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Fuzzy Mixed Variational-Like and Integral Inequalities for Strongly Preinvex Fuzzy Mappings

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Abstract: It is a familiar fact that convex and non-convex fuzzy mappings play a critical role in the study of fuzzy optimization. Due to the behavior of its definition, the idea of convexity plays a significant role in the subject of inequalities. The concepts of convexity and symmetry have a tight connection. We may use whatever we learn from one to the other, thanks to the significant correlation that has developed between both in recent years. Our aim is to consider a new class of fuzzy mappings (FMs) is known as strongly preinvex fuzzy mappings (strongly preinvex-FMs) on the invex set. These FMs are more general than convex fuzzy mappings (convex-FMs) and preinvex fuzzy mappings (preinvex-FMs), and when generalized differentiable (briefly, G-differentiable) strongly preinvex-FMs are strongly invex fuzzy mappings (strongly invex-FMs). Some new relationships among various concepts of strongly preinvex-FMs are established and verify with the support of some useful examples. We have also shown that optimality conditions of G-differentiable strongly preinvex-FMs, and fuzzy functional, where fuzzy functional is sum of G-differentiable preinvex-FMs and non G-differentiable strongly preinvex-FMs, can be distinguished by strongly fuzzy variational-like inequalities and strongly fuzzy mixed variational-like inequalities, respectively. In the end, we have established and verified a strong relationship between Hermite-Hadamard inequality and strongly preinvex-FM. Several exceptional cases are also discussed. These inequalities are very interesting outcome of our main results and appear to be new ones. The results in this research can be seen as refinements and improvements to previously published findings.

Keywords: Preinvex fuzzy mappings; strongly preinvex fuzzy mappings; strongly invex fuzzy mappings; fuzzy strongly monotonicity; strongly fuzzy mixed variational like-inequalities

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1. Introduction

Recently, many generalizations and extensions have been studied for classical convexity. Polyak [1], introduced and studied the idea of strongly convex functions on the convex set, which have a significant impact on optimization theory and related fields. Karmardian [2] discussed how strongly convex functions can be used to solve nonlinear complementarity problems for the first time. Qu and Li [3] and Nikodem and Pales [4] developed the convergence analysis for addressing equilibrium issues and variational inequalities using strongly convex functions. For further study, we refer to reader about applications and properties of the strongly convex functions, see [5-10], and the references therein. For differentiable functions, invex functions were introduced by Hanson [11], which played significant role in mathematical programing. The concept of invex sets and preinvex functions were introduced and studied by Israel and Mond [12]. It is well known that differential preinvex function are invex functions. The converse also holds under Condition C, [13]. Furthermore, Noor [14], studied the optimality conditions of differentiable preinex functions and proved that minimum can be characterized by variational-like inequalities. Noor et al. [15, 16] studied the properties of strongly preinvex function and investigated its applications. For more applications and properties of strongly preinvex functions, see [17-19], and the references therein.

In [20], a large amount of research work on fuzzy sets and systems has been devoted to the advancement of various fields, and it plays an important role in the analysis of broad class problems emerging in pure and applied sciences, such as operation research, computer science, decision sciences, control engineering, artificial intelligence, and management sciences,. Convex analysis has made significant contributions to the improvement of several practical and pure science domains. In the same way, fuzzy convex analysis fundamental principle in fuzzy optimization and it is worthwhile to explore some basic principles of convex sets in fuzzy set theory. Many scholars have addressed fuzzy convex sets. Liu [21] investigated some properties of convex fuzzy sets and updated the definition of shadow of fuzzy sets with the support of useful examples. Lowen [22], gathered some well-known convex sets results and proved separation theorem for convex fuzzy sets. Ammar and Metz [23, 24] investigated forms of convexity and established generalized convexity of fuzzy sets. Furthermore, they used the principle of convexity to formulate a general fuzzy nonlinear programming problem.

A fuzzy number is a generalized version of an interval that can be discussed (in crisp set theory). Zadeh [20] defined fuzzy numbers, while Dubois and Prade [25] built on Zadeh's work by adding new fuzzy number conditions. Furthermore, Goetschel and Voxman [26] adjusted many conditions on fuzzy numbers to make them easier to handle. For example, in [25], one of the conditions for a fuzzy number is that it is a continuous function, whereas in [26], the fuzzy number is upper semi continuous. The purpose is to establish a metric for a collection of fuzzy numbers using the relaxation of requirements on fuzzy numbers, and then use this metric to examine some basic features of topological space. Nanda, and Kar [27], Syau [28] and Furukawa [29], introduced the concept of convex-FMs from $\mathbb{R}^n$ to the set of fuzzy numbers. Furthermore, they also defined different type of convex-FMs like logarithmic convex-FMs and quasi-convex-FMs, as well as they studied Lipschitz continuity of fuzzy valued mappings. Yan and Xu [31] provided the notions of epigraphs and convexity of FMs, as well as the characteristics of convex-FMs and quasi-convex-FMs, based on Goetschel and Voxman's concept of ordering [30]. The concept of fuzzy preinvex mapping on the invex set was introduced and studied by Noor [32]. He also demonstrated that variational inequalities may be used to specify the fuzzy optimality conditions of differentiable fuzzy preinex mappings. Syau [33], introduced notions of (ϕ_1, ϕ_2) −convexity, ϕ_1-B-vexity and ϕ_1-convexity-FMs through the so called fuzzy max" order among the fuzzy numbers, and proved that ϕ_1-B-vexity and ϕ_1-convexity, B-vexity, convexity and preinvexity of FMs are the subclasses. Syau and Lee [34] examined various aspects of fuzzy optimization and discussed continuity and convexity through linear ordering and metric defined on fuzzy integers. They also extended the Weirstrass theorem from real-valued functions to FMs. For recent applications, see [35-39], and the references therein.

On the other hand, integral inequalities have various applications in linear programing, combinatory, orthogonal polynomials, quantum theory, number theory, optimization theory, dynamics, and in the theory of relativity, see [40, 41] and the references therein. The HH-inequality is a familiar, supreme and broadly useful inequality. This inequality has fundamental significance [42, 43] due to other classical inequalities such as the Oslen and Gagliardo-Nirenberg, Hardy, Oslen, Opial, Young, Linger, Arithmetic's-Geometric, Ostrowski, levison, Minkowski, Beckenbach-Dresher, Ky-fan and Holer inequality [44-49], which are closely linked to the classical HH-inequality. It can be stated as follows:

Let $\mathcal{H}: K \to \mathbb{R}$ be a convex function on a convex set K and $, \nu \in K$ with $u \leq \nu$. Then,

$$\mathcal{H}\left(\frac{u+\nu}{2}\right) \leq \frac{1}{\nu-u}\int_u^\nu \mathcal{H}(z)dz \leq \frac{\mathcal{H}(u)+\mathcal{H}(\nu)}{2}. \quad (1)$$

If $\mathcal{H}$ is a concave function, then ineuality (1) is reversed.

There are several integrals that deal with FMs and have FMs as integrands. For FMs, Oseuna-Gomez et al. [50] and Costa et al. [51] constructed Jensen's integral inequality. Costa and Floures [52] used the same method to present Minkowski and Beckenbach's inequalities, where the integrands are fuzzy- mappings. Costa et al established a relationship between elements of fuzzy-interval space and interval space, and introduced level-wise fuzzy order relation on fuzzy-interval space through Kulisch-Miranker order relation defined on interval space. This was motivated by [48-53] and particularly [54], because Costa et al established a relationship between elements of fuzzy-interval space and interval space, and introduced level-wise fuzzy order relation on fuzzy-interval space through Kulisch-Miranker order relation defined on interval space. By using this relation on fuzzy-interval space, we generalize integral inequality (1) by constructing fuzzy integral inequalities for strongly preinvex-FMs, where the integrands are strongly preinvex-FMs. Recently, Khan et al. [55] introduced the new class of convex-FMs which is known as (h_1, h_2)-convex-FMs by means fuzzy order relation and presented the following new version of HH-type inequality for (h_1, h_2)-convex-FM involving fuzzy-interval Riemann integrals:

Theorem 1.1. Let $\mathcal{H}: [u, \nu] \to \mathbb{F}_0$ be a (h_1, h_2)-convex-FM with $h_1, h_2: [0,1] \to \mathbb{R}^+$ and $h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \neq 0$. If $\mathcal{H}$ is fuzzy Riemann integrable (in sort, FR-integrable), then

$$\frac{1}{2\,h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\,\mathcal{H}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{\nu-u}\int_u^\nu \mathcal{H}(z)dz \preccurlyeq [\mathcal{H}(u)\,\tilde{\mp}\,\mathcal{H}(\nu)]\int_0^1 h_1(\tau)\,h_2(1-\tau)\,d\tau. \quad (2)$$

If $h_1(\tau) = \tau$ and $h_2(\tau) \equiv 1$, then Theorem 1.1 reduces to the result for convex fuzzy-IVF:

$$\mathcal{H}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{\nu-u}\int_u^\nu \mathcal{H}(z)dz \preccurlyeq \frac{\mathcal{H}(u)\,\tilde{\mp}\,\mathcal{H}(\nu)}{2}. \quad (3)$$

For further informations related to fuzzy integral inequalities, see [56-63].

Motivated by ongoing studies as well as the relevance of the concepts invexity and preinvexity of FMs. In section 2, we go through some fundamental concepts, preliminary notations, and findings that will be useful in further research. In the parts that follow, the key results are considered and discussed. Section 3 introduces the concepts of strongly preinvex FMs and discusses some of their properties. Moreover, new relationships among various concepts of strongly preinvex-FMs are also investigated in Section 3. In Section 4, we introduce fuzzy variational-like and Hermite-Hadamrd inequalities for strongly preinvex-FMs.

2. Preliminaries

In this section, we first give some definitions, preliminary notations and results which will be helpful for further study.

A fuzzy set of $\mathbb{R}$ is a mapping $\psi: \mathbb{R} \to [0,1]$, for each fuzzy set and $\gamma \in (0,1]$, then γ-level sets of ψ is denoted and defined as follows $\psi_\gamma = \{u \in \mathbb{R} |\, \psi(u) \geq \gamma\}$. The support of ψ is denoted by $\text{supp}(\psi)$ and is defined as $\text{supp}(\psi) = \{u \in \mathbb{R} |\, \psi(u) \geq \gamma\}$. A fuzzy set is normal if there exist $u \in \mathbb{R}$ such that $\psi(u) = 1$. A fuzzy set is convex and concave if $\psi\big((1-\tau)u + \tau\nu\big) \geq min\big(\psi(u), \psi(\nu)\big)$ and $\psi((1-\tau)u + \tau\nu) \leq max\big(\psi(u), \psi(\nu)\big)$ for $u, \nu \in \mathbb{R}$, $\tau \in [0,1]$, respectively. A fuzzy convex set is a generalization of classical convex set.

A fuzzy set is said to be fuzzy number with the following properties

(a) ψ is normal. (b) ψ is convex fuzzy set. (c) ψ is upper semicontinious. (d) ψ_0 is compact.

$\mathbb{F}_0$ denotes the set of all fuzzy numbers. For fuzzy number, it is convenient to distinguish followings γ-levels,

$$\psi_\gamma = \{u \in \mathbb{R} |\, \psi(u) \geq \gamma\},$$

from these definitions, we have

$$\psi_\gamma = [\psi_*(\gamma), \psi^*(\gamma)]$$

where

$$\psi_*(\gamma) = inf\{u \in \mathbb{R} |\, \psi(u) \geq \gamma\}, \psi^*(\gamma) = sup\{u \in \mathbb{R} |\, \psi(u) \geq \gamma\}.$$

Since each $r \in \mathbb{R}$ is also a fuzzy number, defined as

$$\tilde{r}(u) = \begin{cases} 1 & \text{if } u = r \\ 0 & \text{if } u \neq r \end{cases}.$$

It is also well known that for any $\psi, \phi \in \mathbb{F}_0$ and $r \in \mathbb{R}$

$$\psi \tilde{\mp} \phi = \{(\psi_*(\gamma) + \phi_*(\gamma), \psi^*(\gamma) + \phi^*(\gamma), \gamma) : \gamma \in [0,1]\}, \quad (4)$$

$$r\psi = \{(r\psi_*(\gamma), r\psi^*(\gamma), \gamma) : \gamma \in [0,1]\}. \quad (5)$$

Obviously, $\mathbb{F}_0$ is closed under addition and nonnegative scaler multiplication. Furthermore, for each scaler number $r \in \mathbb{R}$,

$$\psi \tilde{\mp} r = \{(\psi_*(\gamma) + r, \psi^*(\gamma) + r, \gamma) : \gamma \in [0,1]\}. \quad (6)$$

For any $\psi, \phi \in \mathbb{F}_0$, we say that $\psi \preccurlyeq \phi$ ("$\preccurlyeq$" relation between fuzzy numbers ψ and ϕ) if for all $\gamma \in (0,1]$, $\psi^*(\gamma) \leq \phi^*(\gamma)$ ("$\leq$" relation $\psi^*(\gamma)$ and $\phi^*(\gamma)$) and $\psi_*(\gamma) \leq \phi_*(\gamma)$. We say comparable if for any $\psi, \phi \in \mathbb{F}_0$, we have $\psi \preccurlyeq \phi$ or $\psi \succcurlyeq \phi$ otherwise they are non-comparable.

