SHANNON–WHITTAKER–KOTEL'NIKOV'S THEOREM GENERALIZED REVISITED

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ABSTRACT. In [MATCH Commun. Math. Chem. **73**, (2015), 385–396] was proved that if $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is a bounded sequence of positive real numbers holding the property $\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \left|\frac{\log \lambda_k}{k}\right| < \infty$ then the function $\sigma_\lambda(t) := \prod_{k\in\mathbb{Z}} \lambda_k^{\mathrm{sinc}(t-k)}$ holds the Shannon–Whittaker–Kotel'nikov's Theorem Generalized (SWKTG) and it can be recomposed in the way $\sigma_\lambda(t) = \lim_{n\to\infty} \left(\sum_{k\in\mathbb{Z}} \lambda_k^{\frac{1}{n}} \mathrm{sinc}(t-k)\right)^n$ for every $t \in \mathbb{R}$. The aim of the present work is to analyze the algebraic structure of the set of sequences of positive real numbers holding $\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \left|\frac{\log \lambda_k}{k}\right| < \infty$. It will allow to apply SWKTG in a more effective way in its many applications, in particular to the chemical ones.

1. INTRODUCTION AND DESCRIPTION OF THE MAIN RESULTS

It is very well–known that one of the main results of the signal theory is so-called Shannon–Whittaker–Kotel'nikov's theorem (see for example [9], [12] or [14]) working for band–limited maps of $L^2(\mathbb{R})$ (i.e., for Paley–Wiener signals), and based on the normalized cardinal sinus map $\mathrm{sinc}(t)$ defined by

$$\mathrm{sinc}(t) = \begin{cases} 1 & \text{if } t = 0, \\ \frac{\sin(\pi t)}{\pi t} & \text{if } t \neq 0. \end{cases}$$

It is known too, in particular at the signal processing theory, that another main result is the Middleton's sampling theorem for band step functions (see [11]). This result was one of the first modifications of the classical Sampling Theorem which only works for band–limited maps,

Key words and phrases. Shannon–Whittaker–Kotel'nikov's Theorem; recomposition of chemical products, signal theory.

2010 Mathematics Subject Classification. 41A60, 41A46.

see [13]. After this starting point many different extensions and generalizations of this theorem appeared in the literature trying to obtain approximations of non band–limited signals (see for instance [4] or [7]). Good surveys on these extensions are [5] or [14].

Inspired by the results of [6] and [8] some papers trying to obtain approximations of non band–limited signals by using band–limited ones by means of the increasing of the band size appeared. See for instance [1], [2] and [3] where an asymptotic generalization of the classical Shannon–Whittaker–Kotel'nikov's Theorem is stated. These works had very deep impact in chemistry because of their possible applications to recompose functions which models different chemical procedures, see for instance [10] or [15].

In [3] was proved that if $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is a bounded sequence of positive real numbers holding the property $\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \left|\frac{\log \lambda_k}{k}\right| < \infty$ then the function $\sigma_\lambda(t) := \prod_{k\in\mathbb{Z}} \lambda_k^{\operatorname{sinc}(t-k)}$ holds the Shannon–Whittaker–Kotel'nikov's Theorem Generalized (SWKTG) and it can be recomposed in the way

$$\sigma_\lambda(t) = \lim_{n\to\infty} \left(\sum_{k\in\mathbb{Z}} \lambda_k^{\frac{1}{n}} \operatorname{sinc}(t-k) \right)^n$$

for every $t \in \mathbb{R}$. In the present work our objective is to analyze the algebraic structure of the set of sequences of positive real numbers holding $\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \left|\frac{\log \lambda_k}{k}\right| < \infty$.

By $\mathcal{L}$ we define the set

$$\mathcal{L} = \left\{ \lambda : \mathbb{Z} \longrightarrow \mathbb{R}^+ ; \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \left|\frac{\log \lambda_k}{k}\right| < \infty \right\}.$$

We will understand the convergence according to the main value of Cauchy, although in this case, being a series of non-negative terms, coincides with the convergence ordinary of the two branches separately.

