$\mathcal{C}^1$ SELF–MAPS ON SOME COMPACT MANIFOLDS WITH INFINITELY MANY HYPERBOLIC PERIODIC ORBITS

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Abstract. The aim of the present work is to provide sufficient conditions for having infinitely many periodic points for $\mathcal{C}^1$ self–maps having all their periodic orbits hyperbolic and defined on a compact manifold without boundary. The tool used for proving our results is the Lefschetz fixed point theory.

1. Introduction and statement of the main results

Let $\mathbb{M}$ be a topological space and $f : \mathbb{M} \to \mathbb{M}$ be a continuous map. A point x is called *fixed* if $f(x) = x$ and *periodic* of period k if $f^k(x) = x$ and $f^i(x) \neq x$ if $1 \leq i < k$. By $\text{Per}(f)$ we denote the set of periods of all the periodic points of f.

If $x \in \mathbb{M}$ the set $\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$ is called the *orbit* of the point x. Here f^n means the composition of n times of f with itself. To study the dynamics of the map f is to study all the different kind of orbits of f. Of course, if x is a periodic point of f of period k, then its orbit is $\{x, f(x), f^2(x), \ldots, f^{k-1}(x)\}$, and it is called a *periodic orbit*.

In this paper we study the periodic structure of $\mathcal{C}^1$ self–maps f defined on a given compact manifold $\mathbb{M}$ without boundary. Often the periodic orbits play an important role in the general dynamics of a map, for studying them we can use topological information. Perhaps the best known example in this direction are the results contained in the seminal paper entitle *Period three implies chaos* for continuous self–maps on the interval, see [11].

For continuous self–maps on compact manifolds one of the most useful tools for proving the existence of fixed points and in general of periodic points, is the *Lefschetz Fixed Point Theorem* and its improvements, see for instance [1, 2, 3, 4, 6, 7, 9, 13, 14]. The *Lefschetz zeta function* $\mathcal{Z}_f(t)$ simplifies the study of the periodic points of f. This is a generating function for the Lefschetz numbers of all iterates of f, see section 2 for a precise definition and results about it.

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Let f be a $\mathcal{C}^1$ map defined on a compact manifold $\mathbb{M}$ without boundary. Recall that if $f(x) = x$ and the Jacobian matrix $Df(x)$ of f at x has all its eigenvalues disjoint from the unit circle in the complex plain, then x is called a *hyperbolic fixed point*. Moreover, if y is a periodic point of period k, then y is a *hyperbolic periodic point* if y is a hyperbolic fixed point of f^k. If the points of a periodic orbit are hyperbolic we say that the periodic orbit is *hyperbolic*. Of course if a point of a periodic orbit is hyperbolic, then all the points of the orbit are hyperbolic.

If γ is a hyperbolic periodic orbit of period p, then for each $x \in \gamma$ let E_x^u denotes the subspace of $T_x\mathbb{M}$ generated by the eigenvectors of $Df^p(x)$ corresponding to the eigenvalues whose moduli are greater than one. Let E_x^s be the subspace of $T_x\mathbb{M}$ generated by the remaining eigenvectors. We define the *orientation type* Δ of γ to be $+1$ if $Df^p(x) : E_x^u \to E_x^u$ preserves orientation, and -1 if it reverses orientation. The *index* u of γ is the dimension of E_x^u for some $x \in \gamma$. We note that the definitions of Δ and u do not depend on the point x, only depend of the periodic orbit γ. Finally we associated the triple (p, u, Δ) to the periodic orbit γ. For f the *periodic data* is defined as the collection composed by all triples (p, u, Δ), where a same triple can occur more than once provided it corresponds to different periodic orbits.

In this work we put our attention on $\mathcal{C}^1$ self–maps having all their periodic orbits hyperbolic, and such that they are defined on a compact manifold without boundary. We will provide some sufficient conditions for having infinitely many periodic points.

