)!!}{2^s s!}2\pi = 2\pi\beta_{2s},\\
\int_0^{2\pi}\cos^{2s+1}\theta d\theta &= 0,\\
\int_0^{2\pi}\cos^p\Theta\sin^{2q+1}\theta d\theta &= 0.
\end{aligned} \tag{9}$$

where in the first and second formula p, q and s are positive integers, in the third one s is a non-negative integer, and in the fourth one p is a positive integer and q is a non-negative integer. For more details of these four integrals see [19, pages 152-153].

Proof of Theorem 1 *when n is even, and l and m are not both even.* We consider the following three cases with n even.

Case 1: m and l are odd. Then we compute tha averaged function (5), and we obtain

$$\begin{aligned}
f(\rho) &= \frac{1}{2\pi}\int_0^{2\pi}\mathcal{F}(\rho,\Theta)\,d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{i+j=0}^{n}(a_{ij}\cos^{i+1}\theta\sin^j\theta + a_{ij}\cos^{i+l+1}\theta\sin^j\theta\\
&\quad + b_{ij}\cos^i\theta\sin^{j+1}\theta + b_{ij}\cos^{i+m}\theta\sin^{j+1}\theta)\rho^{i+j}d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{i+2q=2}^{n+1}((a_{i,2q-1}\cos^{i+1}\theta + a_{i,2q-1}\cos^{i+l+1}\theta)\sin^{2q-1}\theta \\
&\quad + \left(b_{i,2q-1}\cos^{i}\theta + b_{i,2q-1}\cos^{i+m}\theta\right)\sin^{2q}\theta)\rho^{i+2q-1} \\
&\quad + \sum_{i+2q=2}^{n}((a_{i,2q}\cos^{i+1}(\theta) + a_{i,2q}\cos^{i+l+1}\theta)\sin^{2q}(\theta) \\
&\quad + \left(b_{i,2q}\cos^{i}\theta + b_{i,2q}\cos^{i+m}\theta\right)\sin^{2q+1}\theta)\rho^{i+2q}d\theta \\
&= \frac{1}{2\pi}\left[\sum_{2s+1+2q=3}^{n+1}\int_0^{2\pi} b_{2s+1,2q-1}\cos^{2s+1+m}\theta\sin^{2q}\theta\rho^{2s+2q}d\theta\right. \\
&\quad + \sum_{2s+2q=2}^{n}\int_0^{2\pi} b_{2s,2q-1}\cos^{2s}\theta\sin^{2q}\theta\rho^{2s+2q-1}d\theta \\
&\quad + \sum_{2s+1+2q=3}^{n+1}\int_0^{2\pi}(a_{2s+1,2q}\cos^{2s+2}\theta\sin^{2q}(\theta)\,\rho^{2s+1+2q}d\theta \\
&\quad \left. + \sum_{2s+2q=2}^{n}\int_0^{2\pi} a_{2s,2q}\cos^{2s+l+1}\theta\sin^{2q}(\theta)\,\rho^{2s+2q}d\theta\right] \\
&= \sum_{s+q=1}^{n/2} b_{2s+1,2q-1}\alpha_{2s+1+m,2q}\rho^{2s+2q} + \sum_{s+q=1}^{n/2} b_{2s,2q-1}\alpha_{2s,2q}\rho^{2s+2q-1} \\
&\quad + \sum_{s+q=1}^{n/2} a_{2s+1,2q}\alpha_{2s+2,2q}\rho^{2s+1+2q} + \sum_{s+q=1}^{n/2} a_{2s,2q}\alpha_{2s+l+1,2q}\rho^{2s+2q} \\
&= \sum_{k=1}^{n+1} A_k\rho^k.
\end{aligned}
$$

Since $f(\rho)$ is a polynomial generated by a linear combination of the monomials $\left\{\rho,\rho^2,...,\rho^{n+1}\right\}$. Using Descartes Theorem it follows that the polynomial $f(\rho)$ have at most n simple positive zeros, and consequently from subsection 2.1 we get using the averaging theory of first order that for $\varepsilon > 0$ sufficiently small the differential system (3) has at most n limit cycles bifurcating from the periodic solutions of the linear center (3) with $\varepsilon = 0$.

Case 2: m is odd and l is even. Working as in the case 1 we obtain

$$
\begin{aligned}
f(\rho) &= \frac{1}{2\pi}\int_0^{2\pi}\mathcal{F}(\rho,\theta)\,d\theta \\
&= \sum_{s+q=1}^{n/2} b_{2s+1,2q-1}\alpha_{2s+1+m,2q}\rho^{2s+2q} + \sum_{s+q=1}^{n/2} b_{2s,2q-1}\alpha_{2s,2q}\rho^{2s+2q-1} + \\
&\quad \sum_{s+q=1}^{n/2} a_{2s+1,2q}\alpha_{2s+2,2q}\rho^{2s+1+2q} + \sum_{s+q=1}^{n/2} a_{2s+1,2q}\alpha_{2s+l+2,2q}\rho^{2s+1+2q} = \sum_{k=1}^{n+1}\tilde{A}_k\rho^k.
\end{aligned}
$$

Again $f(\rho)$ is a polynomial generated by the monomials $\left\{\rho,\rho^2,...,\rho^{n+1}\right\}$, and as in case 1 by Descartes Theorem it follows that for $\varepsilon > 0$ sufficiently small the differential system (3) has at most n limit cycles bifurcating from the periodic solutions of the linear center (3) with $\varepsilon = 0$.