As the study for the branch with positive indexes is analogous to study negative indexes one, we will assume without loss of generality that $\lambda : \mathbb{N} \longrightarrow \mathbb{R}^+$.

Definition 1. *Let $Q = \{q_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ be a contable set strictly increasing. By $\mathcal{L}_Q$ we denote the set*

$$\mathcal{L}_Q = \left\{ \lambda : \mathbb{N} \longrightarrow \mathbb{R}^+; \sum_{n\in\mathbb{N}} \left| \frac{\log \lambda_n}{q_n} \right| < \infty \right\}.$$

Remark 2. *Note that according to the previous definition, a sequence $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}} \in \mathcal{L}$ if and only if $\{\lambda_k\}_{k\in\mathbb{N}}, \{\lambda_{-k}\}_{k\in\mathbb{N}} \in \mathcal{L}_{\mathbb{N}}$ and $\lambda_0 \in \mathbb{R}^+$. In other words, $\mathcal{L}$ is the cartesian product of $\mathcal{L}_{\mathbb{N}} \times \mathbb{R}^+ \times \mathcal{L}_{\mathbb{N}}$.*

We shall prove that fixed $Q = \{q_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$, every set $\mathcal{L}_Q$, is a Banach space with a very particular structure based on $Q^{-1} = \{q_n^{-1}\}_{n\in\mathbb{N}}$ be $l^1(\mathbb{N})$ or not. Recall that $l^1(\mathbb{N})$ are convergent sequences of natural numbers in $\|\cdot\|_1$. Moreover we shall define an order relation in $\mathcal{L}_Q$ and we will prove that if $\lambda \in \mathcal{L}_Q$ then each sequence with terms located between λ y λ^{-1} is an element of $\mathcal{L}_Q$ too.

2. Definitions and auxilliary results

Definition 3. *Given $Q = \{q_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$, let $\mu_Q : \mathcal{P}(\mathbb{N}) \longrightarrow [0,\infty]$ be the positive measure defined on the parts of $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$, given by*

$$\mu_Q(n) = \frac{1}{q_n}.$$

If $\beta = \{\beta_k\}_{k\in\mathbb{N}}$ is a sequence of real terms, let

$$\|\beta\|_{1,Q} := \sum_{n\in\mathbb{N}} |\beta_n| \, \mu_Q(n) = \sum_{n\in\mathbb{N}} \frac{|\beta_n|}{q_n}.$$

It is defined $l^1(\mathbb{N}, \mu_Q)$ as the set of all sequences β of real terms such that

$$\|\beta\|_{1,Q} < \infty.$$

Remark 4. *It is easy to show that $\|\cdot\|_{1,Q}$ is a norm in $l^1(\mathbb{N}, \mu_Q)$. Moreover, $(l^1(\mathbb{N}, \mu_Q), +, \cdot_{\mathbb{R}})$ with $+$ the sum element to element is a normed linear space over the reals with norm equal to $\|\cdot\|_{1,Q}$.*

Proposition 5. *$\mathcal{L}_Q$ has a $l^1(\mathbb{N}, \mu_Q)$ structure of Banach space.*

Proof. We shall use the exponential map (note that exponential maps hold SWKGT, see [1, 2] and [3] for more details) for our aim. Indeed, let be T the map between $\mathcal{L}_Q$ and $l^1(\mathbb{N}, \mu_Q)$ given by

$$\begin{array}{rcl} T : l^1(\mathbb{N}, \mu_Q) & \longrightarrow & \mathcal{L}_Q \\ \beta & \longmapsto & e^{\beta} = \{e^{\beta_n}\}_{n\in\mathbb{N}}. \end{array}$$