We can state that $\mathbb{F}_0$ is a partial ordered set under the relation $\preccurlyeq$ if we write $\psi \preccurlyeq \phi$ instead of $\phi \succcurlyeq \psi$. If $\psi, \phi \in \mathbb{F}_0$, there exist $\omega \in \mathbb{F}_0$ such that $\psi = \phi \tilde{+} \omega$, then we have the existence of the Hukuhara difference (in short, H-difference) of ψ and ϕ, and we say that ω is the H-difference of ψ and ϕ, and denoted by $\psi \tilde{-} \phi$, see [37]. If this fuzzy operation exist, then

$$(\omega)^*(\gamma) = (\psi \tilde{-} \phi)^*(\gamma) = \psi^*(\gamma) - \phi^*(\gamma),\ (\omega)_*(\gamma) = (\psi \tilde{-} \phi)_*(\gamma) = \psi_*(\gamma) - \phi_*(\gamma).$$

A mapping $\mathcal{H}: K \to \mathbb{F}_0$ is called fuzzy mapping (FM). For each $\gamma \in [0,1]$, denote $[\mathcal{H}(u)]^\gamma = [\mathcal{H}_*(u,\gamma), \mathcal{H}^*(u,\gamma)]$ and in parameterized form, denote $\mathcal{H}(u) = \{(\mathcal{H}_*(u,\gamma), \mathcal{H}^*(u,\gamma), \gamma): \gamma \in [0,1]\}$.

Definition 2.1. [35] Let's say $I = (m,n)$ and $u \in (m,n)$. Then FM $\mathcal{H}: (m,n) \to \mathbb{F}_0$ is said to be a generalized differentiable (briefly, G-differentiable) at u if there exists an element $\mathcal{H}'(u) \in \mathbb{F}_0$ such that

for any $0 < \tau$, sufficiently small, there exist $\mathcal{H}(u+\tau) \tilde{-} \mathcal{H}(u)$, $\mathcal{H}(u) \tilde{-} \mathcal{H}(u-\tau)$ and the limits are (in the metric D)

$$\lim_{\tau \to 0^+} \frac{\mathcal{H}(u+\tau) \tilde{-} \mathcal{H}(u)}{\tau} = \lim_{\tau \to 0^+} \frac{\mathcal{H}(u) \tilde{-} \mathcal{H}(u-\tau)}{\tau} = \mathcal{H}'(u)$$

$$\text{or } \lim_{\tau \to 0^+} \frac{\mathcal{H}(u) \tilde{-} \mathcal{H}(u+\tau)}{-\tau} = \lim_{\tau \to 0^+} \frac{\mathcal{H}(u-\tau) \tilde{-} \mathcal{H}(u)}{-\tau} = \mathcal{H}'(u)$$

$$\text{or } \lim_{\tau \to 0^+} \frac{\mathcal{H}(u+\tau) \tilde{-} \mathcal{H}(u)}{\tau} = \lim_{\tau \to 0^+} \frac{\mathcal{H}(u-\tau) \tilde{-} \mathcal{H}(u)}{-\tau} = \mathcal{H}'(u)$$

$$\text{or } \lim_{\tau \to 0^+} \frac{\mathcal{H}(u) \tilde{-} \mathcal{H}(u+\tau)}{-\tau} = \lim_{\tau \to 0^+} \frac{\mathcal{H}(u) \tilde{-} \mathcal{H}(u-\tau)}{\tau} = \mathcal{H}'(u),$$

where the limits are taken in the metric space (E, D), for $\psi, \phi \in \mathbb{F}_0$

$$D(\psi, \phi) = \sup_{0 \leq \gamma \leq 1} H(\psi_\gamma, \phi_\gamma),$$

and H denote the well-known Hausdorff metric on space of intervals.

Definition 2.2. [27] A FM $\mathcal{H}: K \to \mathbb{F}_0$ is said to be convex on the convex set K if

$$\mathcal{H}((1-\tau)u + \tau\nu) \preccurlyeq (1-\tau)\mathcal{H}(u) \tilde{+} \tau\mathcal{H}(\nu), \forall\, u, \nu \in K,\ \ \tau \in [0,1]. \quad (7)$$

Similarly, $\mathcal{H}$ is said to be concave-FM on K if inequality (7) is reversed.

Definition 2.3. [12] The set K_ξ in $\mathbb{R}$ is said to be invex set with respect to (w.r.t.) arbitrary bifunction $\xi(.,.)$, if

$$u + \tau\xi(\nu, u) \in K_\xi, \forall\, u, \nu \in K_\xi,\ \ \tau \in [0,1].$$

The invex set K_ξ is also known as ξ-connected set. Note that, each convex set with $\nu - u = \xi(\nu, u)$ is an invex set in classical sense, but the reverse is not true. For instance, the following set $K_\xi = [-7,-2] \cup [2,10]$ is an invex set w.r.t. non-trivial bi-function $\xi: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ given as

$$\xi(\nu, u) = \nu - u, \nu \geq 0, u \geq 0,$$

$$\xi(\nu, u) = \nu - u, 0 \geq \nu, 0 \geq u,$$
$$\xi(\nu, u) = -7 - u, \nu \geq 0 \geq u,$$
$$\xi(\nu, u) = 2 - u, u \geq 0 \geq \nu.$$

Definition 2.4. [32] A FM $\mathcal{H}: K_\xi \to \mathbb{F}_0$ is said to be preinvex on the invex set K_ξ w.r.t. bi-function ξ if

$$\mathcal{H}(u + \tau\xi(\nu, u)) \preccurlyeq (1-\tau)\mathcal{H}(u) \tilde{+} \tau\mathcal{H}(\nu), \quad (8)$$

for all $u, \nu \in K_\xi$, $\tau \in [0,1]$, where $\xi: K_\xi \times K_\xi \to \mathbb{R}$. $\mathcal{H}$ is said to be preconcave-FM on K_ξ if inequality (8) is reversed.

Lemma 2.5. [21] Let K_ξ be an invex set w.r.t. ξ and let $\mathcal{H}: K_\xi \to \mathbb{F}_0$ be a FM, parameterized by

$$\mathcal{H}(u) = \{(\mathcal{H}_*(u,\gamma), \mathcal{H}^*(u,\gamma), \gamma): \gamma \in [0,1]\}, \forall\, u \in K_\xi.$$

Then $\mathcal{H}$ is preinvex on K_ξ if and only if, for all $\gamma \in [0,1]$,
$\mathcal{H}_*(u,\gamma)$ and $\mathcal{H}^*(u,\gamma)$ are preinvex w.r.t. ξ on K_ξ.

If $\xi(\nu, u) = \nu - u$, then Lemma 2.5, reduce to following result:
"Let K_ξ be a convex set and let $\mathcal{H}: K_\xi \to \mathbb{F}_0$ be a FM parameterized by

$$\mathcal{H}(u) = \{(\mathcal{H}_*(u,\gamma), \mathcal{H}^*(u,\gamma), \gamma): \gamma \in [0,1]\}, \forall\, u \in K_\xi$$

Then $\mathcal{H}$ is convex on K_ξ if and only if, for all $\gamma \in [0,1]$, $\mathcal{H}_*(u,\gamma)$ and $\mathcal{H}^*(u,\gamma)$ are convex w.r.t. ξ on K_ξ.

Theorem 2.6. [54] If $\mathcal{H}: [c,d] \subset \mathbb{R} \to \mathcal{K}_C$ is an interval valued function on $: [c,d]$ such that $[\mathcal{H}_*, \mathcal{H}^*]$. Then $\mathcal{H}$ is Riemann integrable over $: [c,d]$ if and only if, $\mathcal{H}_*$ and $\mathcal{H}^*$ both are Riemann integrable over $: [c,d]$ such that

$$(IR)\int_c^d \mathcal{H}(z)dz = \left[(R)\int_c^d \mathcal{H}_*(u)dz, (R)\int_c^d \mathcal{H}^*(u)dz\right]. \quad (9)$$

From above literature review, following results can be concluded, see [31, 32, 53, 54]:

Definition 2.7. [47] Let $\mathcal{H}: [c,d] \subset \mathbb{R} \to \mathbb{F}_0$ be a FM. The fuzzy Riemann integral of $\mathcal{H}$ over $[c,d]$, denoted by $(FR)\int_c^d \mathcal{H}(z)dz$, it is defined by

$$\left[(FR)\int_c^d \mathcal{H}(z)dz\right]^\gamma = (IR)\int_c^d \mathcal{H}_\gamma(z)dz = \left\{\int_c^d \mathcal{H}(z,\gamma)dz : \mathcal{H}(z,\gamma) \in \mathcal{R}_{[c,d]}\right\}, \quad (10)$$

for all $\gamma \in [0,1]$, where $\mathcal{R}_{[c,d]}$ is the collection of end point functions of IVFs. $\mathcal{H}$ is (FR)-integrable over $[c,d]$ if $(FR)\int_c^d \mathcal{H}(z)dz \in \mathbb{F}_0$. Note that, if both end point functions are Lebesgue-integrable, then $\mathcal{H}$ is fuzzy Aumann-integrable.

Let K_ξ be a nonempty invex set in $\mathbb{R}$ for future investigation. Let $\xi: K_\xi \times K_\xi \to \mathbb{R}$ be an arbitrary bifunction and $\mathcal{H}: K_\xi \to \mathbb{F}_0$ be an FM. We denote $\|.\|$ and $\langle .,. \rangle$ be the norm and inner product, respectively. Furthermore, throughout in this article FMs are discussed through the so-called "fuzzy-max" order among fuzzy numbers. As it is well-known, the fuzzy-max order is a partial order relation " $\preccurlyeq$ " on the set of fuzzy numbers.

3. Strongly preinvex fuzzy mappings

In this section, we propose and study the class of strongly preinvex-FMs. WE also establish the relationship between strongly preinvex-FMs, strongly monotone operators and strongly invex-FMs. Firstly, we will define the following notion of strongly preinvex-FM.

Definition 3.1. Let K_ξ be an invex set and ω be a positive number. Then FM $\mathcal{H}: K_\xi \to \mathbb{F}_0$ is said to be strongly preinvex-FM on K_ξ w.r.t. bi-function $\xi(.,.)$ if

$$\mathcal{H}(u+\tau\xi(\nu,u)) \preccurlyeq (1-\tau)\mathcal{H}(u)\tilde{+}\tau\mathcal{H}(\nu)\tilde{-}\omega\tau(1-\tau)\|\xi(\nu,u)\|^2, \quad (11)$$

for all $u,\nu \in K_\xi, \tau \in [0,1]$. $\mathcal{H}$ is said to be strongly preconcave-FM on K_ξ if inequality (11) is reversed. $\mathcal{H}$ is said to be strongly affine preinvex-FM on K_ξ if

$$\mathcal{H}(u+\tau\xi(\nu,u)) = (1-\tau)\mathcal{H}(u)\tilde{+}\tau\mathcal{H}(\nu)\tilde{-}\omega\tau(1-\tau)\|\xi(\nu,u)\|^2, \quad (12)$$

for all $u,\nu \in K_\xi, \tau \in [0,1]$.

Remark 3.2. Strongly preinvex-FMs, like preinvex-FMs, have some really desirable features.

1) $\Upsilon\mathcal{H}$ is also strongly preinvex for $\Upsilon \geq 0$, if $\mathcal{H}$ is strongly preinvex-FM.

2) $\max(\mathcal{H}(u), \varpi(u))$ is also strongly preinvex-FM if $\mathcal{H}$ and ϖ both are strongly preinvex-FMs.

Now we discuss some special cases of strongly preinvex-FMs:

If $\xi(\nu,u) = \nu - u$, then strongly preinvex-FM becomes strongly convex-FM, that is

$$\mathcal{H}\big((1-\tau)u+\tau\nu\big) \preccurlyeq (1-\tau)\mathcal{H}(u)\tilde{+}\tau\mathcal{H}(\nu)\tilde{-}\omega\tau(1-\tau)\|\nu-u\|^2, \forall\ u,\nu \in K_\xi, \tau \in [0,1].$$

If $\omega = 0$ then, inequality (11) reduces to the inequality (8).

If $\omega = 0$ and $\xi(\nu,u) = \nu - u$, then inequality (11) reduces to the inequality (7).

Following result characterizes the definition of strongly preinvex-FMs and establishes the relationship between strongly preinvex-FMs and endpoint functions. With the help of this theorem, we can easily handle upcoming results.