First we study the $\mathcal{C}^1$ self–maps defined on the n–dimensional sphere $\mathbb{S}^n$.

Theorem 1. *For every positive integer n let f be a $\mathcal{C}^1$ self-map defined on $\mathbb{S}^n$ of degree D with all its periodic orbits hyperbolic. Then the map f has infinitely many periodic orbits if one of the following conditions holds.*

(a) $D \notin \{-1, 0, 1\}$.
(b) $D = 1$, *n is even and there exists at most one periodic orbit reversing orientation with even index.*
(c) $D \in \{-1, 0\}$ *and there is no periodic orbits reversing orientation with even index.*

Second we deal with the $\mathcal{C}^1$ self–maps defined on $\mathbb{S}^n \times \mathbb{S}^n$.

Theorem 2. *For every positive integer n let f be a $\mathcal{C}^1$ self-map defined on $\mathbb{S}^n \times \mathbb{S}^n$ of degree D with all its periodic orbits hyperbolic. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the induced linear map on n–th dimensional homological space of $\mathbb{S}^n \times \mathbb{S}^n$ with rational coefficients, and let the polynomial $p(t) = 1 - (a + d)t + (ad - bc)t^2$. Then the map f has infinitely many periodic orbits if one of the following conditions holds.*

(a) *Either* $D \notin \{-1,0,1\}$, *or* $p(t)$ *has a root which is not a root of the unity.*
(b) n *is odd,* 1 *is not a root of the polynomial* $p(t)$, *and there are no periodic orbits reversing orientation with even index.*
(c) n *is even and there are no periodic orbits reversing orientation with even index.*

Third we study the $\mathcal{C}^1$ self–maps defined on $\mathbb{S}^n \times \mathbb{S}^m$ with $n \neq m$.

Theorem 3. *Let* n *and* m *two distinct positive integers, and let* f *be a* $\mathcal{C}^1$ *self-map defined on* $\mathbb{S}^n \times \mathbb{S}^m$ *of degree* D *with all its periodic orbits hyperbolic, and let* (a) *and* (b) *be the induced linear maps on its homological spaces of dimension* n *and* m *with rational coefficients, respectively. Then the map* f *has infinitely many periodic orbits if one of the following conditions holds.*

(a) *The three integers* a, b, D *do not belong to the set* $\{-1,0,1\}$.
(b) *Only two integers of the set* $\{a, b, D\}$ *do not belong to the set* $\{-1,0,1\}$, *and either these two integers are different, or they are equal and the* q*'s of their corresponding homological spaces* H_q*'s are both even or both odd.*
(c) *Only one integer of the set* $\{a, b, D\}$ *do not belong to the set* $\{-1,0,1\}$.
(d) n *and* m *are even and there are no periodic orbits reversing orientation with even index.*
(e) n *and* m *are odd,* $1 \notin \{a, b\}$, *and there are no periodic orbits reversing orientation with even index.*
(f) n *is even,* m *is odd,* $1 \notin \{b, D\}$, *and there are no periodic orbits reversing orientation with even index.*
(g) n *is odd,* m *is even,* $1 \notin \{a, D\}$, *and there are no periodic orbits reversing orientation with even index.*

Now, we shall study the product of a finite number of spheres.

Theorem 4. *Let* $X = \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_l}$ *with* $n_1 < \cdots < n_l$ *and let* f *be a* $\mathcal{C}^1$ *self-map defined on* X *with all its periodic orbits hyperbolic. Then the map* f *has infinitely many periodic orbits if there is no periodic orbits reversing orientation with even index and* $1-t$ *does not divide* $q_k(t) = \det(Id_{*k} - tf_{*k})$ *with* $0 \le k \le n_1 + \cdots + n_k$ *odd.*