Since the exponential map is biyective, the new set $\mathcal{L}_Q$ is endowed by the operations of $l^1(\mathbb{N}, \mu_Q)$ via the previous map. Hence, for every $\beta, \widetilde{\beta} \in l^1(\mathbb{N}, \mu_Q)$ and every $\alpha \in \mathbb{R}$ we have that

$$\begin{aligned} T(\beta + \widetilde{\beta}) &= e^{\beta + \widetilde{\beta}} = e^{\beta} e^{\widetilde{\beta}} = T(\beta) T(\widetilde{\beta}), \\ T(\alpha\beta) &= e^{\alpha\beta} = \left(e^{\beta}\right)^{\alpha} = (T(\beta))^{\alpha}. \end{aligned}$$

Therefore we can claim that $\mathcal{L}_Q$ has a linear space structure over $\mathbb{R}$ with these operations: the product element by element as internal operation ($\lambda\beta = \{\lambda_n \beta_n\}_{n\in\mathbb{N}}$) and the power ($\lambda^{\alpha} = \{\lambda_n^{\alpha}\}_{n\in\mathbb{N}}$) as external operation over $\mathbb{R}$. The neutral element is the constant sequel equal 1, $\mathbf{u}$ ($u_n = 1\ \forall n \in \mathbb{N}$) and the inverse element of $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ is the sequence $\lambda^{\mathbf{-1}} = \left\{\dfrac{1}{\lambda_n}\right\}_{n\in\mathbb{N}}$.

Moreover, as norm on $\lambda \in \mathcal{L}_Q$ we consider

$$\|\lambda\|_{\mathcal{L}_Q} = \|T^{-1}(\lambda)\|_{1,Q} = \|\log \lambda\|_{1,Q} = \sum_{n\in\mathbb{N}} \frac{|\log \lambda_n|}{q_n},$$

making T an isometry ($\|T(\beta)\|_{\mathcal{L}_Q} = \|\beta\|_{1,Q}$). Note that with this definition of norm, $\mathcal{L}_Q$ is the set of all sequences of positive real terms such that $\|\lambda\|_{\mathcal{L}_Q} < \infty$.

Thus, like the spaces $\mathcal{L}_Q$ and $l^1(\mathbb{N}, \mu_Q)$ are isometrically isomorphic with the operations and norms previously stated, we have that $\mathcal{L}_Q$ is a linear space over $\mathbb{R}$, normed and completed because of these properties in $l^1(\mathbb{N}, \mu_Q)$, and therefore, $\mathcal{L}_Q$ is a Banach space ending the proof. □

Let we state an order relation in the set of positive real sequences.

Definition 6. *Given two sequences $\lambda, \mu : \mathbb{N} \longrightarrow \mathbb{R}^+$, we say that $\mu << \lambda$ if the following two properties hold for every $n \in \mathbb{N}$:*

a) $(\mu_n - 1)(\lambda_n - 1) \geq 0$,
b) $|\mu_n - 1| \leq |\lambda_n - 1|$.

Geometrically $\mu << \lambda$ means that, for every $n \in \mathbb{N}$, either $1 \leq \mu_n \leq \lambda_n$ or $\lambda_n \leq \mu_n \leq 1$.

Proposition 7. *The relation $<<$ is of partial order.*

Proof. Let $\lambda, \mu : \mathbb{N} \longrightarrow \mathbb{R}^+$. Obviously $\mu << \mu$. Assume that $\lambda << \mu$ and $\mu << \lambda$, then fixed $n \in \mathbb{N}$ we have $(\mu_n - 1)(\lambda_n - 1) \geq 0$ and

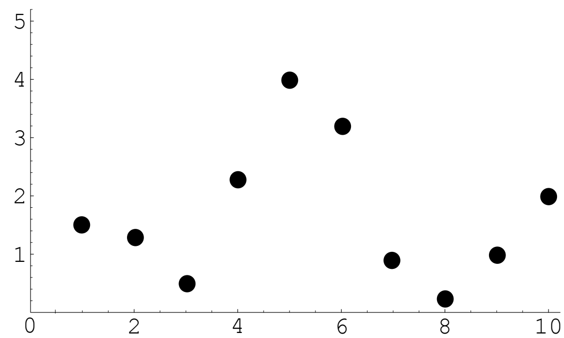

FIGURE 1. Sequence λ.

