Analysis Sequential Fractional Differences and Related Monotonicity Results

Pshtiwan Othman Mohammed[1,2], Carlos Lizama[3], Eman Al-Sarairah[4,5], Juan L.G. Guirao[6], Nejmeddine Chorfi[7], and Miguel Vivas-Cortez*[8]

[1]Department of Mathematics, College of Education, University of Sulaimani, Sulaymaniyah 46001, Iraq
E-Mail: pshtiwansangawi@gmail.com
[2]Research and Development Center, University of Sulaimani, Sulaymaniyah 46001, Iraq
[3]Departamento de Matemática y Ciencia de la Computación, Facultad de Ciencias, Universidad de Santiago de Chile, Las Sophoras 173, Estación Central, Santiago, Chile
E-Mail: carlos.lizama@usach.cl
[4]Department of Mathematics, Khalifa University, P.O.Box 127788, Abu Dhabi, United Arab Emirates
E-Mail: eman.alsarairah@ku.ac.ae
[5]Department of Mathematics, Al-Hussein Bin Talal University, P.O. Box 20, Ma'an 71111, Jordan
[6]Department of Applied Mathematics and Statistics, Technical University of Cartagena, Hospital de Marina 30203-Cartagena, Spain
E-Mail: juan.garcia@upct.es
[7]Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
E-Mail: nchorfi@ksu.edu.sa
[8]Faculty of Exact and Natural Sciences, School of Physical Sciences and Mathematics, Pontifical Catholic University of Ecuador, Av. 12 de octubre 1076 y Roca, Apartado Postal 17-01-2184, Sede Quito, Ecuador
E-Mail: mjvivas@puce.edu.ec

Abstract

This article employs the monotonicity analysis for nonnegativity to derive a class of sequential fractional backward differences of Riemann-Liouville type $\left({}^{\mathrm{RL}}_{a+1}\nabla^{\nu}\ {}^{\mathrm{RL}}_{a}\nabla^{\alpha}u\right)(t)$ based on a certain subspace in the parameter space $(0,1)\times(0,1)$. Auxiliary and restriction conditions are included in the monotonicity results obtained in this paper and they confirm the monotonicity of the function on $\{a+2, a+3, \ldots\}$. A non-monotonicity result is also established based on the main conditions together with further dual conditions, and this confirms that the main theorem is almost sharp. Furthermore, we recast the dual conditions in a sing condition, and then we represent the sharpness result in a new corollary. Finally, numerical results via MATLAB software are used to illustrate the main mathematical results for some special cases.

1 Introduction

Discrete fractional calculus theory has been considered as one of the greatest discovery on applied science with mathematical analysis and fractional calculus, attracting ever increasing attention of most of scientists and researchers from diverse disciplines, see e.g. [1–4]. Particularly in many fields there is an increasing number of publications which are interested on fractional sums and differences contributing to the development

Keywords: Sequential type operators; Riemann-Liouville difference operators; monotonicity results
MSC (2020) : 39A12; 39B62; 33B10; 26A48; 26A51
*Corresponding author. E-Mail: mjvivas@puce.edu.ec

of discrete fractional operators to deal with such fields, and thereafter, numerous mechanisms facilitating cooperative behavior, like mathematical modelling [5–7], stability analyses [8–10], uncertainty theory [11–13], mathematical transformations [14, 15], medical models [16, 17], other mathematical models [18–20] and so on, have been proposed by many researchers.

Analysis of discrete fractional operators for positivity or/and monotonicity is an established and active field of research in the applied mathematical analysis and fractional calculus. Recently, several monotonicity analysis or positivity analysis results have been developed based on different fractional difference and sum operators defined using singular or/and nonsingular kernels, see e.g. [21–23] including standard, exponential and Mittag-Leffler function in kernels. In particular, in the mathematical analysis and numerical computation for both discrete Riemann-Liuoville and Caputo fractional models, a significant challenge is due to non-singularity of their kernels, see e.g. [24–27].

On the other hands, the research work on monotonicity analysis has continued to deepen with the advance in sequential fractional operators or mixed order fractional operators. Depending on the difference and sum properties and auxiliary conditions, the modeling of this process is quite challenging and interesting in the scientific community. Besides, various investigations on fundamental problems of sequential fractional operator of Riemann-Liuoville type have lately been published, see e.g. [28–33]. However, some of these results are sharp in the sense of the main monotonicity results. Moreover, such sharpness models have been considered in the sense of Riemann-Liuoville operators by e.g. [34–37].

Motivated by the above considerations, the aim of this paper is to present monotonicity and non-monotonicity analysis for the following class of sequential fractional backward differences of Riemann-Liouville type

$$\left({}_{a+1}^{\mathrm{RL}}\nabla^{\nu}\ {}_{a}^{\mathrm{RL}}\nabla^{\alpha}u\right)(t), \tag{1.1}$$

on $\mathbb{N}_{a+2} := \{a+2, a+3, \ldots\}$. The sharpness of the main result is another aim of our study. Finally, some example experiments confirm the applicability of the main theorem.