Given topological spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the *wedge sum* $X \vee Y$ is the quotient of the disjoint union X and Y obtained by identifying x_0 and y_0 to a single point (see for more details pp. 10 of [10]). The wedge sum is also known as "one point union". For example, $\mathbb{S}^1 \vee \mathbb{S}^1$ is homeomorphic to the figure "8," two circles touching at a point. We can think the wedge sums of spheres as generalization of graphs in higher dimensions. In the following theorem provide some results for the wedge sum of spheres

Theorem 5. *Let* $X = X_1 \vee \cdots \vee X_l$ *where* $X_i = \mathbb{S}^{n_i} \vee \overset{s_i-times}{\cdots} \vee \mathbb{S}^{n_i}$, $l > 0$ *and* $1 \leq n_1 < \cdots < n_l$, *i.e.* X *is a wedge sum of spheres, and let* f *be a* $\mathcal{C}^1$ *self-map defined on* X *with all its periodic orbits hyperbolic. Then the map* f *has infinitely many periodic orbits if there is no periodic orbits reversing orientation with even index and* $1-t$ *does not divide* $q_k(t) = \det(Id_{*k} - tf_{*k})$ *and* n_k *is an odd number.*

Finally we consider the $\mathcal{C}^1$ self–maps defined on the n–dimensional complex projective space $\mathbb{C}P^n$ and on the n–dimensional quaternion projective space $\mathbb{H}P^n$.

Theorem 6. *Let* n *be a positive integer, and let* f *be a* $\mathcal{C}^1$ *self-map defined on either* $\mathbb{C}P^n$ *or* $\mathbb{H}P^n$ *with all its periodic orbits hyperbolic. It is known that the degree of* f *is equal to* a^n *witn* $a \in \mathbb{Z}$. *Then the map* f *has infinitely many periodic orbits if one of the following conditions holds.*

(a) $a \notin \{-1, 1\}$.
(b) $a = 1$ *and there are at most* $n-1$ *periodic orbits reversing orientation with even index.*
(c) *If* $a = -1$, n *odd and there are at most* $n/2-1$ *periodic orbits reversing orientation with even index.*
(d) *If* $a = -1$, n *even and there are at most* $(n-1)/2-1$ *periodic orbits reversing orientation with even index.*

2. Frank's Theorem

One of the main contribution of the Lefschetz's work in 1920's was to link the homology class of a given map with an earlier work on the indices of Brouwer on the continuous self–maps on compact manifolds. These two notions provide equivalent definitions for the Lefschetz numbers, and from their comparison, can be obtained information on the existence of fixed points.

Let $\mathbb{M}$ be an n–dimensional manifold. We denote by $H_k(\mathbb{M}, \mathbb{Q})$ for $k = 0, 1, ..., n$ the homological groups with coefficients in $\mathbb{Q}$. Each of these groups is a finite linear space over $\mathbb{Q}$.

Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ there exist $n+1$ induced linear maps $f_{*k} : H_k(\mathbb{M}, \mathbb{Q}) \to H_k(\mathbb{M}, \mathbb{Q})$ for $k = 0, 1, \ldots, n$ by f. Every linear map f_{*k} is given by an $n_k \times n_k$ matrix with integer entries, where n_k is the dimension of the linearspace $H_k(\mathbb{M}, \mathbb{Q})$.

Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ on a compact n–dimensional manifold $\mathbb{M}$, its *Lefschetz number* $L(f)$ is defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \mathrm{trace}(f_{*k}). \tag{1}$$

One of the main results connecting the algebraic topology with the fixed point theory is the *Lefschetz Fixed Point Theorem* which establishes the existence of a fixed point if $L(f) \neq 0$, see for instance [3].