$|\mu_n - 1| = |\lambda_n - 1|$. Therefore,

$$\begin{aligned}
&\text{si } \lambda_n, \mu_n > 1 \text{ then } \mu_n - 1 = \lambda_n - 1,\\
&\text{si } \lambda_n, \mu_n < 1 \text{ then } 1 - \mu_n = 1 - \lambda_n,\\
&\text{si } \lambda_n = 1 \text{ then } 1 - \mu_n = 0,\\
&\text{si } \mu_n = 1 \text{ then } 0 = 1 - \lambda_n.
\end{aligned}$$

In any case, we have that $\lambda_n = \mu_n$ for every $n \in \mathbb{N}$ and therefore $\mu = \lambda$.

Assume now that $\mu << \lambda$ y $\lambda << \beta$, then

$$(\mu_n - 1)(\lambda_n - 1)^2(\beta_n - 1) \geq 0.$$

If $\lambda_n = 1$, since $\mu << \lambda$, we have that $\mu_n = 1$ and if $\lambda_n \neq 1$ then $(\lambda_n - 1)^2 > 0$. Thus, in any case, we have that

$$(\mu_n - 1)(\beta_n - 1) \geq 0.$$

Now, since by hypothesis

$$|\mu_n - 1| \leq |\lambda_n - 1| \leq |\beta_n - 1|,$$

we have $\mu << \beta$ ending the proof. □

3. MAIN RESULTS

Definition 8. *Let λ be a sequence of positive real numbers. We call box of λ to the set*

$$\mathcal{C}_\lambda = \left\{\mu : \mathbb{N} \longrightarrow \mathbb{R}^+; \mu << \lambda\right\} \cup \left\{\mu : \mathbb{N} \longrightarrow \mathbb{R}^+; \mu << \boldsymbol{\lambda^{-1}}\right\}.$$

Example 9. *Let us to state an example of the previous notion. Let $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of positive terms. Assume for instance, that it takes values of Figure 1.*

In this case the inverse sequence of λ^{-1} is determined by the values stated in Figure 2.

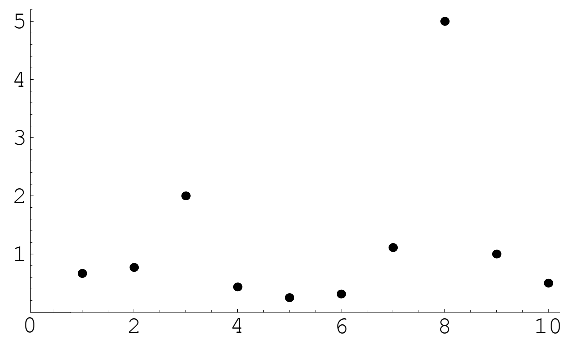

FIGURE 2. Sequence λ^{-1}.

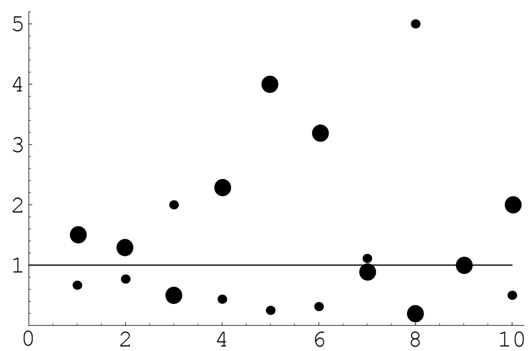

FIGURE 3. Box $\mathcal{C}_\lambda$ linked to λ.