The designation of our study sections is as follows: Section 2 recalls definition of nabla Riemann-Liouville fractional difference operator and an essential lemma concerning the backward difference formula. Next, Section 3 contains our main results including analysis of (1.1), monotonicity, non-monotonicity analysis results and sharpness of the function on a specific region with the auxiliary and dual conditions. Section 4 includes two example illustrations. In Section5, we summarize our proposed sequential with some concluding remarks.

2 Necessary results

Following Theorem 2.1 in [34], we recall the following theorem.

Theorem 2.1. *For u defined on $\mathbb{N}_a$, we have*

$$\left({}_{a+1}^{\mathrm{RL}}\nabla^{\nu}\ {}_{a}^{\mathrm{RL}}\nabla^{\alpha}u\right)(t) = \frac{1}{\Gamma(-\nu-\alpha)}\sum_{\mathsf{s}=a+1}^{t}\frac{\Gamma(t-\nu-\alpha-\mathsf{s})}{\Gamma(t+1-\mathsf{s})}u(\mathsf{s}) - \frac{\Gamma(t-a-\nu-1)}{\Gamma(-\nu)\Gamma(t-a)}u(a+1), \tag{2.1}$$

for $t \in \mathbb{N}_{a+3}$, $1 < \nu \leq 2$ and $0 < \alpha < 1$. Where ${}_{a}^{\mathrm{RL}}\nabla^{\alpha}u$ is given by (see [22, Lemma 2.1]):

$$\left({}_{a}^{\mathrm{RL}}\nabla^{\alpha}u\right)(t) = \frac{1}{\Gamma(-\alpha)}\sum_{\mathsf{s}=a+1}^{t}(t+1-\mathsf{s})^{\overline{-\alpha-1}}u(\mathsf{s}), \tag{2.2}$$

for $t \in \mathbb{N}_{a+T}$, where $\alpha \in (T-1, T)$ with $T \in \mathbb{N}_1$.

Remark 2.1. *It is essential in the context of discrete fractional calculus to note that $t^{\overline{\alpha}}$ tends to zero such that $\Gamma(t)$ becomes undefined, where*

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}. \tag{2.3}$$

Theorem 2.1 helps us to calculate the following lemma.

Lemma 2.1. *Let* $\left({}_{a+1}^{\mathrm{RL}}\nabla^{\nu}\ {}_{a}^{\mathrm{RL}}\nabla^{\alpha}u\right)(t) \geq 0$, *for all* $t \in \mathbb{N}_{a+3}$ *and each* u *to be defined on* $\mathbb{N}_a$. *Then the following inequality can hold*

$$\begin{aligned}(\nabla u)(a+n) \geq &\left[\frac{\Gamma(n-a-\nu-1)}{(n-1)!\Gamma(-\nu)} - \frac{\Gamma(n-\nu-\alpha)}{(n-1)!\Gamma(1-\nu-\alpha)}\right] u(a+1)\\ &- \frac{1}{\Gamma(1-\nu-\alpha)} \sum_{\mathsf{p}=0}^{n-3} \frac{\Gamma(n-\mathsf{p}-1-\nu-\alpha)}{\Gamma(n-\mathsf{p}-1)} (\nabla u)(a+\mathsf{p}+2),\end{aligned} \tag{2.4}$$

for $n \in \mathbb{N}_2$, $0 < \nu \leq 1$ *and* $0 < \alpha < 1$.

Proof. Recasting (2.1) in the following form

$$\begin{aligned}\left({}_{a+1}^{\mathrm{RL}}\nabla^{\nu}\ {}_{a}^{\mathrm{RL}}\nabla^{\alpha}u\right)(t) = &\left[\frac{\Gamma(t-a-\nu-\alpha)}{\Gamma(t-a)\Gamma(1-\nu-\alpha)} - \frac{\Gamma(t-a-\nu-1)}{\Gamma(t-a)\Gamma(-\nu)}\right] u(a+1)\\ &+ \big(\nabla u\big)(t) + \frac{1}{\Gamma(1-\nu-\alpha)} \sum_{\mathsf{s}=a+2}^{t-1} \frac{\Gamma(t-\mathsf{s}+1-\nu-\alpha)}{\Gamma(t-\mathsf{s}+1)} \big(\nabla u\big)(\mathsf{s}).\end{aligned}$$

By rearranging this equality for $\big(\nabla u\big)(t)$ and using the assumption, we see that

$$\begin{aligned}\big(\nabla u\big)(t) = \left({}_{a+1}^{\mathrm{RL}}\nabla^{\nu}\ {}_{a}^{\mathrm{RL}}\nabla^{\alpha}u\right)(t) &- \left[\frac{\Gamma(t-a-\nu-\alpha)}{\Gamma(t-a)\Gamma(1-\nu-\alpha)} - \frac{\Gamma(t-a-\nu-1)}{\Gamma(t-a)\Gamma(-\nu)}\right] u(a+1)\\ &- \frac{1}{\Gamma(1-\nu-\alpha)} \sum_{\mathsf{s}=a+2}^{t-1} \frac{\Gamma(t-\mathsf{s}+1-\nu-\alpha)}{\Gamma(t-\mathsf{s}+1)} \big(\nabla u\big)(\mathsf{s}).\end{aligned}$$

Then, by changing of the variable t to $a+n$, where $n \in \mathbb{N}_3$, we get the required result. □

3 Main Nabla results

To show the sharpness of u in Lemma 2.1 we must prove the monotonicity of u in the following theorem.