Our aim is to obtain information on the set of periods of f. To this purpose it is useful to have information on the whole sequence $\{L(f^m)\}_{m=0}^{\infty}$ of the Lefschetz numbers of all iterates of f. Thus we define the *Lefschetz zeta function* of f as

$$\mathcal{Z}_f(t) = \exp\left(\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m\right).$$

This function generates the whole sequence of Lefschetz numbers, and it may be independently computed through

$$\mathcal{Z}_f(t) = \prod_{k=0}^{n} \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}}, \tag{2}$$

where I_{n_k} is the $n_k \times n_k$ identity matrix, and we take $\det(I_{n_k} - t f_{*k}) = 1$ if $n_k = 0$. Note that the expression (2) is a rational function in t. So the information on the infinite sequence of integers $\{L(f^m)\}_{m=0}^{\infty}$ is contained in two polynomials with integer coefficients, for more details see [7].

Franks in [7] proved the following result which will play a key role for proving our results.

Theorem 7. *Let f be a C^1 self–map defined on a compact manifold without boundary having finitely many periodic orbits all of them hyperbolic, and let Σ be the period data of f. Then the Lefschetz zeta function of f satisfies*

$$\mathcal{Z}_f(t) = \prod_{(p,u,\Delta)\in\Sigma} (1 - \Delta t^p)^{(-1)^{u+1}}. \tag{3}$$

Remark 8. *We note that the polynomials of the form $1 + t^l$ with l a positive integer cannot be factorized in the form*

$$\prod_{i=1}^{s} (1 \pm t^{p_i}).$$

On the other hand, $1 - t^k = (1-t)(1 + t + t^2 + ... + t^{k-1})$ if k is a positive integer.

3. $\mathcal{C}^1$ SELF–MAPS ON $\mathbb{S}^n$

For $n \geq 1$ let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a $\mathcal{C}^1$ map. The homological groups of $\mathbb{S}^n$ over $\mathbb{Q}$ and the induced linear maps are of the form

$$H_q(\mathbb{S}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

where $f_{*0} = (1)$, $f_{*i} = (0)$ for $i = 1, ..., n-1$ and $f_{*n} = (D)$ where D is the degree of the map f, see for more details [5]. From (2) the Lefschetz zeta function of f is

$$\mathcal{Z}_f(t) = \frac{(1 - Dt)^{(-1)^{n+1}}}{1 - t}. \tag{4}$$

Proof of Theorem 1. Assume that $D \neq \pm 1, 0$. Then, since expressions (3) and (4) are not compatible, the map f has infinitely many hyperbolic periodic points by Theorem 7. Consequently statement (a) is proved.

Let n be even and $D = 1$, then by (4) we have that $\mathcal{Z}_f(t) = \frac{1}{(1-t)^2}$. Assume that at most there exists a periodic orbit with even index reversing orientation and that the map f has finitely many periodic points all of them hyperbolic. Therefore by Theorem 7, since f has finitely many periodic points then

$$\mathcal{Z}_f(t) = \frac{\prod_i (1 \pm t^{p_i})}{\prod_j (1 \pm t^{q_j})}.$$

So

$$\prod (1 \pm t^{q_j}) = (1 - t)^2 \prod (1 \pm t^{p_i}). \tag{5}$$

Since there is at most one periodic orbit with even index of even period, by Remark 8 there is at most one factor of the form $1 - t$ in the left hand side of (5) when the special factorization is taken. On the right hand side of (5), the factor $1 - t$ appears with a power greater than or equal to 2. Thus, the equality (5) does not hold and hence f has infinitely many periodic points of all them hyperbolic ending the proof of statement (b).

If $D = 0$, then by (4) we have that $\mathcal{Z}_f(t) = 1/(1 - t)$. So, assuming that f has finitely many periodic points from Theorem 7 we have

$$\prod (1 \pm t^{q_j}) = (1 - t) \prod (1 \pm t^{p_i}), \tag{6}$$

but since there exist no periodic orbits reversing orientation with even index, the left hand side of expression (6) does not contain terms of the form $1 - t$ and the identity does not hold, obtaining that the f has infinitely many periodic points proving one part of statement (c).