Then, the box $\mathcal{C}_\lambda$ linked to λ (see Figure 3) is composed for all sequences $\mu = \{\mu_n\}_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, the value μ_n belongs to the vertical interval determined by the value of λ_n stated in Figure 1 and for the value λ_n^{-1} stated in Figure 2.

Proposition 10. *Let $Q = \{q_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$. If $\lambda \in \mathcal{L}_Q$ then $\mathcal{C}_\lambda \subseteq \mathcal{L}_Q$.*

Proof. Let $\lambda \in \mathcal{L}_Q$ and $\mu \in \mathcal{C}_\lambda$.
If $\mu << \lambda$, for every $n \in \mathbb{N}$ we can have either $1 \leq \mu_n \leq \lambda_n$ or $\lambda_n \leq \mu_n \leq 1$; but in any case, for every $n \in \mathbb{N}$ holds $|\log \mu_n| \leq |\log \lambda_n|$ and therefore

$$\sum_{n\in\mathbb{N}} \frac{|\log \mu_n|}{q_n} \leq \sum_{n\in\mathbb{N}} \frac{|\log \lambda_n|}{q_n} < \infty.$$

So, we have obtained

$$\lambda \in \mathcal{L}_Q, \mu << \lambda \Longrightarrow \mu \in \mathcal{L}_Q. \qquad (1)$$

If $\mu << \lambda^{-1}$, since $\lambda^{\mathbf{-1}} \in \mathcal{L}_Q$ for being $\mathcal{L}_Q$ a linear space, by (1) is stated $\mu \in \mathcal{L}_Q$. □

From now we have stated properties of $\mathcal{L}_Q$ using the product of sequences and the power. In the sequel we endeavour to determine when the ordinary sum is an internal operation in $\mathcal{L}_Q$ and if we multiply an scalar by a sequence we continue having an element of $\mathcal{L}_Q$. The answer to these question will be positive or negative depending if the sequence Q^{-1} is in $l^1(\mathbb{N})$ or not, as we shall state in the next two results.

Proposition 11. *Let $Q = \{q_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that $Q^{-1} \in l^1(\mathbb{N})$, then $(\mathcal{L}_Q, +, \cdot_{\mathbb{R}^+})$ is a cone with $+$ the sum element by element and $\cdot_{\mathbb{R}^+}$ the ordinary product by positive real numbers, i.e., $\forall \lambda, \mu \in \mathcal{L}_Q$ and $\forall \alpha \in \mathbb{R}^+$*

i) $\alpha\lambda \in \mathcal{L}_Q$,
ii) $\lambda + \mu \in \mathcal{L}_Q$.

Proof. For the first part we consider $\alpha \in \mathbb{R}^+$ and $\lambda \in \mathcal{L}_Q$. Obviously $\alpha\lambda_n \in \mathbb{R}^+$ for all $n \in \mathbb{N}$ and since $Q^{-1} \in l^1(\mathbb{N})$,

$$\begin{aligned}\sum_{n\in\mathbb{N}} \frac{|\log(\alpha\lambda_n)|}{q_n} &\leq \sum_{n\in\mathbb{N}} \frac{|\log\alpha|}{q_n} + \sum_{n\in\mathbb{N}} \frac{|\log\lambda_n|}{q_n} \\ &= |\log\alpha| \sum_{n\in\mathbb{N}} \frac{1}{q_n} + \sum_{n\in\mathbb{N}} \frac{|\log\lambda_n|}{q_n} < \infty,\end{aligned}$$

we prove i).
For stating ii) let $\lambda, \mu \in \mathcal{L}_Q$. We define the two sets

$$\begin{aligned} A &= \{n \in \mathbb{N}; 1 < \lambda_n, \mu_n\}, \\ B &= \{n \in \mathbb{N}; \lambda_n, \mu_n < 1\}, \\ C &= \{n \in \mathbb{N}; (\lambda_n - 1)(\mu_n - 1) \leq 0\}. \end{aligned}$$