Theorem 3.1. *If* $u : \mathbb{N}_a \to \mathbb{R}$ *has these four properties*

(i) $u(a+2) \geq u(a+1) \geq 0$;

(ii) $\left({}_{a+1}^{\mathrm{RL}}\nabla^{\nu}\ {}_{a}^{\mathrm{RL}}\nabla^{\alpha}u\right)(t) \geq 0$, *for* $t \in \mathbb{N}_{a+3}$, *where* $0 < \nu, \alpha < 1$ *with* $1 < \nu + \alpha < 2$;

(iii) $\nu \geq 1 - \frac{\alpha}{2}$, *where* $0 < \nu, \alpha < 1$ *with* $1 < \nu + \alpha < 2$,

then $\big(\nabla u\big)(t) \geq 0$, *for all* $t \in \mathbb{N}_{a+2}$.

Proof. Let's recast inequality (2.4), where $n = 3$, to see that

$$\begin{aligned}&- \frac{1}{\Gamma(1-\nu-\alpha)} \sum_{\mathsf{p}=0}^{0} \frac{\Gamma(2-\mathsf{p}-\nu-\alpha)}{\Gamma(2-\mathsf{p})} (\nabla u)(a+\mathsf{p}+2)\\ &= -\frac{\Gamma(2-\nu-\alpha)}{\Gamma(1-\nu-\alpha)} (\nabla u)(a+2)\\ &= -\underbrace{(1-\nu-\alpha)}_{<0} \underbrace{(\nabla u)(a+2)}_{\geq 0 \text{ by (ii)}} \geq 0.\end{aligned}$$

By the same process, we can have

$$-\frac{1}{\Gamma(1-\nu-\alpha)} \sum_{\mathsf{p}=0}^{n-3} \frac{\Gamma(n-\mathsf{p}-1-\nu-\alpha)}{\Gamma(n-\mathsf{p}-1)} (\nabla u)(a+\mathsf{p}+2) \geq 0, \tag{3.1}$$

for each $n \in \mathbb{N}_3$. Next, our goal is by using induction to prove that

$$\frac{\Gamma(n-\nu-1)}{\Gamma(-\nu)(n-1)!} - \frac{\Gamma(n-\nu-\alpha)}{\Gamma(1-\nu-\alpha)(n-1)!} \geq 0. \tag{3.2}$$

Firstly, if $n = 3$ in (3.2), then our proof is holding the following inequality

$$\frac{\Gamma(2-\nu)}{\Gamma(-\nu)2!} \geq \frac{\Gamma(3-\alpha-\nu)}{\Gamma(1-\alpha-\nu)2!}, \tag{3.3}$$

or equivalent to (3.3), we need to show that

$$(-\nu)(1-\nu) \geq (1-\alpha-\nu)(2-\alpha-\nu). \tag{3.4}$$

If we expand (3.4), then we see that our proof is holding the following inequality

$$3\alpha + 2\nu - \alpha^2 - 2\alpha\nu - 2 \geq 0. \tag{3.5}$$

For $\alpha = 1$, (3.5) leads to

$$2\nu - 2\nu \geq 0,$$

and it is clearly true. Next, for fixed $\alpha \in (0,1)$, we can solve (3.5) for ν to have

$$\nu \geq \frac{\alpha^2 - 3\alpha + 2}{2(1-\alpha)} = 1 - \frac{\alpha}{2}, \tag{3.6}$$

which is true for sure by considering the condition (iii), and this is the basic step of the mathematical induction.

In the inductive step, we suppose that (3.4) is correct for $n = n_0$ and we claim that

$$\frac{\Gamma(3-\nu+n_0)}{\Gamma(-\nu)} \geq \frac{\Gamma(4-\alpha-\nu+n_0)}{\Gamma(-\alpha-\nu+1)}, \quad \text{for some integer } n_0 \geq 3. \tag{3.7}$$

Look! (3.7) can be rewrite as follows

$$(n_0-1-\nu)(n_0-2-\nu)\cdots(-\nu) \geq (n_0-\alpha-\nu)(n_0-1-\alpha-\nu)\cdots(1-\alpha-\nu). \tag{3.8}$$

Our induction hypothesis (when $n = n_0$) tells us that the following inequality holds

$$(n_0-2-\nu)\cdots(-\nu) \geq (n_0-1-\alpha-\nu)\cdots(1-\alpha-\nu). \tag{3.9}$$

It is important to observe that

$$\min\left\{n_0-1-\nu, n_0-\alpha-\nu\right\} \geq 0,$$

and

$$\max\left\{\overbrace{(n_0-2-\nu)}^{>0}\cdots\overbrace{(-\nu)}^{<0}, \overbrace{(n_0-1-\alpha-\nu)}^{>0}\cdots\overbrace{(1-\alpha-\nu)}^{<0}\right\} < 0.$$

Now, we put

$$\begin{aligned}
C_0 &:= (n_0-2-\nu)\cdots(-\nu) < 0,\\
D_0 &:= (n_0-1-\alpha-\nu)\cdots(1-\alpha-\nu) < 0,\\
t_M &:= n_0-\alpha-\nu > 0,\\
t_m &:= n_0-1-\nu > 0.
\end{aligned}$$

One can observe that $D_0 < C_0 < 0$ and $0 < t_m \leq t_M$ since $\alpha \leq 1$ from the assumption. Therefore,

$$t_m C_0 \geq t_M D_0$$

which tell us that (3.8) is true in view of inequality (3.9) and hence (3.7) will be true. As a result, by induction (3.2) holds true for all $n \in \mathbb{N}_3$.