Note that for $D = -1$ and n even we have

$$\mathcal{Z}_f(t) = \frac{1}{(1 - t)(1 + t)}$$

and therefore

$$\prod (1 \pm t^{q_j}) = (1 + t)(1 - t) \prod (1 \pm t^{p_i}). \tag{7}$$

On the other hand, for $D = -1$ and n odd

$$\mathcal{Z}_f(t) = \frac{1 + t}{1 - t}$$

and therefore

$$(1+t)\prod(1\pm t^{q_j}) = (1-t)\prod(1\pm t^{p_i}). \tag{8}$$

Now, the proof for the case $D=-1$ (n either even or odd) follows in the same way than the one for the case $D=0$ previously stated. $\square$

4. $\mathcal{C}^1$ SELF–MAPS ON $\mathbb{S}^n \times \mathbb{S}^m$

4.1. **Case $n=m$.** For $n \geq 1$ let $f : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{S}^n \times \mathbb{S}^n$ be a $\mathcal{C}^1$ map. The homological groups of $\mathbb{S}^n \times \mathbb{S}^n$ over $\mathbb{Q}$ and the induced linear maps are of the form

$$H_q(\mathbb{S}^n \times \mathbb{S}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, 2n\}, \\ \mathbb{Q}\oplus\mathbb{Q} & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases}$$

where $f_{*0} = (1)$, $f_{*n} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a,b,c,d \in \mathbb{Z}$, $f_{*2n} = (D)$ where D is the degree of the map f and $f_{*i} = (0)$ for $i \in \{0,...,2n\}$, $i \neq 0, n, 2n$ (see for more details [5]). From (2) the Lefschetz zeta function of f is

$$\mathcal{Z}_f(t) = \frac{p(t)^{(-1)^{n+1}}}{(1-t)(1-Dt)}. \tag{9}$$

where $p(t) = \begin{vmatrix} 1-at & -bt \\ -ct & 1-dt \end{vmatrix}$.

Proof of Theorem 2. Note that if n is odd, then from (9) we have that $\mathcal{Z}_f(t) = p(t)/((1-t)(1-Dt))$. Analogously, if n is even, then form (9) we have that $\mathcal{Z}_f(t) = 1/((1-t)(1-Dt)p(t))$. Assume that the map f has finitely many periodic points all of them hyperbolic. Therefore by Theorem 7, then

$$p(t)\prod(1\pm t^{q_j}) = (1-t)(1-Dt)\prod(1\pm t^{p_i}), \tag{10}$$

in the case n is odd and

$$\prod(1\pm t^{q_j}) = (1-t)(1-Dt)p(t)\prod(1\pm t^{p_i}), \tag{11}$$

in the case n even.

Assume that either $D \notin \{-1,0,1\}$, or the polynomial $p(t)$ has a root which is different from a root of the unity. Then, since expressions (10) and (11) are not compatible, the map f has infinitely many hyperbolic periodic points. Consequently statement (a) is proved.

Assume that n is odd, 1 is not a root of $p(t)$, and there exist no periodic orbits reversing orientation with even index. Then by Remark 8 there is no factor of the form $1-t$ in the left hand side of (10). On the right hand side of (10), the factor $1-t$ appears with a power greater than or equal to 1. Thus, the equality (10) does not hold and hence f has infinitely many periodic points of all them hyperbolic ending the proof of statement (b).

Let now n even and assume that there exist no periodic orbits reversing orientation with even index. Thus, by Remark 8 there is no factor of the form $1-t$ in the left hand side of (11) and the proof follows by the same argument of statement (b). Hence statement (c) is proved ending the proof of the theorem. $\square$