Since the sets A, B, C are disjoint and $A \cup B \cup C = \mathbb{N}$, the sum $\displaystyle\sum_{n\in\mathbb{N}} \frac{|\log(\lambda_n + \mu_n)|}{q_n}$ can be displayed into three sums over each of the three sets.
If $n \in A$ we have $1 < \lambda_n + \mu_n \leq 2\lambda_n\,\mu_n$. Therefore, since $\lambda, \mu \in \mathcal{L}_Q$ and $Q^{-1} \in l^1(\mathbb{N})$, we have that

$$
\begin{aligned}
\sum_{n\in A}\frac{|\log(\lambda_n+\mu_n)|}{q_n} &= \sum_{n\in A}\frac{\log(\lambda_n+\mu_n)}{q_n} \le \sum_{n\in A}\frac{\log(2\lambda_n\mu_n)}{q_n}\\
&\le \log 2\sum_{n\in A}\frac{1}{q_n}+\sum_{n\in A}\frac{\log\lambda_n}{q_n}+\sum_{n\in A}\frac{\log\mu_n}{q_n}\\
&\le \log 2\sum_{n\in \mathbb{N}}\frac{1}{q_n}+\sum_{n\in \mathbb{N}}\frac{\log\lambda_n}{q_n}+\sum_{n\in \mathbb{N}}\frac{\log\mu_n}{q_n}<\infty.
\end{aligned}\tag{2}
$$

If $n\in B$ we can have the following cases: either $\lambda_n+\mu_n>1$ or $\lambda_n+\mu_n<1$. Note that if $\lambda_n+\mu_n=1$ the corresponding terms in the series object of our study becomes zero. Thus, we consider the following subsets

$$
\begin{aligned}
B^+ &= \{n\in B;\lambda_n+\mu_n>1\},\\
B^- &= \{n\in B;\lambda_n+\mu_n<1\}.
\end{aligned}
$$

If $n\in B^+$ then $1<\lambda_n+\mu_n<2$ and, since $Q^{-1}\in l^1(\mathbb{N})$, we have that

$$
\sum_{n\in B^+}\frac{|\log(\lambda_n+\mu_n)|}{q_n}=\sum_{n\in B^+}\frac{\log(\lambda_n+\mu_n)}{q_n}\le\log 2\sum_{n\in B^+}\frac{1}{q_n}<\infty.\tag{3}
$$

On the other hand, if $n\in B^-$ then $|\log(\lambda_n+\mu_n)|\le|\log\lambda_n|$ and therefore, since $\lambda\in\mathcal{L}_Q$,

$$
\sum_{n\in B^-}\frac{|\log(\lambda_n+\mu_n)|}{q_n}\le\sum_{n\in B^-}\frac{|\log\lambda_n|}{q_n}\le\sum_{n\in \mathbb{N}}\frac{|\log\lambda_n|}{q_n}<\infty.\tag{4}
$$

So, using (3) and (4) we obatin that

$$
\begin{aligned}
\sum_{n\in B}\frac{|\log(\lambda_n+\mu_n)|}{q_n} &= \sum_{n\in B^+}\frac{|\log(\lambda_n+\mu_n)|}{q_n}\\
&\quad+\sum_{n\in B^-}\frac{|\log(\lambda_n+\mu_n)|}{q_n}<\infty.
\end{aligned}\tag{5}
$$

Finally, let $n\in C$ and assume, without loss of generality, that $\lambda_n\le\mu_n$ (conversely the proof is similar changing the roles of λ_n and μ_n). Then we have that

$$
1<\lambda_n+\mu_n\le 1+\mu_n\le 2\mu_n
$$

and thus

$$
\sum_{n\in C}\frac{\log(\lambda_n+\mu_n)}{q_n}\le\sum_{n\in C}\frac{\log 2}{q_n}+\sum_{n\in C}\frac{\log\mu_n}{q_n}<\infty.\tag{6}
$$