Theorem 3.3. Let $\mathcal{H}: K_\xi \to \mathbb{F}_0$ be a FM parametrized by

$$\mathcal{H}(u) = \{(\mathcal{H}_*(u,\gamma), \mathcal{H}^*(u,\gamma), \gamma): \gamma \in [0,1]\}, \forall\, u \in K_\xi. \quad (13)$$

Then $\mathcal{H}$ is strongly preinvex on K w.r.t. ξ, with modulus ω if and only if, for all $\gamma \in [0,1]$, $\mathcal{H}_*(u,\gamma)$ and $\mathcal{H}^*(u,\gamma)$ are strongly preinvex w.r.t. ξ and modulus ω. (14)

Proof. Assume that for each $\gamma \in [0,1]$, $\mathcal{H}_*(u,\gamma)$ and $\mathcal{H}^*(u,\gamma)$ are strongly preinvex w.r.t. ξ and modulus ω on K_ξ. Then from (11), for all $u,\nu \in K_\xi, \tau \in [0,1]$, we have

$$\mathcal{H}_*(u+\tau\xi(\nu,u),\gamma) \leq (1-\tau)\mathcal{H}_*(u,\gamma) + \tau\mathcal{H}_*(\nu,\gamma) - \omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$

and

$$\mathcal{H}^*(u+\tau\xi(\nu,u),\gamma) \leq (1-\tau)\mathcal{H}^*(u,\gamma) + \tau\mathcal{H}^*(\nu,\gamma) - \omega\tau(1-\tau)\|\xi(\nu,u)\|^2.$$

Then by (13), (4), (5) and (6), we obtain

$$\mathcal{H}(u+\tau\xi(\nu,u)) = \{(\mathcal{H}_*(u+\tau\xi(\nu,u),\gamma), \mathcal{H}^*(u+\tau\xi(\nu,u),\gamma),\gamma): \gamma \in [0,1]\},$$

$$\preccurlyeq \big\{\big((1-\tau)\mathcal{H}_*(u,\gamma), (1-\tau)\mathcal{H}^*(u,\gamma),\gamma\big): \gamma \in [0,1]\big\}\tilde{+}\{(\tau\mathcal{H}_*(\nu,\gamma), \tau\mathcal{H}^*(\nu,\gamma),\gamma): \gamma \in [0,1]\}$$

$$\tilde{-}\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$

$$= (1-\tau)\mathcal{H}(u)\tilde{+}\mathcal{H}(\nu)\tilde{-}\omega\tau(1-\tau)\|\xi(\nu,u)\|^2.$$

Hence, $\mathcal{H}$ is strongly preinvex-FM on K_ξ with modulus ω.

Conversely, let $\mathcal{H}$ is strongly preinvex-FM on K_ξ with modulus ω. Then for all $u,\nu \in K_\xi$ and $\tau \in [0,1]$, we have $\mathcal{H}(u+\tau\xi(\nu,u)) \preccurlyeq (1-\tau)\mathcal{H}(u)\tilde{+}\tau\mathcal{H}(\nu)\tilde{-}\omega\tau(1-\tau)\|\xi(\nu,u)\|^2$.

From (13), we have

$$\mathcal{H}(u+\tau\xi(\nu,u)) = \{(\mathcal{H}_*(u+\tau\xi(\nu,u),\gamma), \mathcal{H}^*(u+\tau\xi(\nu,u),\gamma),\gamma): \gamma \in [0,1]\}.$$

Again, from (13), (4), (5) and (6), we obtain

$$(1-\tau)\mathcal{H}(u)\tilde{+}\tau\mathcal{H}(u)\tilde{-}\omega\tau(1-\tau)\|\xi(\nu,u)\|^2$$
$$=\{((1-\tau)\mathcal{H}_*(u,\gamma),(1-\tau)\mathcal{H}^*(u,\gamma),\gamma):\gamma\in[0,1]\}$$
$$\tilde{+}\{(\tau\mathcal{H}_*(\nu,\gamma),\ \tau\mathcal{H}^*(\nu,\gamma),\gamma):\gamma\in[0,1]\}\tilde{-}\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$

for all $u,\nu\in K_\xi$ and $\tau\in[0,1]$. Then by strongly preinvexity of $\mathcal{H}$, we have for all $u,\nu\in K_\xi$ and $\tau\in[0,1]$ such that

$$\mathcal{H}_*(u+\tau\xi(\nu,u),\gamma)\le(1-\tau)\mathcal{H}_*(u,\gamma)+\tau\mathcal{H}_*(\nu,\gamma)-\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$

and

$$\mathcal{H}^*(u+\tau\xi(\nu,u),\gamma)\le(1-\tau)\mathcal{H}^*(u,\gamma)+\tau\mathcal{H}^*(\nu,\gamma)-\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$

for each $\gamma\in[0,1]$. Hence, the result follows.

Example 3.4. We consider the FM $\mathcal{H}:[0,1]\to\mathbb{F}_0$ defined by,

$$\mathcal{H}(u)(\sigma)=\begin{cases}\frac{\sigma}{2u^2} & \sigma\in[0,2u^2]\\ \frac{4u^2-\sigma}{2u^2} & \sigma\in(2u^2,4u^2]\\ 0 & otherwise,\end{cases}$$

Then, for each $\gamma\in[0,1]$, we have $\mathcal{H}_\gamma(u)=[2\gamma u^2,(4-2\gamma)u^2\,]$. Since $\mathcal{H}_*(u,\gamma),\mathcal{H}^*(u,\gamma)$ are strongly preinvex functions for each $\gamma\in[0,1]$. Hence $\mathcal{H}(u)$ is strongly preinvex-FM w.r.t.

$$\xi(\nu,u)=\nu-u,$$

with $0<\omega=\gamma\le1$. It can be easily seen that for each $\omega\in(0,1]$, there exist a strongly preinvex-FM and $\mathcal{H}(u)$ is neither convex FM and nor preinvex-FM w.r.t. bifunction $\xi(\nu,u)=\nu-u$ with $0<\omega\le1$.

Now we show that the difference between strongly preinvex-FM and strongly affine preinvex-FM is again a preinvex-FM for strongly preinvex-FM.

Theorem 3.5. Let FM $f:K_\xi\to\mathbb{F}_0$ be a strongly affine preinvex w.r.t. ξ and $0\le\omega$. Then $\mathcal{H}$ is strongly preinvex-FM w.r.t. same bi-function ξ if an only if, $\varpi=\mathcal{H}-f$ is preinvex-FM.

Proof. The "If" part is obvious. To prove the "only if" assume that, $f:K_\xi\to\mathbb{F}_0$ be a strongly fuzzy affine preinvex w.r.t. non-negative bi-function ξ and $0\le\omega$. Then

$$f(u+\tau\xi(\nu,u))=(1-\tau)f(u)\tilde{+}\tau f(\nu)\tilde{-}\omega\tau(1-\tau)\|\xi(\nu,u)\|^2, \quad (15)$$

Therefore, for each $\gamma\in[0,1]$, we have

$$f_*(u+\tau\xi(\nu,u),\gamma)=(1-\tau)f_*(u,\gamma)+\tau f_*(\nu,\gamma)-\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$
$$f^*(u+\tau\xi(\nu,u),\gamma)=(1-\tau)f^*(u,\gamma)+\tau f^*(\nu,\gamma)-\omega\tau(1-\tau)\|\xi(\nu,u)\|^2.$$

Since $\mathcal{H}$ is strongly preinvex-FM w.r.t. same bi-function ξ, then, for each $\gamma\in[0,1]$, we have

$$\begin{gathered}\mathcal{H}_*(u+\tau\xi(\nu,u),\gamma)\le(1-\tau)\mathcal{H}_*(u,\gamma)+\tau\mathcal{H}_*(\nu,\gamma)-\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,\\ \mathcal{H}^*(u+\tau\xi(\nu,u),\gamma)\le(1-\tau)\mathcal{H}^*(u,\gamma)+\tau\mathcal{H}^*(\nu,\gamma)-\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,\end{gathered} \quad (16)$$

from (15) and (16), we have

$$\mathcal{H}_*(u+\tau\xi(\nu,u),\gamma)-f_*(u+\tau\xi(\nu,u),\gamma)\le(1-\tau)\mathcal{H}_*(u,\gamma)+\tau\mathcal{H}_*(\nu,\gamma)$$
$$-(1-\tau)f_*(u,\gamma)-\tau f_*(\nu,\gamma),$$
$$\mathcal{H}^*(u+\tau\xi(\nu,u),\gamma)-f^*(u+\tau\xi(\nu,u),\gamma)\le(1-\tau)\mathcal{H}^*(u,\gamma)+\tau\mathcal{H}^*(\nu,\gamma)$$
$$-(1-\tau)f^*(u,\gamma)-\tau f^*(\nu,\gamma),$$

$$\mathcal{H}_*(u+\tau\xi(\nu,u),\gamma)-f_*(u+\tau\xi(\nu,u),\gamma)\le(1-\tau)\big(\mathcal{H}_*(u,\gamma)-f_*(u,\gamma)\big)$$
$$+\tau\big(\mathcal{H}_*(\nu,\gamma)-f_*(\nu,\gamma)\big),$$
$$\mathcal{H}^*(u+\tau\xi(\nu,u),\gamma)-f^*(u+\tau\xi(\nu,u),\gamma)\le(1-\tau)\big(\mathcal{H}^*(u,\gamma)-f^*(u,\gamma)\big)$$
$$+\tau\big(\mathcal{H}^*(\nu,\gamma)-f^*(\nu,\gamma)\big),$$

from which it follows that

$$\varpi_*(u+\tau\xi(\nu,u),\gamma)=\mathcal{H}_*(u+\tau\xi(\nu,u),\gamma)-f_*(u+\tau\xi(\nu,u),\gamma),$$
$$\varpi^*(u+\tau\xi(\nu,u),\gamma)=\mathcal{H}^*(u+\tau\xi(\nu,u),\gamma)-f^*(u+\tau\xi(\nu,u),\gamma),$$

$$\varpi_*(u+\tau\xi(\nu,u),\gamma)\le(1-\tau)\varpi_*(u,\gamma)+\tau\varpi_*(\nu,\gamma),$$
$$\varpi_*(u+\tau\xi(\nu,u),\gamma)\le(1-\tau)\varpi^*(u,\gamma)+\tau\varpi^*(\nu,\gamma),$$

that is

$$\varpi(u+\tau\xi(\nu,u))\preccurlyeq(1-\tau)\varpi(u)\tilde{+}\tau\varpi(u).$$

Showing that $\varpi=\mathcal{H}-f$ is preinvex-FM.

We know that under certain condition invex-FMs, we get a solution of fuzzy optimization problem because with the help of these FM, we obtain relationship between the fuzzy variational inequalities and optimization problems.

Definition 3.6. The G-differentiable FM $\mathcal{H}:K_\xi\to\mathbb{F}_0$ on K_ξ is said to be strongly invex-FM w.r.t. bi-function ξ if there exist a constant $0\le\omega$ such that

$$\mathcal{H}(\nu)\tilde{-}\mathcal{H}(u)\succcurlyeq\langle F'(u),\xi(\nu,u)\rangle\tilde{+}\omega\|\xi(\nu,u)\|^2,\ \text{for all } u,\nu\in K_\xi. \quad (17)$$

Example 3.7. We consider the FMs $\mathcal{H}:(0,1)\to\mathbb{F}_0$ defined by, $\mathcal{H}_\gamma(u)=[2\gamma u^2,(4-2\gamma)u^2]$, as in Example 3.4, then $\mathcal{H}(u)$ is strongly invex-FM w.r.t. bifunction $\xi(\nu,u)=\nu-u$, with $0<\omega=\gamma\le1$, where $u\le\nu$. We have $\mathcal{H}_*(u,\gamma)=\gamma u^2$ and $\mathcal{H}^*(u,\gamma)=(2-\gamma)u^2$. Now we computing the following

$$\mathcal{H}_*(\nu,\gamma)-\mathcal{H}_*(u,\gamma)=\gamma\nu^2-\gamma u^2,$$

while

$$\langle\mathcal{H}_*{}'(u,\gamma),\xi(\nu,u)\rangle+\omega\|\xi(\nu,u)\|^2=2\gamma(\nu-u)+\omega\|\nu-u\|^2.$$

And $\gamma\nu^2-\gamma u^2\ge2\gamma(\nu-u)+\omega\|\nu-u\|^2$, with $0<\omega\le1$, where $u\le\nu$.
Similarly, it can be easily show that

$$\mathcal{H}^*(\nu,\gamma)-\mathcal{H}^*(u,\gamma)\ge\langle\mathcal{H}^*{}'(u,\gamma),\xi(\nu,u)\rangle+\omega\|\xi(\nu,u)\|^2$$

Hence, $\mathcal{H}(u)$ is strongly invex-FM w.r.t. bifunction $\xi(\nu,u)=\nu-u$, with $0<\omega\le1$. It can be easily seen that $\mathcal{H}(u)$ is not invex-FM w.r.t. bifunction $\xi(\nu,u)=\nu-u$.