4.2. **Case $n \neq m$.** For integers $n, m \geq 1$, let $f : \mathbb{S}^n \times \mathbb{S}^m \to \mathbb{S}^n \times \mathbb{S}^m$ be a $\mathcal{C}^1$ map. The homological groups of $\mathbb{S}^n \times \mathbb{S}^m$ over $\mathbb{Q}$ and the induced linear maps are of the form

$$H_q(\mathbb{S}^n \times \mathbb{S}^m, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, n, m, n+m\}, \\ 0 & \text{otherwise.} \end{cases}$$

where $f_{*0} = (1)$, $f_{*n} = (a)$, $f_{*m} = (b)$ with $a, b \in \mathbb{Z}$, $f_{*n+m} = (D)$ where D is the degree of the map f, and $f_{*i} = (0)$ for $i \in \{0, ..., n+m\}$ and $i \neq 0, n, m, n+m$ (see for more details [5]). From (2) the Lefschetz zeta function of f is of the form

$$\mathcal{Z}_f(t) = \frac{(1-at)^{(-1)^{n+1}}(1-bt)^{(-1)^{m+1}}(1-Dt)^{(-1)^{n+m+1}}}{1-t}. \tag{12}$$

Proof of Theorem 3. The proofs of statements (a), (b) and (c) follow from the fact that the expression of the Lefschetz zeta function (12) is not compatible with the expression (3), and consequently Theorem 7 implies that the map f has infinitely many hyperbolic periodic points.

Let n and m be even positive integers, then by (12), $\mathcal{Z}_f(t) = 1/((1-at)(1-bt)(1-Dt)(1-t))$. Assume that the map f has finitely many periodic points all of them hyperbolic. Therefore by Theorem 7, then

$$\prod(1 \pm t^{q_j}) = (1-at)(1-bt)(1-Dt)(1-t)\prod(1 \pm t^{p_i}). \tag{13}$$

Note that if there are no periodic orbits reversing orientation with even index, then in the left hand side of (13), by Remark 8, there is no terms of the form $1-t$ obtaining a contradiction and proving statement (d).

Let n and m be odd positive integers, then by (12), $\mathcal{Z}_f(t) = (1-at)(1-bt)/((1-Dt)(1-t))$. Assume that the map f has finitely many periodic points all of them hyperbolic. Therefore by Theorem 7, then

$$(1-at)(1-bt)\prod(1 \pm t^{q_j}) = (1-Dt)(1-t)\prod(1 \pm t^{p_i}). \tag{14}$$

Note that if there are no periodic orbits reversing orientation with even index and $1 \notin \{a, b\}$, then in the left hand side of (14), by Remark 8, there is no terms of the form $1-t$ obtaining a contradiction and proving statement (e).

Let n be even and m be odd, then by (12), $\mathcal{Z}_f(t) = (1-bt)(1-Dt)/((1-at)(1-t))$. Assume that the map f has finitely many periodic points all of them hyperbolic. Therefore by Theorem 7, then

$$(1-bt)(1-Dt)\prod(1\pm t^{q_j}) = (1-at)(1-t)\prod(1\pm t^{p_i}). \tag{15}$$

Note that if there are no periodic orbits reversing orientation with even index and $1 \notin \{b, D\}$, then in the left hand side of (15), by Remark 8, there is no terms of the form $1-t$ obtaining a contradiction and proving statement (f).

The proof of statement (g) is equal than the one of statement (d) changing the roles of a and b. $\square$

5. $\mathcal{C}^1$ self–maps on $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_l}$

Let $X = \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_l}$. By the Künneth Theorem (see for example [10]), we obtain the homology groups of X over the rational numbers, i.e.

$$H_k(X,\mathbb{Q}) = \begin{cases} \underbrace{\mathbb{Q}\oplus\cdots\oplus\mathbb{Q}}_{b_k} & \text{if } b_k \neq 0, \\ \{0\} & \text{if } b_k = 0, \end{cases}$$

with $0 \leq k \leq n_1 + \cdots + n_k$; where b_k is the cardinality of the set

$$\left\{ S \subset \{1,\ldots,l\} \mid \sum_{j\in S} n_j = k \right\}.$$

Recall that the numbers b_k are called the *Betti numbers of* X.