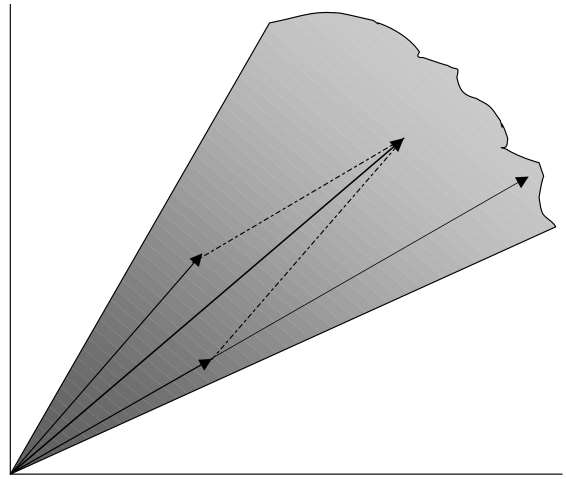

FIGURE 4. $\mathcal{L}_Q$ with $Q^{-1} \in l^1(\mathbb{N})$.

Using (2), (5) and (6) in the sum display done before we have

$$\sum_{n\in\mathbb{N}} \frac{|\log(\lambda_n + \mu_n)|}{q_n} < \infty,$$

and therefore ii). ☐

Remark 12. *The condition stated in the previous proposition $Q^{-1} \in l^1(\mathbb{N})$ is not only sufficient one, it is necessary too. Moreover, in the case of not holding it the algebraic structure of $\mathcal{L}_Q$ is diametrically opposite to a cone as we shall state in the next result.*

Proposition 13. *Let $Q = \{q_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that $Q^{-1} \notin l^1(\mathbb{N})$. Let λ, μ be belonging to $\mathcal{L}_Q$ and $\alpha \in \mathbb{R}^+ \setminus \{1\}$. Then*

i) $\alpha\lambda \notin \mathcal{L}_Q$,
ii) $\lambda + \mu \notin \mathcal{L}_Q$.

Proof. For proving i) let $\alpha \in \mathbb{R}^+$ with $\alpha \neq 1$ and consider the set $I_\alpha = \left\{ n \in \mathbb{N}; |\log \lambda_n| > \frac{|\log \alpha|}{2} \right\}$.
If $n \notin I_\alpha$ we have that

$$|\log(\alpha\lambda_n)| = |\log\alpha + \log\lambda_n| \geq ||\log\alpha| - |\log\lambda_n|| \geq \frac{|\log\alpha|}{2}$$

and consequently

$$\sum_{n\notin I_\alpha}\frac{|\log(\alpha\lambda_n)|}{q_n}\geq\frac{|\log\alpha|}{2}\sum_{n\notin I_\alpha}\frac{1}{q_k}. \tag{7}$$

Moreover, from the definition of the set I_α we have that

$$\sum_{n\in\mathbb{N}}\frac{|\log\lambda_n|}{q_n}\geq\sum_{n\in I_\alpha}\frac{|\log\lambda_n|}{q_n}>\frac{|\log\alpha|}{2}\sum_{n\in I_\alpha}\frac{1}{q_n}.$$

Since $\lambda\in\mathcal{L}_Q$, from the previous bound we have that $\sum_{n\in I_\alpha}\frac{1}{q_n}$ is finite and, therefore since $Q^{-1}\notin l^1(\mathbb{N})$, we can claim that the series $\sum_{n\notin I_\alpha}\frac{1}{q_n}$ diverges. Thus, by (7), the series

$$\sum_{n\in\mathbb{N}}\frac{|\log(\alpha\lambda_n)|}{q_n}$$

diverges too, and therefore i) is proved.