Definition 3.8. The G-differentiable FM $\mathcal{H}:K_\xi\to\mathbb{F}_0$ on K_ξ is said to be strongly pseudo invex-FM w.r.t. bi-function ξ if there exist a constant $0\le\omega$ such that

$$\langle\mathcal{H}'(u),\xi(\nu,u)\rangle\tilde{+}\omega\|\xi(\nu,u)\|^2\succcurlyeq\tilde{0}\Longrightarrow\mathcal{H}(\nu)\tilde{-}\mathcal{H}(u)\succcurlyeq\tilde{0},\ \text{for all } u,\nu\in K_\xi. \quad (18)$$

If $\omega=0$, then from Definition 3.6 and Definition 3.8, we obtain the classical definitions of invex-FM and pseudo invex-FM, respectively. If $\xi(\nu,u)=\nu-u$, then Definitions 11and Definition 3.8 reduce to known ones.

Example 3.9. We consider the FMs $\mathcal{H}:(0,\infty)\to\mathbb{F}_0$ defined by, $\mathcal{H}_\gamma(u)=[\gamma u,(3-2\gamma)u]$, then $\mathcal{H}(u)$ is strongly pseudo invex-FM w.r.t. bifunction $\xi(\nu,u)=\nu-u$, with $0\le\omega=\gamma$, where $u\le\nu$. We have $\mathcal{H}_*(u,\gamma)=\gamma u$ and $\mathcal{H}^*(u,\gamma)=(3-2\gamma)u$. Now we computing the following

$$\langle\mathcal{H}_*{}'(u,\gamma),\xi(\nu,u)\rangle+\omega\|\xi(\nu,u)\|^2=\gamma(\nu-u)+\omega\|\nu-u\|^2\ge0,$$

for all $u,\nu\in K_\xi$ and $\gamma\in[0,1]$ with $u\le\nu$, $0\le\omega$; which implies that

$$\mathcal{H}_*(\nu,\gamma)=\gamma\nu\geq\gamma u=\mathcal{H}_*(u,\gamma),$$
$$\mathcal{H}_*(\nu,\gamma)\geq\mathcal{H}_*(u,\gamma),$$

Similarly, it can be easily show that

$$\langle\mathcal{H}_*{}'(u,\gamma),\xi(\nu,u)\rangle+\omega\|\xi(\nu,u)\|^2=(3-2\gamma)(\nu-u)+\omega\|\nu-u\|^2\geq 0,$$

for all $u,\nu\in K_\xi$ and $\gamma\in[0,1]$ with $u\leq\nu$, $0\leq\omega$; that means

$$\mathcal{H}^*(\nu,\gamma)=(3-2\gamma)\nu\geq\gamma u=\mathcal{H}^*(u,\gamma),$$

From which, It follows that

$$\mathcal{H}^*(\nu,\gamma)\geq\mathcal{H}^*(u,\gamma)$$

Hence, the FM $\mathcal{H}_\gamma(u)=[\gamma u,(3-2\gamma)u]$ is strongly pseudo invex-FM w.r.t. $\xi(\nu,u)=\nu-u$, with $0\leq\omega$, where $u\leq\nu$. It can be easily seen that $\mathcal{H}(u)$ is not a pseudo invex-FM w.r.t. ξ.

Theorem 3.10. Let $\mathcal{H}\colon K_\xi\to\mathbb{F}_0$ be a G-differentiable and strongly preinvex-FM then $\mathcal{H}$ is a strongly invex-FM.

Proof. Let $\mathcal{H}\colon K_\xi\to\mathbb{F}_0$ be G-differentiable strongly preinvex-FM. Since $\mathcal{H}$ is strongly preinvex then, for each $u,\nu\in K_\xi$ and $\tau\in[0,1]$, we have

$$\mathcal{H}(u+\tau\xi(\nu,u))\preccurlyeq(1-\tau)\mathcal{H}(u)\tilde{+}\tau\mathcal{H}(\nu)\tilde{-}\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$
$$\preccurlyeq\mathcal{H}(u)\tilde{+}\tau\big(\mathcal{H}(\nu)\tilde{-}\mathcal{H}(u)\big)\tilde{-}\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$

Therefore, for every $\gamma\in[0,1]$, we have

$$\mathcal{H}_*(u+\tau\xi(\nu,u),\gamma)\leq\mathcal{H}_*(u,\gamma)+\tau\big(\mathcal{H}_*(\nu,\gamma)-\mathcal{H}_*(u,\gamma)\big)-\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$
$$\mathcal{H}^*(u+\tau\xi(\nu,u),\gamma)\leq\mathcal{H}_*(u,\gamma)+\tau\big(\mathcal{H}^*(\nu,\gamma)-\mathcal{H}^*(u,\gamma)\big)-\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$

which implies that

$$\tau\big(\mathcal{H}_*(\nu,\gamma)-\mathcal{H}_*(u,\gamma)\big)\geq\mathcal{H}_*(u+\tau\xi(\nu,u),\gamma)-\mathcal{H}_*(u,\gamma)+\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$
$$\tau\big(\mathcal{H}^*(\nu,\gamma)-\mathcal{H}^*(u,\gamma)\big)\geq\mathcal{H}^*(u+\tau\xi(\nu,u),\gamma)-\mathcal{H}^*(u,\gamma)+\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$

$$\mathcal{H}_*(\nu,\gamma)-\mathcal{H}_*(u,\gamma)\geq\frac{\mathcal{H}_*(u+\tau\xi(\nu,u),\gamma)-\mathcal{H}_*(u,\gamma)}{\tau}+\omega(1-\tau)\|\xi(\nu,u)\|^2,$$

$$\mathcal{H}^*(\nu,\gamma)-\mathcal{H}^*(u,\gamma)\geq\frac{\mathcal{H}^*(u+\tau\xi(\nu,u),\gamma)-\mathcal{H}^*(u,\gamma)}{\tau}+\omega(1-\tau)\|\xi(\nu,u)\|^2.$$

Taking limit in the above inequality as $\tau\to 0$, we have

$$\mathcal{H}_*(\nu,\gamma)-\mathcal{H}_*(u,\gamma)\geq\langle\mathcal{H}_*{}'(u,\gamma),\xi(\nu,u)\rangle+\omega\|\xi(\nu,u)\|^2,$$
$$\mathcal{H}^*(\nu,\gamma)-\mathcal{H}^*(u,\gamma)\geq\langle\mathcal{H}^{*\prime}(u,\gamma),\xi(\nu,u)\rangle+\omega\|\xi(\nu,u)\|^2,$$

that is

$$\mathcal{H}(\nu)\tilde{-}\mathcal{H}(u)\succcurlyeq\langle\mathcal{H}'(u),\xi(\nu,u)\rangle\tilde{+}\omega\|\xi(\nu,u)\|^2.$$

As special case of Theorem 3.16, when $\omega=0$, we have the following:

Corollary 3.11. [32] Let $\mathcal{H}\colon K_\xi\to\mathbb{F}_0$ be a G-differentiable preinvex-FM on K_ξ. Then $\mathcal{H}$ is an invex-FM,

It is well known that the differentiable preinvex functions are invex functions, but the converse is not true. However, Mohan and Neogy [13], have shown that the preinvex functions and invex functions are equivalent under certain Condition C. Similarly, the converse of Theorem 3.16, is not valid; the natural question how to get a strongly preinvex-FM from strongly invex-FM. To prove converse, we need the following assumption regarding the bi-function ξ, which plays an important role in G-differentiation of the main results.

Condition C.

$$\xi\big(\nu, u + \tau\xi(\nu, u)\big) = (1 - \tau)\xi(\nu, u),$$
$$\xi\big(u, u + \tau\xi(\nu, u)\big) = -\tau\xi(\nu, u).$$

Clearly for $\tau = 0$, we have $\xi(\nu, u) = 0$ if and only if, $\nu = u$ for all $u, \nu \in K_\xi$. Additionally, note that from Condition C, we have

$$\xi\big(u + \tau_2\xi(\nu, u), u + \tau_1\xi(\nu, u)\big) = (\tau_2 - \tau_1)\xi(\nu, u)$$

For the application of Condition C, see [13, 14-17].

The following Theorem 3.12 gives the result of the converse of Theorem 3.10.

Theorem 3.12. Let $\mathcal{H}: K_\xi \to \mathbb{F}_0$ be a G-differentiable FM on K_ξ. Let Condition C holds and $\mathcal{H}(u)$ satisfies the following condition

$$\mathcal{H}(u + \tau\xi(\nu, u)) \preccurlyeq \mathcal{H}(\nu), \tag{19}$$

then the followings are equivalent:

(a) $\mathcal{H}$ is strongly preinvex-FM.

(b) $\mathcal{H}(\nu) \tilde{-} \mathcal{H}(u) \succcurlyeq \langle \mathcal{H}'(u), \xi(\nu, u)\rangle \tilde{+} \omega\|\xi(\nu, u)\|^2$, for all $u, \nu \in K_\xi$, (20)

(c) $\langle \mathcal{H}'(u), \xi(\nu, u)\rangle \tilde{+} \langle \mathcal{H}'(\nu), \xi(u, \nu)\rangle \preccurlyeq \tilde{-}\omega\{\|\xi(\nu, u)\|^2 + \|\xi(u, \nu)\|^2\}$, (21)

for all $u, \nu \in K_\xi$.

Proof (a) implies (b)

The demonstration is analogous to the demonstration of Theorem 3.10.

(b) implies (c). Let (b) holds. Then, for every $\gamma \in [0, 1]$, we have

$$\begin{aligned} \mathcal{H}_*(\nu, \gamma) - \mathcal{H}_*(u, \gamma) &\geq \langle \mathcal{H}_*{}'(u, \gamma), \xi(\nu, u)\rangle + \omega\|\xi(\nu, u)\|^2, \\ \mathcal{H}^*(\nu, \gamma) - \mathcal{H}^*(u, \gamma) &\geq \langle \mathcal{H}^{*\prime}(u, \gamma), \xi(\nu, u)\rangle + \omega\|\xi(\nu, u)\|^2, \end{aligned} \tag{22}$$

Then, by replacing ν by u and u by ν in (22), we get

$$\begin{aligned} \mathcal{H}_*(u, \gamma) - \mathcal{H}_*(\nu, \gamma) &\geq \langle \mathcal{H}_*{}'(\nu, \gamma), \xi(u, \nu)\rangle + \omega\|\xi(u, \nu)\|^2, \\ \mathcal{H}^*(u, \gamma) - \mathcal{H}^*(\nu, \gamma) &\geq \langle \mathcal{H}^{*\prime}(\nu, \gamma), \xi(u, \nu)\rangle + \omega\|\xi(u, \nu)\|^2. \end{aligned} \tag{23}$$

Adding (22) and (23), we have

$$\begin{aligned} \langle \mathcal{H}_*{}'(u, \gamma), \xi(\nu, u)\rangle + \langle \mathcal{H}_*{}'(\nu, \gamma), \xi(u, \nu)\rangle &\leq -\omega(\|\xi(\nu, u)\|^2 + \|\xi(u, \nu)\|^2), \\ \langle \mathcal{H}^{*\prime}(u, \gamma), \xi(\nu, u)\rangle + \langle \mathcal{H}^{*\prime}(\nu, \gamma), \xi(u, \nu)\rangle &\leq -\omega(\|\xi(\nu, u)\|^2 + \|\xi(u, \nu)\|^2), \end{aligned}$$

That is

$$\langle \mathcal{H}'(u), \xi(\nu, u)\rangle \tilde{+} \langle \mathcal{H}'(\nu), \xi(u, \nu)\rangle \preccurlyeq \tilde{-}\omega\{\|\xi(\nu, u)\|^2 + \|\xi(u, \nu)\|^2\}.$$

(c) implies (b). Assume that (21) holds. Then, for every $\gamma \in [0, 1]$, we have

$$\begin{aligned} \langle \mathcal{H}_*{}'(\nu, \gamma), \xi(u, \nu)\rangle &\leq -\langle \mathcal{H}_*{}'(u, \gamma), \xi(\nu, u)\rangle - \omega(\|\xi(\nu, u)\|^2 + \|\xi(u, \nu)\|^2), \\ \langle \mathcal{H}^{*\prime}(\nu, \gamma), \xi(u, \nu)\rangle &\leq -\langle \mathcal{H}^{*\prime}(u, \gamma), \xi(\nu, u)\rangle - \omega(\|\xi(\nu, u)\|^2 + \|\xi(u, \nu)\|^2). \end{aligned} \tag{24}$$