In the final step, we use (2.4) with $n = 3$ to get

$$\begin{aligned}(\nabla u)(a+3) &\geq \left[\frac{\Gamma(2-a-\nu)}{2!\Gamma(-\nu)} - \frac{\Gamma(3-\nu-\alpha)}{2!\Gamma(1-\nu-\alpha)}\right] u(a+1) \\ &\quad - \frac{1}{\Gamma(1-\nu-\alpha)} \sum_{\mathsf{p}=0}^{0} \frac{\Gamma(2-\mathsf{p}-\nu-\alpha)}{\Gamma(2-\mathsf{p})} (\nabla u)(a+\mathsf{p}+2) \\ &\geq 0,\end{aligned}$$

where all of the conditions (i), (ii) and (iii) have been used. As a result,

$$\big(\nabla u\big)(t) \geq 0, \quad \text{for each } t \in \mathbb{N}_{a+2},$$

by iterating (2.4) inductively and using the initial result that $\big(\nabla u\big)(a+3) \geq 0$. This finishes our result. □

Remark 3.1. *By an assumption of Theorem 3.1, we are in need of*

$$2 > \alpha + \nu > 1. \tag{3.10}$$

It is important to discuss the restriction

$$\nu \geq 1 - \frac{\alpha}{2}, \tag{3.11}$$

which appears in the assertion of Theorem 3.1. *First, we see that*

$$\lim_{\alpha \to 0^+} \left(1 - \frac{\alpha}{2}\right) = 1.$$

This makes (3.11) tend to $\nu > 1$ as $\alpha \to 0^+$. However, this is really no great loss because we need (3.10). Therefore, we can deduce that the range of ν values is only weakly affected, for $\alpha \approx 0$, while slightly truncated.

Next, we consider the case that

$$\lim_{\alpha \to 1^-} \left(1 - \frac{\alpha}{2}\right) = \frac{1}{2}.$$

This makes (3.11) tend to $\nu > \frac{1}{2}$ as $\alpha \to 1^-$. Thus, an importantly less trivial restriction is imposed in this case. Particularly, considering (3.10), if $\alpha \approx 1$, then $\nu \gtrapprox 0$. Otherwise, ν could importantly be any number in $(0,1)$ without the auxiliary restriction. However, the range $0 < \nu < \frac{1}{2}$ is disallowed with the auxiliary restriction. Therefore, a much more substantial restriction is effected.

To begin the almost sharpness of Theorem 3.1, we need the following preliminary lemma.

Lemma 3.1. *Let $1 < \alpha + \nu < 2$, and conditions* (i) *and* (ii) *of* Theorem 3.1 *be hold and the condition* (iii) *be hold for $t = a + 3$. In addition, let η_0 be a number such that*

$$\eta_0 > \frac{\alpha^2}{2}.$$

Then there is a function u defined on $\mathbb{N}_{a+1}$ that is not a monotone increasing function on $\mathbb{N}_{a+2}$, for all number ν satisfying

$$0 < \nu < 1 - \left(\eta_0 + \frac{\alpha}{2}\right).$$

Proof. We need to find a function $u : \mathbb{N}_{a+1} \to \mathbb{R}$ combined with $t_0 \in \mathbb{N}_{a+1}$ with $(\nabla u)(t_0) < 0$. Without loss of generality, we only establish the result in case $a = 0$. Now, assume that $\nu \in \left(0, 1 - \left(\eta_0 + \frac{\alpha}{2}\right)\right)$ is fixed. Assume that $\varepsilon > 0$ is a constant (sufficiently small), which will be set on later. Then we define the function u on $\mathbb{N}_1^3$ by

$$\begin{aligned} u(1) &= 1, \\ u(2) &= \varepsilon + 1, \\ u(3) &= 1. \end{aligned}$$

It is clear that u is not a monotone increasing function on $\mathbb{N}_1$ since $(\nabla u)(3) = -\varepsilon < 0$. Thus, it is enough to prove that conditions (i)–(iii) nonetheless hold.

It is obvious that conditions (i) and (ii) hold since $u(1) \geq 0$ and $(\nabla u)(2) = \varepsilon > 0$, respectively. So, it only remains to prove that

$$\left({}^{\mathrm{RL}}_{1}\nabla^{\nu}\ {}^{\mathrm{RL}}_{0}\nabla^{\alpha} u\right)(a+3) \geq 0. \tag{3.12}$$