Case 3: m is even and l is odd. Then

$$\begin{aligned} f(\rho) &= \frac{1}{2\pi}\int_0^{2\pi}\mathcal{F}(\rho,\theta)\,d\theta \\ &= \sum_{s+q=1}^{n/2} b_{2s,2q-1}\alpha_{2s+m,2q}\rho^{2s+2q-1} + \sum_{s+q=1}^{n/2} b_{2s,2q-1}\alpha_{2s,2q}\rho^{2s+2q-1} + \\ &\quad \sum_{s+q=1}^{n/2} a_{2s+1,2q}\alpha_{2s+2,2q}\rho^{2s+1+2q} + \sum_{s+q=1}^{n/2} a_{2s,2q}\alpha_{2s+l+1,2q}\rho^{2s+2q} = \sum_{k=1}^{n+1}\hat{A}_k\rho^k. \end{aligned}$$

As in the previous two cases we conclude that for $\varepsilon > 0$ sufficiently small the differential system (3) has at most n limit cycles bifurcating from the periodic solutions of the linear center (3) with $\varepsilon = 0$. □

Proof of Theorem 1 *when* n*,* l *and* m *are even.* Working as in case 1 we compute the averaged function and we get

$$\begin{aligned} f(\rho) &= \frac{1}{2\pi}\int_0^{2\pi}\mathcal{F}(\rho,\theta)\,dt \\ &= \sum_{s+q=1}^{n/2} b_{2s,2q-1}\alpha_{2s+m,2q}\rho^{2s+2q-1} + \sum_{s+q=1}^{n/2} b_{2s,2q-1}\alpha_{2s,2q}\rho^{2s+2q-1} + \\ &\quad \sum_{s+q=1}^{n/2} a_{2s+1,2q}\alpha_{2s+2,2q}\rho^{2s+1+2q} + \sum_{s+q=1}^{n/2} a_{2s+1,2q}\alpha_{2s+l+2,2q}\rho^{2s+1+2q} = \sum_{k=1}^{(n/2)+1}\tilde{B}_k\rho^{2k-1}. \end{aligned}$$

Since $f(\rho)$ is a polynomial generated by a linear combination of the monomials $\{\rho,\rho^3,...,\rho^{n+1}\}$. Using Descartes Theorem it follows that the polynomial $f(\rho)$ have at most $n/2$ simple positive zeros, and consequently from subsection 2.1 we get using the averaging theory of first order that for $\varepsilon > 0$ sufficiently small the differential system (3) has at most $n/2$ limit cycles bifurcating from the periodic solutions of the linear center (3) with $\varepsilon = 0$. □

Proof of Theorem 1 *if* n*,* l *and* m *are odd.* Again working as in case 1 we compute the averaged function and we obtain

$$\begin{aligned} f(\rho) &= \frac{1}{2\pi}\int_0^{2\pi}\mathcal{F}(\rho,\theta)\,d\theta \\ &= \sum_{s+q=1}^{(n-1)/2} b_{2s+1,2q-1}\alpha_{2s+1+m,2q}\rho^{2s+2q} + \sum_{s+q=1}^{(n+1)/2} b_{2s,2q-1}\alpha_{2s,2q}\rho^{2s+2q-1} + \\ &\quad \sum_{s+q=1}^{(n-1)/2} a_{2s+1,2q}\alpha_{2s+2,2q}\rho^{2s+1+2q} + \sum_{s+q=1}^{(n+1)/2} a_{2s,2q}\alpha_{2s+l+1,2q}\rho^{2s+2q} = \sum_{k=1}^{n+1}C_k\rho^k. \end{aligned}$$

Since the polynomial $f(\rho)$ has the monomials $\{\rho,\rho^2,...,\rho^{n+1}\}$. Using Descartes theorem it follows that the polynomial $f(\rho)$ has at most n simple zeros. Therefore for $\varepsilon > 0$ sufficiently small the differential system (3) has at most n limit cycles bifurcating from the periodic solutions of the linear center (3) with $\varepsilon = 0$. □

Proof of Theorem 1 *if* n *is odd and* l *and* m *are one odd and the other even.* We distinguish two cases with n odd.