Since, $\nu_\tau = u + \tau\xi(\nu, u) \in K_\xi$ for all $u, \nu \in K_\xi$ and $\tau \in [0, 1]$. Taking $\nu = \nu_\tau$ in (24), we get

$$\begin{aligned} \langle \mathcal{H}_*{}'(u + \tau\xi(\nu, u), \gamma), \xi\big(u, u + \tau\xi(\nu, u)\big)\rangle &\leq -\langle \mathcal{H}_*{}'(u, \gamma), \xi(u + \tau\xi(\nu, u), u)\rangle \\ &\quad -\omega(\|\xi(u + \tau\xi(\nu, u), u)\|^2 + \|\xi(u, u + \tau\xi(\nu, u))\|^2), \\ \langle \mathcal{H}^{*\prime}(u + \tau\xi(\nu, u), \gamma), \xi\big(u, u + \tau\xi(\nu, u)\big)\rangle &\leq -\langle \mathcal{H}^{*\prime}(u, \gamma), \xi(u + \tau\xi(\nu, u), u)\rangle \\ &\quad -\omega(\|\xi(u + \tau\xi(\nu, u), u)\|^2 + \|\xi(u, u + \tau\xi(\nu, u))\|^2), \end{aligned}$$

by using Condition C, we have

$$\begin{aligned} \langle \mathcal{H}_*{}'(u + \tau\xi(\nu, u), \gamma), \tau\xi(\nu, u)\rangle &\geq \langle \mathcal{H}_*{}'(u, \gamma), \tau\xi(\nu, u)\rangle + 2\omega\tau^2\|\xi(\nu, u)\|^2, \\ \langle \mathcal{H}^{*\prime}(u + \tau\xi(\nu, u), \gamma), \tau\xi(\nu, u)\rangle &\geq \langle \mathcal{H}^{*\prime}(u, \gamma), \tau\xi(\nu, u)\rangle + 2\omega\tau^2\|\xi(\nu, u)\|^2, \end{aligned}$$

$$\begin{aligned} \langle \mathcal{H}_*{}'(u + \tau\xi(\nu, u), \gamma), \xi(\nu, u)\rangle &\geq \langle \mathcal{H}_*{}'(u, \gamma), \xi(\nu, u)\rangle + 2\omega\tau\|\xi(\nu, u)\|^2, \\ \langle \mathcal{H}^{*\prime}(u + \tau\xi(\nu, u), \gamma), \xi(\nu, u)\rangle &\geq \langle \mathcal{H}^{*\prime}(u, \gamma), \xi(\nu, u)\rangle + 2\omega\tau\|\xi(\nu, u)\|^2, \end{aligned} \tag{25}$$

Let

$$H_*(\tau) = \mathcal{H}_*(u + \tau\xi(\nu,u),\gamma),$$
$$H^*(\tau) = \mathcal{H}^*(u + \tau\xi(\nu,u),\gamma).$$

Taking derivative w.r.t. τ, we get

$$H_*{}'(\tau) = \mathcal{H}_*{}'(u + \tau\xi(\nu,u),\gamma).\xi(\nu,u) = \langle \mathcal{H}_*{}'(u + \tau\xi(\nu,u),\gamma), \xi(\nu,u)\rangle,$$
$$H^*{}'(\tau) = \mathcal{H}^*{}'(u + \tau\xi(\nu,u),\gamma).\xi(\nu,u) = \langle \mathcal{H}^*{}'(u + \tau\xi(\nu,u),\gamma), \xi(\nu,u)\rangle,$$

from which, using (25), we have

$$H_*{}'(\tau) \geq \langle \mathcal{H}_*{}'(u,\gamma), \xi(\nu,u)\rangle + 2\omega\tau\|\xi(\nu,u)\|^2,$$
$$H^*{}'(\tau) \geq \langle \mathcal{H}^*{}'(u,\gamma), \xi(\nu,u)\rangle + 2\omega\tau\|\xi(\nu,u)\|^2. \quad (26)$$

By integrating (26) between 0 to 1, w.r.t. τ, we get

$$H_*(1) - H_*(0) \geq \langle \mathcal{H}_*{}'(u,\gamma), \xi(\nu,u)\rangle + \omega\|\xi(\nu,u)\|^2,$$
$$H^*(1) - H^*(0) \geq \langle \mathcal{H}^*{}'(u,\gamma), \xi(\nu,u)\rangle + \omega\|\xi(\nu,u)\|^2.$$

$$\mathcal{H}_*(u + \xi(\nu,u),\gamma) - \mathcal{H}_*(u,\gamma) \geq \langle \mathcal{H}_*{}'(u,\gamma), \xi(\nu,u)\rangle + \omega\|\xi(\nu,u)\|^2,$$
$$\mathcal{H}^*(u + \xi(\nu,u),\gamma) - \mathcal{H}^*(u,\gamma) \geq \langle \mathcal{H}^*{}'(u,\gamma), \xi(\nu,u)\rangle + \omega\|\xi(\nu,u)\|^2.$$

Using (19), we have

$$\mathcal{H}_*(\nu,\gamma) - \mathcal{H}_*(u,\gamma) \geq \langle \mathcal{H}_*{}'(u,\gamma), \xi(\nu,u)\rangle + \omega\|\xi(\nu,u)\|^2,$$
$$\mathcal{H}^*(\nu,\gamma) - \mathcal{H}^*(u,\gamma) \geq \langle \mathcal{H}^*{}'(u,\gamma), \xi(\nu,u)\rangle + \omega\|\xi(\nu,u)\|^2,$$

that is

$$\mathcal{H}(\nu) \tilde{-} \mathcal{H}(u) \succcurlyeq \langle \mathcal{H}'(u), \tau\xi(\nu,u)\rangle \tilde{+} \omega\|\xi(\nu,u)\|^2, \text{ for all } u,\nu \in K_\xi.$$

(b) implies (a). Assume that (20) holds. Since K_ξ, $\nu_\tau = u + \tau\xi(\nu,u) \in K_\xi$ for all $u,\nu \in K_\xi$ and $\tau \in [0,1]$. Taking $\nu = \nu_\tau$ in (20), we get

$$\mathcal{H}\big(u + \tau\xi(\nu,u)\big) \tilde{-} \mathcal{H}(u) \succcurlyeq \langle \mathcal{H}'(u), \xi(u + \tau\xi(\nu,u),u)\rangle \tilde{+} \omega\|\xi(u + \tau\xi(\nu,u),u)\|^2.$$

Therefore, for every $\gamma \in [0,1]$, we have

$$\mathcal{H}_*(u + \tau\xi(\nu,u),\gamma) - \mathcal{H}_*(u,\gamma) \geq \langle \mathcal{H}_*{}'(u,\gamma), \xi(u + \tau\xi(\nu,u),u)\rangle + \omega\|\xi(u + \tau\xi(\nu,u),u)\|^2,$$
$$\mathcal{H}^*(u + \tau\xi(\nu,u),\gamma) - \mathcal{H}^*(u,\gamma) \geq \langle \mathcal{H}^*{}'(u,\gamma), \xi(u + \tau\xi(\nu,u),u)\rangle + \omega\|\xi(u + \tau\xi(\nu,u),u)\|^2.$$

Using Condition C, we have

$$\mathcal{H}_*(u + \tau\xi(\nu,u),\gamma) - \mathcal{H}_*(u,\gamma) \geq (1-\tau)\langle \mathcal{H}_*{}'(u,\gamma), \xi(\nu,u)\rangle + \omega(1-\tau)^2\|\xi(\nu,u)\|^2,$$
$$\mathcal{H}^*(u + \tau\xi(\nu,u),\gamma) - \mathcal{H}^*(u,\gamma) \geq (1-\tau)\langle \mathcal{H}^*{}'(u,\gamma), \xi(\nu,u)\rangle + \omega(1-\tau)^2\|\xi(\nu,u)\|^2. \quad (27)$$

In a similar way, we have

$$\mathcal{H}_*(u,\gamma) - \mathcal{H}_*(u + \tau\xi(\nu,u),\gamma) \geq -\tau\langle \mathcal{H}_*{}'(u,\gamma), \xi(\nu,u)\rangle + \omega\tau^2\|\xi(\nu,u)\|^2,$$
$$\mathcal{H}^*(u,\gamma) - \mathcal{H}^*(u + \tau\xi(\nu,u),\gamma) \geq -\tau\langle \mathcal{H}^*{}'(u,\gamma), \xi(\nu,u)\rangle + \omega\tau^2\|\xi(\nu,u)\|^2. \quad (28)$$

Multiplying (27) by τ and (28) by $(1-\tau)$, and adding the resultant, we have

$$\mathcal{H}_*(u + \tau\xi(\nu,u),\gamma) \leq (1-\tau)\mathcal{H}_*(u,\gamma) + \tau\mathcal{H}_*(\nu,\gamma) - \omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$
$$\mathcal{H}^*(u + \tau\xi(\nu,u),\gamma) \leq (1-\tau)\mathcal{H}^*(u,\gamma) + \tau\mathcal{H}^*(\nu,\gamma) - \omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$

That is

$$\mathcal{H}\big(u + \tau\xi(\nu,u)\big) \preccurlyeq (1-\tau)\mathcal{H}(u) \tilde{+} \tau\mathcal{H}(\nu) \tilde{-} \omega\tau(1-\tau)\|\xi(\nu,u)\|^2.$$

Hence, $\mathcal{H}$ is strongly preinvex-FM w.r.t. ξ.

Theorem 3.10 and Theorem 3.12, enable us to define the followings new definitions.

Definition 3.13. A G-differentiable FM $\mathcal{H}: K_\xi \to \mathbb{F}_0$ is said to be:

(i) Strongly monotone w.r.t. bi-function ξ if and only if, there exist a constant $0 \leq \omega$ such that

$$\langle \mathcal{H}'(u), \xi(\nu,u)\rangle \tilde{+} \langle \mathcal{H}'(\nu), \xi(u,\nu)\rangle \preccurlyeq \tilde{-}\omega\{\|\xi(\nu,u)\|^2 + \|\xi(u,\nu)\|^2\}, \text{ for all } u,\nu \in K_\xi.$$

(ii) Strongly pseudo monotone w.r.t. bi-function ξ if and only if, there exist a constant $0 \leq \omega$ such that

$$\langle \mathcal{H}'(u), \xi(\nu, u)\rangle \tilde{\mp} \omega \|\xi(\nu, u)\|^2 \succcurlyeq \tilde{0} \Longrightarrow \tilde{-}\langle \mathcal{H}'(\nu), \xi(u, \nu)\rangle \succcurlyeq \tilde{0}, \text{for all } u, \nu \in K_\xi.$$

If $\xi(\nu, u) = -\xi(u, \nu)$, then Definition 3.13, reduce to new one.

Example 3.14. We consider the FMs $\mathcal{H}: (0, \infty) \to \mathbb{F}_0$ defined by,

$$\mathcal{H}(u)(\sigma) = \begin{cases} \frac{\sigma}{2u^2} & \sigma \in [0, 2u^2] \\ \frac{5u^2 - \sigma}{3u^2} & \sigma \in (2u^2, 5u^2] \\ 0 & otherwise. \end{cases}$$

Then, for each $\gamma \in [0,1]$, we have $\mathcal{H}_\gamma(u) = [2\gamma u^2, (5 - 3\gamma)u^2]$, $\mathcal{H}(u)$ is fuzzy strongly pseudomonotone w.r.t. bifunction $\xi(\nu, u) = u - \nu$, with $1 \leq \omega$, where $\nu \leq u$. We have $\mathcal{H}_*(u, \gamma) = 2\gamma u^2$ and $\mathcal{H}^*(u, \gamma) = (5 - 3\gamma)u^2$. Now we computing the following

$$\langle \mathcal{H}_*'(u, \gamma), \xi(\nu, u)\rangle + \omega\|\xi(\nu, u)\|^2 = 4\gamma u(u - \nu) + \omega\|u - \nu\|^2 \geq 0,$$

for all $u, \nu \in K_\xi$ and $\gamma \in [0,1]$ with $\nu \leq u$, $1 \leq \omega$; which implies that

$$-\langle \mathcal{H}_*'(\nu, \gamma), \xi(u, \nu)\rangle = -4\gamma u(\nu - u) = 4\gamma\nu(u - \nu) \geq 0, \forall\, u, \nu \in K_\xi,$$

$$-\langle \mathcal{H}^{*\prime}(\nu, \gamma), \xi(u, \nu)\rangle \geq 0.$$

Similarly, it can be easily show that

$$\langle \mathcal{H}^{*\prime}(u, \gamma), \xi(\nu, u)\rangle + \omega\|\xi(\nu, u)\|^2 = 2(5 - 3\gamma)u(u - \nu) + \omega\|u - \nu\|^2 \geq 0,$$

for all $u, \nu \in K_\xi$ and $\gamma \in [0,1]$ with $\nu \leq u$, $1 \leq \omega$; that means

$$-\langle \mathcal{H}^{*\prime}(\nu, \gamma), \xi(u, \nu)\rangle = -2(5 - 3\gamma)u(\nu - u) = 2(5 - 3\gamma)\nu(u - \nu) \geq 0, \forall\, u, \nu \in K_\xi,$$

From which, It follows that

$$-\langle \mathcal{H}^{*\prime}(\nu, \gamma), \xi(u, \nu)\rangle \geq 0.$$

Hence, the G-differentiable FM $\mathcal{H}_\gamma(u) = [\gamma u, (5 - 4\gamma)u]$ is fuzzy strongly pseudo monotone w.r.t. $\xi(\nu, u) = u - \nu$, with $1 \leq \omega$, where $\nu \leq u$. it can be easily note that $\mathcal{H}'(u)$ is neither fuzzy pseudomonotone nor fuzzy quasimonotone w.r.t. ξ.