To prove (3.12), we observe that

$$\begin{aligned} \left({}^{\mathrm{RL}}_{1}\nabla^{\nu}\ {}^{\mathrm{RL}}_{0}\nabla^{\alpha} u\right)(a+3) &= \frac{1}{\Gamma(-\alpha-\nu)} \sum_{\mathsf{s}=1}^{3} \frac{\Gamma(t-\alpha-\nu-\mathsf{s})}{\Gamma(t-\mathsf{s}+1)} u(\mathsf{s}) - \frac{\Gamma(2-\nu)}{2\Gamma(-\nu)} u(1) \\ &= \left[\frac{(-\alpha-\nu)(-\alpha-\nu+1)}{2} + \frac{\nu(1-\nu)}{2}\right] u(1) + (-\alpha-\nu)u(2) + u(3) \\ &= \alpha\left(-\frac{1}{2} + \frac{1}{2}\alpha + \nu\right) u(1) + (-\alpha-\nu)u(2) + u(3) \\ &= \left[\alpha\left(-\frac{1}{2} + \frac{1}{2}\alpha + \nu\right) + (-\alpha-\nu) + 1\right] + (-\alpha-\nu)\varepsilon, \end{aligned} \tag{3.13}$$

where the values of $u(1)$, $u(2)$, and $u(3)$ are used. By making use of the supposition that $\nu < 1 - \left(\eta_0 + \dfrac{\alpha}{2}\right)$ we arrive at

$$-\alpha - \nu > -\alpha - 1 + \left(\eta_0 + \frac{\alpha}{2}\right) = \eta_0 - \frac{\alpha}{2} - 1. \tag{3.14}$$

On the other hand, from our requirement $\alpha + \nu > 1$, we have

$$\nu > 1 - \alpha. \tag{3.15}$$

Therefore, by simplifying the right-hand side of (3.13) and by using inequalities (3.14)–(3.15) in the resulting inequality we find that

$$\left({}^{\mathrm{RL}}_{1}\nabla^{\nu}\ {}^{\mathrm{RL}}_{0}\nabla^{\alpha} u\right)(a+3) = \left[\alpha\left(-\frac{1}{2} + \frac{1}{2}\alpha + \nu\right) + (-\alpha-\nu) + 1\right] + (-\alpha-\nu)\varepsilon \tag{3.16}$$

$$> \left[\frac{\alpha(1-\alpha)}{2} + \eta_0 - \frac{\alpha}{2}\right] + \left(\eta_0 - \frac{\alpha}{2} - 1\right)\varepsilon \tag{3.17}$$

$$= \underbrace{\left[\eta_0 - \frac{\alpha^2}{2}\right]}_{>0 \text{ by assumption}} + \left(\eta_0 - \frac{\alpha}{2} - 1\right)\varepsilon. \tag{3.18}$$

By taking limits on both sides of (3.16), it follows that

$$\begin{aligned} \lim_{\varepsilon\to 0^+}\left[\left({}^{\mathrm{RL}}_{1}\nabla^{\nu}\ {}^{\mathrm{RL}}_{0}\nabla^{\alpha} u\right)(a+3)\right] &> \lim_{\varepsilon\to 0^+}\left\{\left[\eta_0 - \frac{\alpha^2}{2}\right] + \left(-\frac{1}{2}\alpha - 1 + \eta_0\right)\varepsilon\right\} \\ &= \eta_0 - \frac{\alpha^2}{2} > 0, \end{aligned}$$

which implies that $\left({}^{\mathrm{RL}}_{1}\nabla^{\nu}\ {}^{\mathrm{RL}}_{0}\nabla^{\alpha} u\right)(a+3) > 0$ for $\varepsilon > 0$ small enough. Consequently, the condition (iii) holds. Hence, the proof is done. $\square$

Remark 3.2.

1. *In the statement of* Lemma 3.1, *we can recast the dual conditions*

$$\eta_0 > \frac{\alpha^2}{2},$$
$$\nu < 1 - \left(\eta_0 + \frac{\alpha}{2}\right),$$

as the following single condition

$$0 < \nu < 1 - \left(\eta_0 + \frac{\alpha}{2}\right) < 1 - \frac{\alpha}{2} - \frac{\alpha^2}{2}.$$

Therefore, we conclude that the result of Lemma 3.1 *can be hold such that we press the restriction* $\nu < 1 - \frac{\alpha}{2} - \frac{\alpha^2}{2}$ *on the space of* $(\alpha, \nu)-$*parameter.*

2. *It is worth mentioning that the plot of the function* $\alpha \mapsto 1 - \frac{\alpha}{2}$ *lies above the plot of the function* $\alpha \mapsto 1 - \frac{\alpha}{2} - \frac{\alpha^2}{2}$ *– that is,*

$$1 - \frac{\alpha}{2} - \frac{\alpha^2}{2} < 1 - \frac{\alpha}{2},$$

which is true for $0 \leq \alpha \leq 1$.

In view of Lemma 3.1 together with Remark 3.2(1), we can deduce Corollary 3.1 directly.

Corollary 3.1. *Let* $1 < \alpha + \nu < 2$, *and conditions* (i) *and* (ii) *of* Theorem 3.1 *be hold and the condition* (iii) *be hold for* $t = a + 3$. *Then for all numbers* $\alpha, \nu \in (0, 1)$ *satisfying*

$$\nu < 1 - \frac{\alpha}{2} - \frac{\alpha^2}{2}, \tag{3.19}$$

there is a function u *defined on* $\mathbb{N}_{a+1}$ *and it is not a monotone increasing function on* $\mathbb{N}_{a+1}$.