From (2) the Lefschetz zeta function of a C^1 function $f : X \to X$ is

$$\mathcal{Z}_f(t) = (1-t)^{-1}\prod_{k=1}^{l} q_k(t)^{(-1)^{k+1}} \tag{16}$$

where $q_k(t) = \det(Id_{*k} - tf_{*k})$.

Proof of Theorem 4. Assume that there are no periodic orbits with even index reversing orientation, 1 is not a root of $q_k(t)$ for k odd and the map f has finitely many periodic points all of them hyperbolic. Therefore by Theorem 7, if f has finitely many periodic points then

$$\mathcal{Z}_f(t) = \frac{\prod_i(1\pm t^{p_i})}{\prod_j(1\pm t^{q_j})}.$$

So, by (16) we have that

$$\left(\prod_{k\equiv 1(mod2)}^{l} q_k(t)\right)\left(\prod(1\pm t^{q_j})\right) = (1-t)\left(\prod(1\pm t^{p_i})\right)\left(\prod_{k\equiv 0(mod2)}^{l} q_k(t)\right).$$

Note that in the right hand side of the previous equation there is at least one factor of the form $1-t$ but on the left hand side since there exists no periodic orbits with even index reversing orientation and 1 is not a root of $q_k(t)$ for k odd no factor of the form $1-t$ appears, a contradiction. So the map f has infinitely many hyperbolic periodic points. □

6. $\mathcal{C}^1$ SELF–MAPS ON $X = X_1 \vee \cdots \vee X_l$ WHERE $X_i = \mathbb{S}^{n_i} \vee \overset{s_i-\text{TIMES}}{\cdots} \vee \mathbb{S}^{n_i}$

Let $X = X_1 \vee \cdots \vee X_l$ where $X_i = \mathbb{S}^{n_i} \vee \overset{s_i-\text{times}}{\cdots} \vee \mathbb{S}^{n_i}$, $l > 0$ and $1 \leq n_1 < \cdots < n_l$. The homology groups of X over the rational numbers $\mathbb{Q}$ are

$$H_k(X,\mathbb{Q}) = \begin{cases} \mathbb{Q} \oplus \overset{m_k}{\cdots} \oplus \mathbb{Q} & \text{for } k = 0, m_0 = 1, \text{ and for } k = n_i, m_{n_i} = s_i, \\ \{0\} & \text{otherwise.} \end{cases}$$

The computation of these homology groups follows from the facts $H_k(X,\mathbb{Q}) = \oplus_{i=1}^{l} H_k(X_i,\mathbb{Q})$, $H_k(X_i,\mathbb{Q}) = \oplus_{j=1}^{s_i} H_k(\mathbb{S}^{n_i},\mathbb{Q})$, and $H_k(\mathbb{S}^{n_i},\mathbb{Q}) = \mathbb{Q}$ for $k = 0, n_i$ and trivial otherwise, (*cf.* [10]).

Let $f : X \to X$ be a continuous self-map on X, its Lefschetz zeta function has the form:

$$\mathcal{Z}_f(t) = (1-t)^{-1} q_1(t)^{(-1)^{n_1+1}} \cdots q_l(t)^{(-1)^{n_l+1}}, \tag{17}$$

where $q_k(t) = \det(Id_{*n_k} - t f_{*n_k})$,

Proof of Theorem 5. The proof of this theorem follows in a similar way to the proof of Theorem 4 due to the Lefschetz zeta function morphology given by (17). □

7. $\mathcal{C}^1$ SELF–MAPS ON $\mathbb{C}P^n$ AND $\mathbb{H}P^n$

7.1. **Case $\mathbb{C}P^n$.** For $n \geq 1$ let $f : \mathbb{C}P^n \to \mathbb{C}P^n$ be a $\mathcal{C}^1$ map. The homological groups of $\mathbb{C}P^n$ over $\mathbb{Q}$ and the induced linear maps are of the form

$$H_q(\mathbb{C}P^n,\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0,2,4,...,2n\}, \\ 0 & \text{otherwise.} \end{cases}$$

where $f_{*q} = (a^{\frac{q}{2}})$ where $a \in \mathbb{Z}$ and $q \in \{0,2,4,...,2n\}$, $f_{*q} = (0)$ otherwise (see for more details [15, Corollary 5.28]).