If $\omega = 0$, then from Theorem 3.12 we obtain following result.

Corollary 3.15. [36] Let $\mathcal{H}: K_\xi \to \mathbb{F}_0$ be a G-differentiable FM on K_ξ. Let Condition C holds and $\mathcal{H}(u)$ satisfies the following condition

$$\mathcal{H}(u + \tau\xi(\nu, u)) \preccurlyeq \mathcal{H}(\nu),$$

then the followings are equivalent:

(a) $\mathcal{H}$ is invex-FM.

(b) $\mathcal{H}'$ is monotone

Theorem 3.16. Let $\mathcal{H}: K_\xi \to \mathbb{F}_0$ be FM on K_ξ w.r.t. ξ and Condition C hold. Let $\mathcal{H}(u)$ is G-differentiable on K_ξ with following conditions:

(a) $\mathcal{H}(u + \tau\xi(\nu, u)) \preccurlyeq \mathcal{H}(\nu)$.

(b) $\mathcal{H}'(u)$ is a fuzzy strongly pseudo monotone.

Then $\mathcal{H}$ is a strongly pseudo invex-FM.

Proof. Let $\mathcal{H}'$ be a strongly pseudo monotone. Then for all $u, \nu \in K_\xi$, we have

$$\langle \mathcal{H}'(u), \xi(\nu, u)\rangle \tilde{\mp} \omega\|\xi(\nu, u)\|^2 \succcurlyeq \tilde{0}.$$

Therefore, for every $\gamma \in [0,1]$, we have

$$\langle \mathcal{H}_*'(u, \gamma), \xi(\nu, u)\rangle + \omega\|\xi(\nu, u)\|^2 \geq 0,$$
$$\langle \mathcal{H}^{*\prime}(u, \gamma), \xi(\nu, u)\rangle + \omega\|\xi(\nu, u)\|^2 \geq 0,$$

which implies that

$$-\langle \mathcal{H}_*{}'(\nu,\gamma), \xi(u,\nu)\rangle \geq 0,$$
$$-\langle \mathcal{H}^*{}'(\nu,\gamma), \xi(u,\nu)\rangle \geq 0. \quad (29)$$

Since, $\nu_\tau = u + \tau\xi(\nu,u) \in K_\xi$ for all $u,\nu \in K_\xi$ and $\tau \in [0,1]$. Taking ν=ν_τ in (29), we get

$$-\langle \mathcal{H}_*{}'(u+\tau\xi(\nu,u),\gamma), \xi(u, u+\tau\xi(\nu,u))\rangle \geq 0,$$
$$-\langle \mathcal{H}^*{}'(u+\tau\xi(\nu,u),\gamma), \xi(u, u+\tau\xi(\nu,u))\rangle \geq 0.$$

by using Condition C, we have

$$\langle \mathcal{H}_*{}'(u+\tau\xi(\nu,u),\gamma), \xi(\nu,u)\rangle \geq 0,$$
$$\langle \mathcal{H}^*{}'(u+\tau\xi(\nu,u),\gamma), \xi(\nu,u)\rangle \geq 0. \quad (30)$$

Assume that

$$H_*(\tau) = \mathcal{H}_*(u+\tau\xi(\nu,u),\gamma),$$
$$H^*(\tau) = \mathcal{H}^*(u+\tau\xi(\nu,u),\gamma),$$

taking G-derivative w.r.t. τ, then using (30), we have

$$H_*{}'(\tau) = \langle \mathcal{H}_*{}'(u+\tau\xi(\nu,u),\gamma), \xi(\nu,u)\rangle \geq 0,$$
$$H^*{}'(\tau) = \langle \mathcal{H}^*{}'(u+\tau\xi(\nu,u),\gamma), \xi(\nu,u)\rangle \geq 0, \quad (31)$$

Integrating (31) between 0 to 1 w.r.t. τ, we get

$$H_*(1) - H_*(0) \geq 0,$$
$$H^*(1) - H^*(0) \geq 0,$$

which implies that

$$\mathcal{H}_*(u+\xi(\nu,u),\gamma) - \mathcal{H}_*(u,\gamma) \geq 0,$$
$$\mathcal{H}^*(u+\xi(\nu,u),\gamma) - \mathcal{H}^*(u,\gamma) \geq 0.$$

From condition (i), we have

$$\mathcal{H}_*(\nu,\gamma) - \mathcal{H}_*(u,\gamma) \geq 0,$$
$$\mathcal{H}^*(\nu,\gamma) - \mathcal{H}^*(u,\gamma) \geq 0,$$

that is

$$\mathcal{H}(\nu) \tilde{-} \mathcal{H}(u) \succcurlyeq \tilde{0}, \forall\, u,\nu \in K_\xi.$$

Hence, $\mathcal{H}$ is a strongly pseudo invex-FM.

If $\omega = 0$, then from Theorem 3.16 reduces to the following result:

Corollary 3.17. [36] Let $\mathcal{H}: K_\xi \to \mathbb{F}_0$ be a FM on K_ξ w.r.t. ξ and Condition C hold. Let $\mathcal{H}(u)$ is G-differentiable on K_ξ with following conditions

(a) $\mathcal{H}(u+\tau\xi(\nu,u)) \preccurlyeq \mathcal{H}(\nu)$.

(b) $\mathcal{H}'(.)$ is a fuzzy pseudomonotone.

Then $\mathcal{H}$ is a pseudo invex-FM.

The fuzzy optimality requirement for G-differentiable strongly preinvex-FMs, which is the fundamental impetus for our findings, is now discussed.

4. Fuzzy mixed variational-like and integral inequalities

The variational inequality problem has a close relationship with the optimization problem, which is a well-known fact in mathematical programming. Similarly, the fuzzy variational inequality problem and the fuzzy optimization problem have a strong link. Consider the unconstrained fuzzy optimization problem

$$\min_{u \in K_\xi} \mathcal{H}(u),$$

where K_ξ is a subset of $\mathbb{R}$, $\mathcal{H}: K_\xi \to \mathbb{F}_0$ is a FM.

A feasible point is defined as $u \in K_\xi$ is called an optimal solution, a global optimal solution, or simply a solution to the fuzzy optimization problem if $u \in K_\xi$ and no $v \in K_\xi$, $\mathcal{H}(u) \preccurlyeq \mathcal{H}(v)$.

The fuzzy optimality criterion for G-differentiable preinvex-FMs is discussed in the following theorems, and this is the fundamental rationale for the results.

Theorem 4.1. Let $\mathcal{H}$ be a G-differentiable strongly preinvex-FM modulus $0 \le \omega$. If $u \in K_\xi$ is the minimum of the FM $\mathcal{H}$, then

$$\mathcal{H}(v) \tilde{-} \mathcal{H}(u) \succcurlyeq \omega \|\xi(v,u)\|^2, \text{ for all } u, v \in K_\xi. \tag{32}$$

Proof: Let $u \in K$ be a minimum of $\mathcal{H}$**.** Then

$$\mathcal{H}(u) \preccurlyeq \mathcal{H}(v), \text{ for all } v \in K_\xi.$$

Therefore, for every $\gamma \in [0,1]$, we have

$$\begin{gathered} \mathcal{H}_*(u,\gamma) \le \mathcal{H}_*(v,\gamma), \\ \mathcal{H}^*(u,\gamma) \le \mathcal{H}^*(v,\gamma). \end{gathered} \tag{33}$$

For all $u, v \in K_\xi$, $\tau \in [0,1]$, we have

$$v_\tau = u + \tau\xi(v,u) \in K_\xi.$$

Taking $v = v_\tau$ in (33), and dividing by "τ", we get

$$0 \le \frac{\mathcal{H}_*(u+\tau\xi(v,u),\gamma) - \mathcal{H}_*(u,\gamma)}{\tau},$$
$$0 \le \frac{\mathcal{H}^*(u+\tau\xi(v,u),\gamma) - \mathcal{H}^*(u,\gamma)}{\tau}.$$

Taking limit in the above inequality as $\tau \to 0$**,** we get

$$\begin{gathered} 0 \le \langle \mathcal{H}_*'(u,\gamma), \xi(v,u) \rangle, \\ 0 \le \langle \mathcal{H}^{*\prime}(u,\gamma), \xi(v,u) \rangle. \end{gathered} \tag{34}$$

Since $\mathcal{H}: K_\xi \to \mathbb{F}_0$ is a G-differentiable strongly preinvex-FM, so

$$\mathcal{H}_*(u+\tau\xi(v,u),\gamma) \le (1-\tau)\mathcal{H}_*(u,\gamma) + \tau\mathcal{H}_*(v,\gamma) - \omega\tau(1-\tau)\|\xi(v,u)\|^2,$$
$$\mathcal{H}^*(u+\tau\xi(v,u),\gamma) \le (1-\tau)\mathcal{H}^*(u,\gamma) + \tau\mathcal{H}^*(v,\gamma) - \omega\tau(1-\tau)\|\xi(v,u)\|^2,$$
$$\mathcal{H}_*(v,\gamma) - \mathcal{H}_*(u,\gamma) \ge \frac{\mathcal{H}_*(u+\tau\xi(v,u),\gamma) - \mathcal{H}_*(u,\gamma)}{\tau} + \omega(1-\tau)\|\xi(v,u)\|^2,$$
$$\mathcal{H}^*(v,\gamma) - \mathcal{H}^*(u,\gamma) \ge \frac{\mathcal{H}^*(u+\tau\xi(v,u),\gamma) - \mathcal{H}^*(u,\gamma)}{\tau} + \omega(1-\tau)\|\xi(v,u)\|^2,$$

again taking limit in the above inequality as $\tau \to 0$, we get

$$\mathcal{H}_*(v,\gamma) - \mathcal{H}_*(u,\gamma) \ge \langle \mathcal{H}_*'(u,\gamma), \xi(v,u) \rangle + \omega\|\xi(v,u)\|^2,$$
$$\mathcal{H}^*(v,\gamma) - \mathcal{H}^*(u,\gamma) \ge \langle \mathcal{H}^{*\prime}(u,\gamma), \xi(v,u) \rangle + \omega\|\xi(v,u)\|^2,$$

from which, using (34), we have

$$\mathcal{H}_*(v,\gamma) - \mathcal{H}_*(u,\gamma) \ge \omega\|\xi(v,u)\|^2 \ge 0,$$
$$\mathcal{H}^*(v,\gamma) - \mathcal{H}^*(u,\gamma) \ge \omega\|\xi(v,u)\|^2 \ge 0,$$

that is

$$\mathcal{H}(v) \tilde{-} \mathcal{H}(u) \succcurlyeq \tilde{0}.$$

Hence, the result follows.

Theorem 4.2. Let $\mathcal{H}$ be a G-differentiable strongly preinvex-FM modulus $0 \le \omega$, and

$$\langle \mathcal{H}'(u), \xi(v,u) \rangle \tilde{+} \omega\|\xi(v,u)\|^2 \succcurlyeq \tilde{0}, \text{ for all } u, v \in K_\xi, \tag{35}$$

then $u \in K_\xi$ is the minimum of the FM $\mathcal{H}$.