Remark 3.3. *In a particular sense as in* Corollary 3.1, *we can observe that* Theorem 3.1 *is "almost sharp" on the subset of* $(0, 1) \times (0, 1)$ *for which the inequality* (3.19) *holds.*

4 Clarification examples

We consider two examples to confirm the validity of our main Theorem 3.1.

Example 4.1. Let us consider u defined on $\mathbb{N}_0$ as follows:

$$u(t) = t^{\overline{\nu+\alpha}}.$$

Then for $\nu = 0.7$, $\alpha = 0.8$, and $t = 3$, we have

$$\begin{aligned}\left({}_{1}^{\mathrm{RL}}\nabla^{\nu}\ {}_{0}^{\mathrm{RL}}\nabla^{\alpha}u\right)(3) &= \frac{1}{\Gamma(-\nu-\alpha)}\sum_{\mathsf{s}=1}^{3}(4-\mathsf{s})^{\overline{-\nu-\alpha-1}}u(\mathsf{s}) - \frac{(3)^{\overline{-\nu-1}}}{\Gamma(-\nu)}u(1)\\ &= 1.4689 \geq 0,\end{aligned}$$

and for $t = 4$, we have

$$\begin{aligned}\left({}_{1}^{\mathrm{RL}}\nabla^{\nu}\ {}_{0}^{\mathrm{RL}}\nabla^{\alpha}u\right)(4) &= \frac{1}{\Gamma(-\nu-\alpha)}\sum_{\mathsf{s}=1}^{4}(5-\mathsf{s})^{\overline{-\nu-\alpha-1}}u(\mathsf{s}) - \frac{(4)^{\overline{-\nu-1}}}{\Gamma(-\nu)}u(1)\\ &= 1.3898 \geq 0.\end{aligned}$$

Similarly, we can deduce that

$$\left({}^{\mathrm{RL}}_{\ 1}\nabla^{\nu}\ {}^{\mathrm{RL}}_{\ 0}\nabla^{\alpha}u\right)(t) \geq 0,$$

for each $\mathbb{N}_{a+2}$. Furthermore, $u(1) \geq 0$ and $(\nabla u)(2) = 1.9940 \geq 0$. Consequently, we see that $t^{\overline{\nu+\alpha}}$ is increasing on $\mathbb{N}_2$ in view of Theorem 3.1. On the other hand, it has been drown in Figure 1.

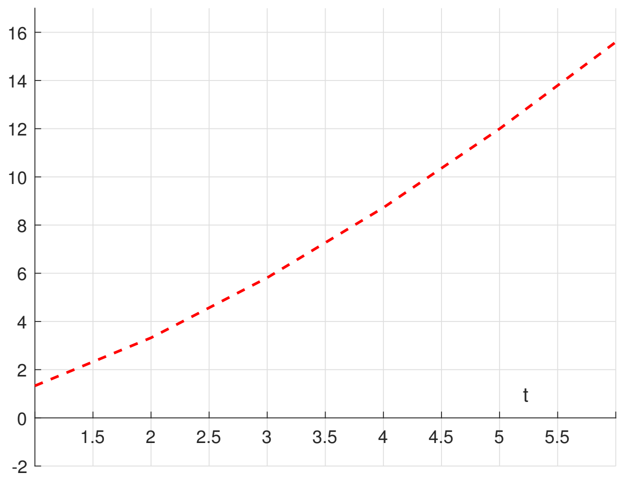

Figure 1: Graph of $u(t)$ in Example 4.1.

Example 4.2. For this reason, we consider the following data

t	1	2	3	4	5
$u(t)$	0	1	1.5	2.2	3.5

Let $\nu = 0.7$, $\alpha = 0.8$. Then, for $t = 3$, $t = 4$ and $t = 4$, we have

$$\begin{aligned}\left({}^{\mathrm{RL}}_{\ 1}\nabla^{\nu}\ {}^{\mathrm{RL}}_{\ 0}\nabla^{\alpha}u\right)(3) &= \frac{1}{\Gamma(-\nu-\alpha)}\sum_{\mathsf{s}=1}^{3}(4-\mathsf{s})^{\overline{-\nu-\alpha-1}}u(\mathsf{s}) - \frac{(3)^{\overline{-\nu-1}}}{\Gamma(-\nu)}u(1)\\ &= 0 \geq 0,\end{aligned}$$

$$\begin{aligned}\left({}^{\mathrm{RL}}_{\ 1}\nabla^{\nu}\ {}^{\mathrm{RL}}_{\ 0}\nabla^{\alpha}u\right)(4) &= \frac{1}{\Gamma(-\nu-\alpha)}\sum_{\mathsf{s}=1}^{4}(5-\mathsf{s})^{\overline{-\nu-\alpha-1}}u(\mathsf{s}) - \frac{(4)^{\overline{-\nu-1}}}{\Gamma(-\nu)}u(1)\\ &= 0.3250 \geq 0,\end{aligned}$$

and

$$\begin{aligned}\left({}^{\mathrm{RL}}_{\ 1}\nabla^{\nu}\ {}^{\mathrm{RL}}_{\ 0}\nabla^{\alpha}u\right)(5) &= \frac{1}{\Gamma(-\nu-\alpha)}\sum_{\mathsf{s}=1}^{5}(6-\mathsf{s})^{\overline{-\nu-\alpha-1}}u(\mathsf{s}) - \frac{(5)^{\overline{-\nu-1}}}{\Gamma(-\nu)}u(1)\\ &= 0.8250 \geq 0,\end{aligned}$$

respectively. Furthermore, $u(1) = 0 \geq 0$ and $\big(\nabla u\big)(1) = 1 \geq 0$. Hence u is a convex function on $\{2, 3, 4, 5\}$ according to Theorem 3.1.