From (2) the Lefschetz zeta function of f has the form

$$\mathcal{Z}_f(t) = \left(\prod_q (1 - a^{q/2} t) \right)^{-1}, \tag{18}$$

where q runs over $\{0,2,4,...,2n\}$.

7.2. **Case $\mathbb{H}P^n$.** For $n \geq 1$ let $f : \mathbb{H}P^n \to \mathbb{H}P^n$ be a $\mathcal{C}^1$ map. The homological groups of $\mathbb{H}P^n$ over $\mathbb{Q}$ and the induced linear maps are of the form

$$H_q(\mathbb{H}P^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, 4, 8, ..., 4n\}, \\ 0 & \text{otherwise.} \end{cases}$$

where $f_{*q} = (a^{\frac{q}{4}})$ where $a \in \mathbb{Z}$ and $q \in \{0, 4, 8, ..., 4n\}$, $f_{*q} = (0)$ otherwise (see for more details [15, Corollary 5.33]).

From (2) the Lefschetz zeta function of f has the form

$$\mathcal{Z}_f(t) = \left(\prod_q (1 - a^{q/4} t) \right)^{-1}, \tag{19}$$

where q runs over $\{0, 4, 8, ..., 4n\}$.

Proof of Theorem 6. Suppose that $a \neq \pm 1$, therefore the expression of the Lefschetz zeta function given by (18) and (19) is not compatible with the one of (3) and, like in previous proofs, Theorem 7 implies that the map f has infinitely many hyperbolic periodic points.

Assume that $a = 1$, then $\mathcal{Z}_f(t) = (1 - t)^{-n}$. Then, if f has finitely many periodic points from Theorem (7) we have

$$\prod (1 \pm t^{q_j}) = (1 - t)^n \prod (1 \pm t^{p_i}). \tag{20}$$

Since there are at most $n - 1$ periodic orbits reversing orientation with even index, the left hand side of expression (20) contains at most $n-1$ terms of the form $1-t$, and the identity does not hold because the right hand side contains at least n terms of the form $1 - t$, obtaining that the f has infinitely many periodic points proving statement (b).

Assume that $a = -1$, then the Lefschetz zeta function of f has the form

$$\mathcal{Z}_f(t) = \begin{cases} (1 - t^2)^{\frac{-n}{2}} & \text{if } n \text{ is odd,} \\ (1 - t^2)^{\frac{1-n}{2}} (1 + t)^{-1} & \text{if } n \text{ is even.} \end{cases}$$

Thus, if the map f has finitely many periodic points, Theorem 7 states

$$\prod (1 \pm t^{q_j}) = (1 - t^2)^{\frac{n}{2}} \prod (1 \pm t^{p_i}), \tag{21}$$

if n is odd and

$$\prod (1 \pm t^{q_j}) = (1 - t^2)^{\frac{n-1}{2}} (1 + t) \prod (1 \pm t^{p_i}), \tag{22}$$

if n is even.

In the case n odd, since there are at most $n/2 - 1$ periodic orbits reversing orientation with even index, the left hand side of expression (21) contains at mots $n/2 - 1$ terms of the form $1 - t$, and the identity does not hold because the right hand side contains at least $n/2$ terms of the form $1 - t$, obtaining that the f has infinitely many periodic points proving statement (c).

On the other hand, if n is even since there are at most $(n-1)/2-1$ periodic orbits reversing orientation with even index, the left hand side of expression (22) contains at mots $(n-1)/2-1$ terms of the form $1-t$, and the identity does not hold because the right hand side contains at least $(n-1)/2$ terms of the form $1-t$, obtaining that the f has infinitely many periodic points proving statement (d) and ending the proof of the theorem. $\square$

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