Proof. Let $\mathcal{H}: K_\xi \to \mathbb{F}_0$ be a G-differentiable strongly preinvex-FM and $u \in K_\xi$ satisfies (35). Then, by Theorem 3.10, we have

$$\mathcal{H}(\nu)\tilde{-}\mathcal{H}(u) \succcurlyeq \langle \mathcal{H}'(u), \xi(\nu,u)\rangle \tilde{+}\omega\|\xi(\nu,u)\|^2,$$

Therefore, for every $\gamma \in [0,1]$, we have

$$\mathcal{H}_*(\nu,\gamma) - \mathcal{H}_*(u,\gamma) \geq \langle \mathcal{H}_*{}'(u,\gamma), \xi(\nu,u)\rangle + \omega\|\xi(\nu,u)\|^2,$$
$$\mathcal{H}^*(\nu,\gamma) - \mathcal{H}^*(u,\gamma) \geq \langle \mathcal{H}^*{}'(u,\gamma), \xi(\nu,u)\rangle + \omega\|\xi(\nu,u)\|^2,$$

from which, using (35), we have

$$\mathcal{H}_*(\nu,\gamma) - \mathcal{H}_*(u,\gamma) \geq 0,$$
$$\mathcal{H}^*(\nu,\gamma) - \mathcal{H}^*(u,\gamma) \geq 0,$$

that is

$$\mathcal{H}(u) \preccurlyeq \mathcal{H}(\nu).$$

If $\omega = 0$ then, Theorem 4.2 reduces to the following result:

Corollary 4.3 [32] Let $\mathcal{H}$ be a G-differentiable preinvex-FM w.r.t. ξ. Then $u \in K_\xi$ is the minimum of $\mathcal{H}$ if and only if, $u \in K_\xi$ satisfies

$$\langle \mathcal{H}'(u), \xi(\nu,u)\rangle \succcurlyeq \tilde{0}, \text{for all } u,\nu \in K_\xi.$$

Remark 4.4. The inequality of the type (35) is called strongly variational like-inequality. It is very important to note that the optimality condition of preinvex-FMs can't be obtained with the help of (35). So this idea inspires us to introduce a more general form of fuzzy variational-like inequality of which (35) is a special case. To be more unambiguous, for given FM Ψ, bi function $\xi(.,.)$ and a $0 \leq \omega$, consider the problem of finding $u \in K_\xi$, such that

$$\langle \Psi(u), \xi(\nu,u)\rangle \tilde{+}\omega\|\xi(\nu,u)\|^2 \succcurlyeq \tilde{0}, \forall\, \nu \in K_\xi. \quad (36)$$

This inequality is called strongly fuzzy variational-like inequality.

We look at the functional $I(\nu)$, which is defined as

$$I(\nu) = \mathcal{H}(\nu)\tilde{+}\mathcal{J}(\nu), \forall\, \nu \in \mathbb{R}, \quad (37)$$

where $\mathcal{H}$ is a G-differentiable preinvex-FM and $\mathcal{J}$ is a strongly preinvex-FM which is non G-differentiable.

The following theorem shows that the functional $I(\nu)$ minimum can be distingusihed by a class of variational-like inequalities.

Theorem 4.5. Let $\mathcal{H}: K_\xi \to \mathbb{F}_0$ be a G-differentiable preinvex-FM and $\mathcal{J}: K_\xi \to \mathbb{F}_0$ be a non G-differentiable strongly preinvex-FM. Then the functional $I(\nu)$ has minimum $u \in K_\xi$, if and only if $u \in K_\xi$ satisfies

$$\langle \mathcal{H}'(u), \xi(\nu,u)\rangle \tilde{+}\mathcal{J}(\nu)\tilde{-}\mathcal{J}(u)\tilde{+}\omega\|\xi(\nu,u)\|^2 \succcurlyeq \tilde{0}, \forall\, \nu \in K_\xi. \quad (38)$$

Proof: Let $u \in K_\xi$ is the smallest value of I, then for all $\nu \in K_\xi$ we have

Therefore, for every $\gamma \in [0,1]$, we have

$$I_*(u,\gamma) \leq I_*(\nu,\gamma),$$
$$I^*(u,\gamma) \leq I^*(\nu,\gamma). \quad (39)$$

Since, $\nu_\tau = u + \tau\xi(\nu,u)$, for all $u,\nu \in K_\xi$ and $\tau \in [0,1]$. Replacing ν by ν_τ in (39), we get

$$I_*(u,\gamma) \leq I_*(u + \tau\xi(\nu,u),\gamma),$$
$$I^*(u,\gamma) \leq I^*(u + \tau\xi(\nu,u),\gamma).$$

which implies that, using (37)

$$\mathcal{H}_*(u,\gamma)+\mathcal{J}_*(u,\gamma)\le\mathcal{H}_*(u+\tau\xi(\nu,u),\gamma)+\mathcal{J}_*(u+\tau\xi(\nu,u),\gamma),$$
$$\mathcal{H}^*(u,\gamma)+\mathcal{J}^*(u,\gamma)\le\mathcal{H}^*(u+\tau\xi(\nu,u),\gamma)+\mathcal{J}^*(u+\tau\xi(\nu,u),\gamma).$$

Since $\mathcal{J}$ is strongly preinvex-FM then,

$$\mathcal{H}_*(u,\gamma)+\mathcal{J}_*(u,\gamma)\le\mathcal{H}_*(u+\tau\xi(\nu,u),\gamma)+(1-\tau)\mathcal{J}_*(u,\gamma)+\tau\mathcal{J}_*(\nu,\gamma)$$
$$+\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$
$$\mathcal{H}^*(u,\gamma)+\mathcal{J}^*(u,\gamma)\le\mathcal{H}^*(u+\tau\xi(\nu,u),\gamma)+(1-\tau)\mathcal{J}^*(u,\gamma)+\tau\mathcal{J}^*(\nu,\gamma)$$
$$+\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$

that is

$$0\le\mathcal{H}_*(u+\tau\xi(\nu,u),\gamma)-\mathcal{H}_*(u,\gamma)+\tau\big(\mathcal{J}_*(\nu,\gamma)-\mathcal{J}_*(u,\gamma)\big)+\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$
$$0\le\mathcal{H}^*(u+\tau\xi(\nu,u),\gamma)-\mathcal{H}^*(u,\gamma)+\tau\big(\mathcal{J}^*(\nu,\gamma)-\mathcal{J}^*(u,\gamma)\big)+\omega\tau(1-\tau)\|\xi(\nu,u)\|^2,$$

Now dividing by "τ" and taking $\lim_{\tau\to0}$, we have

$$0\le\lim_{\tau\to0}\left\{\frac{\mathcal{H}_*(u+\tau\xi(\nu,u),\gamma)-\mathcal{H}_*(u,\gamma)}{\tau}+\mathcal{J}_*(\nu,\gamma)-\mathcal{J}_*(u,\gamma)+\omega(1-\tau)\|\xi(\nu,u)\|^2\right\},$$
$$0\le\lim_{\tau\to0}\left\{\frac{\mathcal{H}^*(u+\tau\xi(\nu,u),\gamma)-\mathcal{H}^*(u,\gamma)}{\tau}+\mathcal{J}^*(\nu,\gamma)-\mathcal{J}^*(u,\gamma)+\omega(1-\tau)\|\xi(\nu,u)\|^2\right\},$$

then

$$0\le\langle\mathcal{H}_*{}'(u,\gamma),\xi(\nu,u)\rangle+\mathcal{J}_*(\nu,\gamma)-\mathcal{J}_*(u,\gamma)+\omega\|\xi(\nu,u)\|^2,$$
$$0\le\langle\mathcal{H}^*{}'(u,\gamma),\xi(\nu,u)\rangle+\mathcal{J}^*(\nu,\gamma)-\mathcal{J}^*(u,\gamma)+\omega\|\xi(\nu,u)\|^2,$$

that is

$$\tilde{0}\preccurlyeq\langle\mathcal{H}'(u),\xi(\nu,u)\rangle\tilde{+}\mathcal{J}(\nu)\tilde{-}\mathcal{J}(u)\tilde{+}\omega\|\xi(\nu,u)\|^2.$$

Conversely, let (38) be satisfy to prove $u\in K_\xi$ is a minimum of I. Assume that for all $\nu\in K_\xi$ we have

$$I(u)\tilde{-}I(\nu)=\mathcal{H}(u)\tilde{+}\mathcal{J}(u)\tilde{-}\mathcal{H}(\nu)\tilde{-}\mathcal{J}(\nu),$$
$$=\mathcal{H}(u)\tilde{-}\mathcal{H}(\nu)\tilde{+}\mathcal{J}(u)\tilde{-}\mathcal{J}(\nu),$$

Therefore, for every $\gamma\in[0,1]$, we have

$$I_*(u,\gamma)-I_*(\nu,\gamma)=\mathcal{H}_*(u,\gamma)-\mathcal{H}_*(\nu,\gamma)+\mathcal{J}_*(u,\gamma)-\mathcal{J}_*(\nu,\gamma),$$
$$I^*(u,\gamma)-I^*(\nu,\gamma)=\mathcal{H}^*(u,\gamma)-\mathcal{H}^*(\nu,\gamma)+\mathcal{J}^*(u,\gamma)-\mathcal{J}^*(\nu,\gamma).$$

by Corollary 3.11, we have

$$I_*(u,\gamma)-I_*(\nu,\gamma)\le-[\langle\mathcal{H}_*{}'(u,\gamma),\xi(\nu,u)\rangle+\mathcal{J}_*(\nu,\gamma)-\mathcal{J}_*(u,\gamma)],$$
$$I^*(u,\gamma)-I^*(\nu,\gamma)\le-[\langle\mathcal{H}^*{}'(u,\gamma),\xi(\nu,u)\rangle+\mathcal{J}^*(\nu,\gamma)-\mathcal{J}^*(u,\gamma)].$$

from which, using (38), we have

$$I_*(u,\gamma)-I_*(\nu,\gamma)\le-\omega\|\xi(\nu,u)\|^2\le0,$$
$$I^*(u,,\gamma)-I^*(\nu,\gamma)\le-\omega\|\xi(\nu,u)\|^2\le0,$$

that is

$$I(u)\tilde{-}I(\nu)\preccurlyeq\tilde{0},$$

hence, $I(u)\preccurlyeq I(\nu)$.

Note that the (38) is called strongly fuzzy mixed variational-like inequalities. This result shows that the minimum of fuzzy functional $I(\nu)$ can be characterized by strongly fuzzy mixed variational-like inequality. It is very important to observe that optimality conditions of preinvex-FMs and strongly preinvex-FMs can't be obtained with the help of (38). This idea encourage us to introduce a more general type of fuzzy variational-like inequality of which (38) is a particular case. In order to be more precise, for given FMs Ψ, ϖ, bi function $\xi(.,.)$ and a $0\le\omega$, consider problem of finding $u\in K_\xi$, such that

$$\langle \Psi(u), \xi(\nu, u)\rangle \tilde{+} \varpi(\nu) \tilde{-} \varpi(u) \tilde{+} \omega \|\xi(\nu, u)\|^2 \succcurlyeq \tilde{0}, \forall\, \nu \in K_\xi. \quad (40)$$

This inequality is called strongly fuzzy mixed variational-like inequality.

Now we'll look at a few specific types of strongly fuzzy mixed variational-like inequalities:

If $\xi(\nu, u) = \nu - u$, then (40) is called strongly fuzzy mixed variational inequality such as

$$\langle \Psi(u), \nu - u\rangle \tilde{+} \varpi(\nu) \tilde{-} \varpi(u) \tilde{+} \omega \|\nu - u\|^2 \succcurlyeq \tilde{0},\ \forall\, \nu \in K_\xi.$$

If $\omega = 0$, then (40), is called fuzzy mixed variational-like inequality such as

$$\langle \Psi(u), \xi(\nu, u)\rangle \tilde{+} \varpi(\nu) \tilde{-} \varpi(u) \succcurlyeq \tilde{0},\ \forall\, \nu \in K_\xi.$$

If $\xi(\nu, u) = \nu - u$ and $\omega = 0$, then (40) is called fuzzy mixed variational inequality such as

$$\langle \Psi(u), \nu - u\rangle \tilde{+} \varpi(\nu) \tilde{-} \varpi(u) \succcurlyeq \tilde{0},\ \forall\, \nu \in K_\xi.$$

Similarly, we can obtain fuzzy variational inequality and fuzzy variational-like inequality in [32], as special cases of (40). In a similar way, some special cases of strongly fuzzy variational-like inequality (36) can also be discussed.

Remark 4.6. The inequalities (36) and (40), show that the variational-like inequalities arise naturally in connection with the minimization of the G-differentiable preinvex-FMs subject to certain constraints.

The Theorem 4.7 provides Hermite-Hadamard inequality for strongly preinvex-FM. this inequality provides a lower and an upper estimation for the average of strongly preinvex-FM defined on a compact interval.