5 Concluding remarks

In this contribution, we have studied the monotonicity result of discrete fractional operators determined by a nonnegativity of a class of sequential fractional backward differences of Riemann-Liouville type

$$\left({}_{a+1}^{\mathrm{RL}}\nabla^{\nu}\,{}_{a}^{\mathrm{RL}}\nabla^{\alpha}u\right)(t).$$

Then the main conclusion of our study is as follows:

- A backward difference formula associated with the nonnegativity of the above sequential fractional difference has been formulated in Lemma 2.1.
- The monotonicity of u on the subregion $(0,1)\times(0,1)$ such that $0<\nu+\alpha<2$ with the auxiliary restriction $\nu\geq 1-\frac{\alpha}{2}$ has been established in Theorem 3.1.
- A non-monotonicity result has been obtained in Lemma 3.1 based on dual conditions.
- The dual conditions has been recaptured in a new single condition, and thus the sharpness of the function has been observed in Corollary 3.1.
- Finally, to show the applicability and generality of the main theorem, we have applied the theoretical findings to two test examples, considering and impulse strengths.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Funding

Not applicable.

Author contributions

Pshtiwan Othman Mohammed: Formal analysis, Writing – review & editing. **Carlos Lizama:** Software, Supervision. **Eman Al-Sarairah:** Methodology, Writing – review & editing. **Juan L.G. Guirao:** Conceptualization, Writing – original draft. **Nejmeddine Chorfi:** Visualization, Investigation. **Miguel Vivas-Cortez:** Writing – review & editing. All authors reviewed the results and approved the final version of the manuscript.

Acknowledgements

Researchers Supporting Project number (RSP2024R153), King Saud University, Riyadh, Saudi Arabia.

Declarations

Ethical approval Not applicable.
Competing interests The authors declare no competing interests.

ORCID

Pshtiwan Othman Mohammed	0000-0001-6837-8075
Carlos Lizama	0000-0002-9277-8092
Eman Al-Sarairah	0000-0002-0223-4711
Juan L.G. Guirao	0000-0003-2788-809X
Nejmeddine Chorfi	0000-0001-8833-6585
Miguel Vivas-Cortez	0000-0002-1567-0264

References

[1] Mohammed, P.O.; Srivastava, H.M.; Baleanu, D.; Al-Sarairah, E.; Sahoo, S.K.; Chorfi, N. Monotonicity and positivity analyses for two discrete fractional-order operator types with exponential and Mittag-Leffler kernels. *J. King Saud Univ. Sci.* **2023**, *35*, 102794.

[2] Atıcı, F.M.; Chang, S.; Jonnalagadda, J.M. Mittag-Leffler Functions in Discrete Time. *Fractal Fract.* **2023**, *7*, 254.

[3] Ostalczyk, P. *Discrete Fractional Calculus: Applications in Control and Image Processing*; World Scientific: Singapore, 2015.

[4] Mohammed, P.O.; Srivastava, H.M.; Baleanu, D.; Abualnaja, K.M. Modified fractional difference operators defined using Mittag-Leffler kernels. *Symmetry* **2022**, *14*, 1519.

[5] Atici, F.; Uyanik, M. Analysis of discrete fractional operators. *Appl. Anal. Discr. Math.* **2015**, *9*, 139–149.

[6] Li, A.; Wei, Y.; Li, Z.; Wang, Y. The numerical algorithms for discrete Mittag-Leffler functions approximation. *Fract. Calc. Appl. Anal.* **2019**, *22*, 95–112.

[7] Podlubny, I. Matrix approach to discrete fractional calculus. *Fract. Calc. Appl. Anal.* **2000**, 3, 359–386.

[8] Goodrich, C.S. On discrete sequential fractional boundary value problems. *J. Math. Anal. Appl.* **2012**, *385*, 111–124.

[9] Chen, C.R.; Bohner M., Jia, B.G. Ulam-hyers stability of Caputo fractional difference equations. *Math. Meth. Appl. Sci.* **2019**, *42*, 7461–7470.

[10] Jonnalagadda, J.M.; Alzabut, J. Numerical computation of exponential functions in frame of nabla fractional calculus. *Comput. Methods Differ. Equ.* **2023**, *11*, 291–302.

[11] Lizama, C. The Poisson distribution, abstract fractional difference equations, and stability. *P. Am. Math. Soc.* **2017**, *145*, 3809–3827.

[12] Mohammed, P.O.; Almusawa, M.Y. On analysing discrete sequential operators of fractional order and their monotonicity results. *AIMS Math.* **2023**, *8*, 12872–12888.

[13] Atıcı, F.M.; Dadashova, K.; Jonnalagadda, J. Linear fractional order h-difference equations. *Int. J. Differ. Equ. (Special Issue Honor. Profr. Johnny Henderson)* **2020**, *15*, 281–300.