Case 1: m is odd and l is even. Computing the averaged function we get

$$\begin{aligned} f(\rho) &= \frac{1}{2\pi}\int_0^{2\pi} \mathcal{F}(\rho,\theta)\, d\theta \\ &= \sum_{s+q=1}^{(n-1)/2} b_{2s+1,2q-1}\alpha_{2s+1+m,2q}\rho^{2s+2q} + \sum_{s+q=1}^{(n+1)/2} b_{2s,2q-1}\alpha_{2s,2q}\rho^{2s+2q-1} + \\ & \sum_{s+q=1}^{(n-1)/2} a_{2s+1,2q}\alpha_{2s+2,2q}\rho^{2s+1+2q} + \sum_{s+q=1}^{(n-1)/2} a_{2s+1,2q}\alpha_{2s+l+2,2q}\rho^{2s+1+2q} = \sum_{k=1}^{n} \tilde{C}_k\rho^k. \end{aligned}$$

Case 2: m is even and l is odd. Then the averaged function

$$\begin{aligned} f(\rho) &= \frac{1}{2\pi}\int_0^{2\pi} \mathcal{F}(\rho,\theta)\, d\theta \\ &= \sum_{s+q=1}^{(n+1)/2} b_{2s,2q-1}\alpha_{2s+m,2q}\rho^{2s+2q-1} + \sum_{s+q=1}^{(n+1)/2} b_{2s,2q-1}\alpha_{2s,2q}\rho^{2s+2q-1} + \\ & \sum_{s+q=1}^{(n-1)/2} a_{2s+1,2q}\alpha_{2s+2,2q}\rho^{2s+1+2q} + \sum_{s+q=1}^{(n-1)/2} a_{2s,2q}\alpha_{2s+l+1,2q}\rho^{2s+2q} = \sum_{k=1}^{n} \hat{C}_k\rho^k. \end{aligned}$$

Now the polynomials $f(\rho)$ in cases 1 and 2 are generated by the monomials $\{\rho, \rho^2, ..., \rho^n\}$. Using Descartes theorem it follows that the polynomial $f(\rho)$ has at most $n-1$ simple zeros. Therefore for $\varepsilon > 0$ sufficiently small the differential system (3) has at most $n-1$ limit cycles bifurcating from the periodic solutions of the linear center (3) with $\varepsilon = 0$. □

Proof of Theorem 1 *if n is odd, and l and m are even.* Computing the averaged function we obtain

$$\begin{aligned} f(\rho) &= \frac{1}{2\pi}\int_0^{2\pi} \mathcal{F}(\rho,\theta)\, d\theta \\ &= \sum_{s+q=1}^{(n+1)/2} b_{2s,2q-1}\alpha_{2s+m,2q}\rho^{2s+2q-1} + \sum_{s+q=1}^{(n+1)/2} b_{2s,2q-1}\alpha_{2s,2q}\rho^{2s+2q-1} + \\ & \sum_{s+q=1}^{(n-1)/2} a_{2s+1,2q}\alpha_{2s+2,2q}\rho^{2s+1+2q} + \sum_{s+q=1}^{(n-1)/2} a_{2s+1,2q}\alpha_{2s+l+2,2q}\rho^{2s+1+2q} \\ &= \sum_{k=1}^{(n+1)/2} D_k\rho^{2k-1} + \sum_{k=1}^{(n-1)/2} \hat{D}_k\rho^{2k+1}. \end{aligned}$$

The polynomial $f(\rho)$ is generated by the monomials $\{\rho, \rho^3, ..., \rho^n\}$. Using Descartes theorem it follows that the polynomial $f(\rho)$ has at most $(n-1)/2$ simple zeros. Therefore for $\varepsilon > 0$ sufficiently small the differential system (3) has at most $(n-1)/2$ limit cycles bifurcating from the periodic solutions of the linear center (3) with $\varepsilon = 0$. □

Proof of Theorem 1 *if n, l and m are odd.* Again working as in case 1 we compute the averaged function and we obtain

$$\begin{aligned} f(\rho) &= \frac{1}{2\pi}\int_0^{2\pi} \mathcal{F}(\rho,\theta)\, d\theta \\ &= \sum_{s+q=1}^{(n-1)/2} b_{2s+1,2q-1}\alpha_{2s+1+m,2q}\rho^{2s+2q} + \sum_{s+q=1}^{(n+1)/2} b_{2s,2q-1}\alpha_{2s,2q}\rho^{2s+2q-1} + \\ & \sum_{s+q=1}^{(n-1)/2} a_{2s+1,2q}\alpha_{2s+2,2q}\rho^{2s+1+2q} + \sum_{s+q=1}^{(n+1)/2} a_{2s,2q}\alpha_{2s+l+1,2q}\rho^{2s+2q} = \sum_{k=1}^{n+1} C_k\rho^k. \end{aligned}$$

Since the polynomial $f(\rho)$ has the monomials $\left\{\rho, \rho^2, ..., \rho^{n+1}\right\}$. Using Descartes theorem it follows that the polynomial $f(\rho)$ has at most n simple zeros. Therefore for $\varepsilon > 0$ sufficiently small the differential system (3) has at most n limit cycles bifurcating from the periodic solutions of the linear center (3) with $\varepsilon = 0$. □

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