Theorem 4.7 Let $\mathcal{H}: [u, u + \xi(\nu, u)] \to \mathbb{F}_0$ be a strongly preinvex-FM with $\mathcal{H}(z) \succcurlyeq \tilde{0}$. If $\mathcal{H}$ is fuzzy integrable and $\xi(.,.)$ satisfies the Condition C, then

$$\mathcal{H}\left(\frac{2u+\xi(\nu,u)}{2}\right) \tilde{+} \frac{\omega}{12}\|\xi(\nu, u)\|^2 \preccurlyeq \frac{1}{\xi(\nu,u)}\ (FR) \int_u^{u+\xi(\nu,u)} \mathcal{H}(z)dz \preccurlyeq \frac{\mathcal{H}(u) \tilde{+} \mathcal{H}(\nu)}{2} \tilde{-} \frac{\omega}{6}\|\xi(\nu, u)\|^2. \quad (41)$$

If $\mathcal{H}$ is preconcave FM then, we inequality (41) reduces to the following inequality:

$$\mathcal{H}\left(\frac{2u+\xi(\nu,u)}{2}\right) \tilde{+} \frac{\omega}{12}\|\xi(\nu, u)\|^2 \succcurlyeq \frac{1}{\xi(\nu,u)}\ (FR) \int_u^{u+\xi(\nu,u)} \mathcal{H}(z)dz \succcurlyeq \frac{\mathcal{H}(u) \tilde{+} \mathcal{H}(\nu)}{2} \tilde{-} \frac{\omega}{6}\|\xi(\nu, u)\|^2.$$

Proof. Let $\mathcal{H}: [u, u + \xi(\nu, u)] \to \mathbb{F}_0$ be a strongly preinvex-FM. Then, by hypothesis, we have

$$2\mathcal{H}\left(\frac{2u+\xi(\nu,u)}{2}\right) \preccurlyeq \mathcal{H}\big(u + (1-\tau)\xi(\nu, u)\big) \tilde{+} \mathcal{H}(u + \tau\xi(\nu, u)) \tilde{-} \frac{\omega}{2}(1-2\tau)^2\|\xi(\nu, u)\|^2.$$

Therefore, for every $\gamma \in (0, 1]$, we have

$$\begin{aligned} 2\mathcal{H}_*\left(\frac{2u+\xi(\nu,u)}{2}, \gamma\right) &\leq \mathcal{H}_*(u + (1-\tau)\xi(\nu, u), \gamma) + \mathcal{H}_*(u + \tau\xi(\nu, u), \gamma) \\ &\quad - \frac{\omega}{2}(1-2\tau)^2\|\xi(\nu, u)\|^2 \\ 2\mathcal{H}^*\left(\frac{2u+\xi(\nu,u)}{2}, \gamma\right) &\leq \mathcal{H}^*(u + (1-\tau)\xi(\nu, u), \gamma) + \mathcal{H}^*(u + \tau\xi(\nu, u), \gamma) \\ &\quad - \frac{\omega}{2}(1-2\tau)^2\|\xi(\nu, u)\|^2. \end{aligned}$$

Then

$$2\int_0^1 \mathcal{H}_*\left(\frac{2u+\xi(\nu,u)}{2},\gamma\right)d\tau \leq \int_0^1 \mathcal{H}_*(u+(1-\tau)\xi(\nu,u),\gamma)d\tau + \int_0^1 \mathcal{H}_*(u+\tau\xi(\nu,u),\gamma)d\tau - \frac{\omega}{6}\|\xi(\nu,u)\|^2,$$

$$2\int_0^1 \mathcal{H}^*\left(\frac{2u+\xi(\nu,u)}{2},\gamma\right)d\tau \leq \int_0^1 \mathcal{H}^*(u+(1-\tau)\xi(\nu,u),\gamma)d\tau + \int_0^1 \mathcal{H}^*(u+\tau\xi(\nu,u),\gamma)d\tau - \frac{\omega}{2}\|\xi(\nu,u)\|^2.$$

It follows that

$$\mathcal{H}_*\left(\frac{2u+\xi(\nu,u)}{2},\gamma\right) + \frac{\omega}{12}\|\xi(\nu,u)\|^2 \leq \frac{1}{\xi(\nu,u)}\int_u^{u+\xi(\nu,u)}\mathcal{H}_*(z,\gamma)dz,$$

$$\mathcal{H}^*\left(\frac{2u+\xi(\nu,u)}{2},\gamma\right) + \frac{\omega}{12}\|\xi(\nu,u)\|^2 \leq \frac{1}{\xi(\nu,u)}\int_u^{u+\xi(\nu,u)}\mathcal{H}^*(z,\gamma)dz.$$

That is

$$\left[\mathcal{H}_*\left(\frac{2u+\xi(\nu,u)}{2},\gamma\right), \mathcal{H}^*\left(\frac{2u+\xi(\nu,u)}{2},\gamma\right)\right] + \frac{\omega}{12}\|\xi(\nu,u)\|^2$$

$$\leq_I \frac{1}{\xi(\nu,u)}\left[\int_u^{u+\xi(\nu,u)}\mathcal{H}_*(z,\gamma)dz, \int_u^{u+\xi(\nu,u)}\mathcal{H}^*(z,\gamma)dz\right].$$

Thus,

$$\mathcal{H}\left(\frac{2u+\xi(\nu,u)}{2}\right) + \frac{\omega}{12}\|\xi(\nu,u)\|^2 \preccurlyeq \frac{1}{\xi(\nu,u)}\,(FR)\int_u^{u+\xi(\nu,u)}\mathcal{H}(z)dz. \quad (42)$$

In a similar way as above, we have

$$\frac{1}{\xi(\nu,u)}\,(FR)\int_u^{u+\xi(\nu,u)}\mathcal{H}(z)dz \preccurlyeq \frac{\mathcal{H}(u)\,\tilde{+}\,\mathcal{H}(\nu)}{2} - \frac{\omega}{6}\|\xi(\nu,u)\|^2. \quad (43)$$

Combining (42) and (43), we have

$$\mathcal{H}\left(\frac{2u+\xi(\nu,u)}{2}\right) \tilde{+} \frac{\omega}{12}\|\xi(\nu,u)\|^2 \preccurlyeq \frac{1}{\xi(\nu,u)}\,(FR)\int_u^{u+\xi(\nu,u)}\mathcal{H}(z)dz \preccurlyeq \frac{\mathcal{H}(u)\,\tilde{+}\,\mathcal{H}(\nu)}{2} \simeq \frac{\omega}{6}\|\xi(\nu,u)\|^2.$$

This completes the proof.

Remark 4.8. If $\omega = 0$, then Theorem 4.7 reduces to the result for preinvex convex-FM

$$\mathcal{H}\left(\frac{2u+\xi(\nu,u)}{2}\right) \preccurlyeq \frac{1}{\xi(\nu,u)}\,(FR)\int_u^{u+\xi(\nu,u)}\mathcal{H}(z)dz \preccurlyeq \frac{\mathcal{H}(u)\,\tilde{+}\,\mathcal{H}(\nu)}{2}.$$

If $\xi(\nu,u) = \nu - u$, then Theorem 4.7 reduces to the result for strongly convex-FM:

$$\mathcal{H}\left(\frac{u+\nu}{2}\right) \tilde{+} \frac{\omega}{12}\|\nu-u\|^2 \preccurlyeq \frac{1}{\nu-u}\,(FR)\int_u^{\nu}\mathcal{H}(z)dz \preccurlyeq \frac{\mathcal{H}(u)\,\tilde{+}\,\mathcal{H}(\nu)}{2} \simeq \frac{\omega}{6}\|\nu-u\|^2.$$

If $\xi(\nu,u) = \nu - u$ and $\omega = 0$, then Theorem 4.7 reduces to the result for convex-FM in [55]

$$\mathcal{H}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{\nu-u}\,(FR)\int_u^{\nu}\mathcal{H}(z)dz \preccurlyeq \frac{\mathcal{H}(u)\,\tilde{+}\,\mathcal{H}(\nu)}{2}. \quad (44)$$

If $\mathcal{H}_*(u,\gamma) = \mathcal{H}^*(\nu,\gamma)$ with $\omega = 0$ and $\gamma = 1$ then Theorem 4.7 reduces to the result for preinvex function, see [36]:

$$\mathcal{H}\left(\frac{2u+\xi(\nu,u)}{2}\right) \leq \frac{1}{\xi(\nu,u)}\,(R)\int_u^{u+\xi(\nu,u)}\mathcal{H}(z)dz \leq \frac{\mathcal{H}(u)+\mathcal{H}(\nu)}{2}. \quad (45)$$

If $\mathcal{H}_*(u,\gamma) = \mathcal{H}^*(\nu,\gamma)$ with $\xi(\nu,u) = \nu - u$, $\omega = 0$ and $\gamma = 1$ then Theorem 4.7 reduces to the result for convex function, see [42, 43]:

$$\mathcal{H}\left(\frac{u+\nu}{2}\right) \leq \frac{1}{\nu-u}\,(R)\int_u^{\nu}\mathcal{H}(z)dz \leq \frac{\mathcal{H}(u)+\mathcal{H}(\nu)}{2}. \quad (46)$$

Example 4.9. We consider the fuzzy-IVF $\mathcal{H}: [u, u+\xi(\nu,u)] = [0, \xi(2,0)] \to \mathbb{F}_0$ defined by,

$$\mathcal{H}(z)(\sigma) = \begin{cases} \frac{\sigma}{2z^2}, & \sigma \in [0, 2z^2], \\ \frac{4z^2-\sigma}{2z^2}, & \sigma \in (2z^2, 4z^2], \\ 0, & otherwise, \end{cases}$$

Then, for each $\gamma \in [0,1]$, we have $\mathcal{H}_\gamma(z) = [2\gamma z^2, (4-2\gamma)z^2]$. Since for each $\gamma \in [0,1]$, $\mathcal{H}_*(z,\gamma) = 2\gamma z^2$, $\mathcal{H}^*(z,\gamma) = (4-2\gamma)z^2$ are preinvex functions w.r.t. $\xi(\nu,u) = \nu - u$ and $\omega = \frac{2}{3}\gamma$. Hence $\mathcal{H}(z)$ is preinvex fuzzy-IVF w.r.t. $\xi(\nu,u) = \nu - u$. We now compute the following:

$$\mathcal{H}_*\left(\frac{2u+\xi(\nu,u)}{2}, \gamma\right) + \frac{\omega}{12}\|\xi(\nu,u)\|^2 = \mathcal{H}_*(1,\gamma) = \frac{8\gamma}{3},$$

$$\frac{1}{\xi(\nu,u)} \int_u^{u+\xi(\nu,u)} \mathcal{H}_*(z,\gamma)dz = \frac{1}{2}\int_0^2 2\gamma z^2 dz = \frac{8\gamma}{3},$$

$$\frac{\mathcal{H}_*(u,\gamma) + \mathcal{H}_*(\nu,\gamma)}{2} - \frac{\omega}{6}\|\xi(\nu,u)\|^2 = \frac{32\gamma}{9},$$

for all $\gamma \in [0,1]$. That means

$$\frac{8\gamma}{3} \le \frac{8\gamma}{3} \le \frac{32\gamma}{9}.$$

Similarly, it can be easily show that

$$\mathcal{H}^*\left(\frac{2u+\xi(\nu,u)}{2}, \gamma\right) \le \frac{1}{\xi(\nu,u)} \int_u^{u+\xi(\nu,u)} \mathcal{H}^*(z,\gamma)dz \le \frac{\mathcal{H}^*(u,\gamma) + \mathcal{H}^*(\nu,\gamma)}{2}.$$

for all $\gamma \in [0,1]$, such that

$$\mathcal{H}^*\left(\frac{2u+\xi(\nu,u)}{2}, \gamma\right) + \frac{\omega}{12}\|\xi(\nu,u)\|^2 = \mathcal{H}_*(1,\gamma) = \frac{36-16\gamma}{9},$$

$$\frac{1}{\xi(\nu,u)} \int_u^{u+\xi(\nu,u)} \mathcal{H}^*(z,\gamma)dz = \frac{1}{2}\int_0^2 (4-2\gamma)z^2 dz = \frac{8(2-\gamma)}{3},$$

$$\frac{\mathcal{H}^*(u,\gamma) + \mathcal{H}^*(\nu,\gamma)}{2} - \frac{\omega}{6}\|\xi(\nu,u)\|^2 = \frac{72-22\gamma}{9}.$$

From which, it follows that

$$\frac{36-16\gamma}{9} \le \frac{8(2-\gamma)}{3} \le \frac{72-22\gamma}{9},$$

that is

$$\left[\frac{8\gamma}{3}, \frac{36-16\gamma}{9}\right] \le_I \left[\frac{8\gamma}{3}, \frac{8(2-\gamma)}{3}\right] \le_I \left[\frac{32\gamma}{9}, \frac{72-22\gamma}{9}\right], \text{ for all } \gamma \in [0,1],$$

hence, the Theorem 4.7 has been verified.

5. Conclusions

In this study, we have introduced and studied a new class of preinvex-FMs is called strongly preinvex-FMs. Using Condition C, we have obtained equivalence relation between strongly preinvex and strongly invex-FMs. To characterize the optimality condition of the sum preinvex-FMs and strongly preinvex-FMs, we have introduced strong fuzzy mixed variational-like inequality. Moreover, we have established strong relationship between strongly preinvex-FM and Hermite-Hadamard inequality. There is much room for further study to explore this concept in fuzzy convex and non-convex theory like, the existence of unique solution of strong fuzzy mixed variational like-inequalities can be conducted and some iterative algorithms can also obtained under some mild conditions. From last two sections, we can conclude that these classes of FMs will play important and significant role in fuzzy optimization and their related areas.

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Authors' Contributions

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