[14] Cabada, A.; Dimitrov, N. Nontrivial solutions of non-autonomous Dirichlet fractional discrete problems. *Fract. Calc. Appl. Anal.* **2020**, *23*, 980–995.

[15] Mohammed, P.O.; Abdeljawad, T. Discrete generalized fractional operators defined using h-discrete Mittag-Leffler kernels and applications to AB fractional difference systems. *Math. Meth. Appl. Sci.* **2020**, 1–26; doi:10.1002/mma.7083.

[16] Higazy, M.; Allehiany, F.M.; Mahmoud, E.E. Numerical study of fractional order COVID-19 pandemic transmission model in context of ABO blood group. *Results Phys.* **2021**, *22*, 103852.

[17] Sun, Q.; Xiao, M.; Tao, B. et al. Hopf bifurcation analysis in a fractional-order survival red blood cells model and PD^{α} control. *Adv. Differ. Equ.* **2018**, *10*, 2018.

[18] Henderson, J.; Neugebauer, J.T. Existence of local solutions for fractional difference equations with left focal boundary conditions. *Fract. Calc. Appl. Anal.* **2021**, *24*, 324–331.

[19] Wu, G.; Baleanu, D. Discrete chaos in fractional delayed logistic maps. *Nonlinear Dyn.* **2015**, *80*, 1697–1703.

[20] He, J.W.; Zhang, L.; Zhou, Y.; Ahmad, B. Existence of solutions for fractional difference equations via topological degree methods. *Adv. Differ. Equ.* **2018**, *2018*, 153.

[21] Abdeljawad, T.; Baleanu, D. Monotonicity analysis of a nabla discrete fractional operator with discrete Mittag-Leffler kernel. *Chaos Solit. Fract.* **2017**, *116*, 1–5.

[22] Liu, X.; Du, F.; Anderson, D.; Jia, B. Monotonicity results for nabla fractional h-difference operators. *Math. Meth. Appl. Sci.* **2021**, *44*, 1207–1218.

[23] Mohammed, P.O.; Abdeljawad, T.; Hamasalh, F.K. On discrete delta Caputo-Fabrizio fractional operators and monotonicity analysis. *Fractal Fract.* **2021**, *5*, 116.

[24] Jia, B.; Erbe, L.; Peterson, A. Two monotonicity results for nabla and delta fractional differences. *Arch. Math.* (Basel) **2015**, *104*, 589–597.

[25] Baoguo, J.; Erbe, L.; Goodrich, C.S.; Peterson, A. Monotonicity results for delta fractional difference revisited. *Math. Slovaca* **2017**, *67*, 895–906.

[26] Mohammed, P.O.; Abdeljawad, T.; Hamasalh, F.K. On Riemann-Liouville and Caputo fractional forward difference monotonicity analysis. *Mathematics* **2021**, *9*, 1303.

[27] Mohammed, P.O.; Srivastava, H.M.; Baleanu, D.; Elattar, E.E.; Hamed, Y.S. Positivity analysis for the discrete delta fractional differences of the Riemann-Liouville and Liouville-Caputo types. *Electronic Research Archive* **2022**, *30*, 3058–3070.

[28] Goodrich, C.S. A sharp convexity result for sequential fractional delta differences. *J. Differ. Equ. Appl.* **2017**, *23*, 1986–2003.

[29] Dahal, R.; Goodrich, C.S.; B. Lyons, Monotonicity results for sequential fractional differences of mixed orders with negative lower bound. *J. Differ. Equ. Appl.* **2021**, *27*, 1574–1593.

[30] Srivastava, H.M.; Mohammed, P.O.; Guirao, J.L.G.; Baleanu, D.; Al-Sarairah, E.; Jan, R. A study of positivity analysis for difference operators in the Liouville-Caputo setting. *Symmetry* **2023**, *15*, 391.

[31] Goodrich, C.S.; Jonnalagadda, J.M. Monotonicity results for CFC nabla fractional differences with negative lower bound. *Analysis* (Berlin) **2021**, *41*, 221–229.

[32] Goodrich, C.S. Monotonicity and non-monotonicity results for sequential fractional delta differences of mixed order. *Analysis* (Berlin) **2021**, *41*, 221–229.

[33] Alzabut, J.; Grace, S.R.; Jonnalagadda, J.M.; Santra, S.S.; Abdalla, B. Higher-Order Nabla Difference Equations of Arbitrary Order with Forcing, Positive and Negative Terms: Non-Oscillatory Solutions. *Axioms* **2023**, *12*, 325.

[34] Jonnalagadda, J. Analysis of a system of nonlinear fractional nabla difference equations. *Int. J. Dyn. Syst. Differ. Equ.* **2015**, *5*, 149–174.

[35] Dahal, R.; Goodrich, C.S. A monotonicity result for discrete fractional difference operators. *Arch. Math.* (Basel) **2014**, *102*, 293–299.

[36] Dahal, R.; Goodrich, C.S. An almost sharp monotonicity result for discrete sequential fractional delta differences. *J. Difference Equ. Appl.* **2017**, *23*, 1190–1203.

[37] Goodrich, C.S. A uniformly sharp monotonicity result for discrete fractional sequential differences. *Arch. Math.* (Basel) **2018**, *110*, 145–154.