For proving ii), given a sequence $\lambda\in\mathcal{L}_Q$, let T_λ the following set

$$T_\lambda=\left\{q_n\in Q;|\log\lambda_n|>\frac{1}{2}\right\}.$$

Then we have, using $\lambda\in\mathcal{L}_Q$, that

$$\frac{1}{2}\sum_{q_n\in T_\lambda}\frac{1}{q_n}\leq\sum_{q_n\in T_\lambda}\frac{|\log\lambda_n|}{q_n}\leq\sum_{q_n\in Q}\frac{|\log\lambda_n|}{q_n}<\infty,$$

from which we obtain that $\sum_{q_n\in T_\lambda}\frac{1}{q_n}$ is finite. Analogously it is obtained that $\sum_{q_n\in T_\mu}\frac{1}{q_n}$ is finite too. Thus, we can claim that $\sum_{q_n\in T_\lambda\cup T_\mu}\frac{1}{q_n}<\infty$. So, since by hypothesis $Q^{-1}\notin l^1(\mathbb{N})$, we have that the sequence $S=Q\setminus(T_\lambda\cup T_\mu)$ holds that the series $\sum_{q_n\in S}\frac{1}{q_n}$ does not converge.

On the other hand, if $n\in S$ then

$$\frac{2}{\sqrt{e}}\leq\lambda_n+\mu_n\leq 2\sqrt{e}.$$

So we can claim that

$$\sum_{q_n\in Q}\frac{|\log(\lambda_n+\mu_n)|}{q_n}\geq\sum_{q_n\in S}\frac{|\log(\lambda_n+\mu_n)|}{q_n}\geq\log\frac{2}{\sqrt{e}}\sum_{q_n\in S}\frac{1}{q_n}$$

and using that $\sum_{q_n\in S}\frac{1}{q_n}$ does not converge, we have that the sequence $\lambda+\mu\notin\mathcal{L}_Q$. □

Remark 14. *Let H_λ be the ray generated by the sequence λ of positive real terms*

$$H_\lambda=\left\{\alpha\lambda;\alpha\in\mathbb{R}^+\right\},$$

and let $Q=\{q_n\}_{n\in\mathbb{N}}\subseteq\mathbb{N}$. We can claim that if $Q^{-1}\in l^1(\mathbb{N})$ then either the ray H_λ is completely inside in $\mathcal{L}_Q$ or there is not element of such ray in that set. On the other hand, if $Q^{-1}\notin l^1(\mathbb{N})$ in each ray H_λ there are at most one element in $\mathcal{L}_Q$. Note that in both cases can occur that there is no element at the intersection $H_\lambda\cap\mathcal{L}_Q$, as for instance happens for the sequence $\lambda=\{e^{q_n}\}_{n\in\mathbb{N}}$.

On the other hand, if we focus our attention in bounded sequences we have that if $Q^{-1}\in l^1(\mathbb{N})$ then all element from the ray H_λ is in $\mathcal{L}_Q$ because there exist C constant such that

$$\sum_{n\in\mathbb{N}}\frac{|\log\lambda_n|}{q_n}\leq C\sum_{n\in\mathbb{N}}\frac{1}{q_n}<\infty.$$

But if $Q^{-1}\notin l^1(\mathbb{N})$ then the intersection can be empty (take as example the sequence $\lambda=\left\{e^{(-1)^n}\right\}_{n\in\mathbb{N}}$).

Proposition 15. *Let $Q=\{q_n\}_{n\in\mathbb{N}}\subseteq\mathbb{N}$.*

a) *If $Q^{-1}\in l^1(\mathbb{N})$ all positive constant sequences are in $\mathcal{L}_Q$.*
b) *If $Q^{-1}\notin l^1(\mathbb{N})$ the unique constant sequence in $\mathcal{L}_Q$ is the unit sequence $\mathbf{u}$.*

Proof. Let $Q\subseteq\mathbb{N}$ and $\mathbf{c}=\{c\}_{n\in\mathbb{N}}\subseteq\mathbb{R}^+$ be a constant sequence in $\mathcal{L}_Q$. Then

$$|\log c|\sum_{n\in\mathbb{N}}\frac{1}{q_n}=\sum_{n\in\mathbb{N}}\frac{|\log c|}{q_n}<\infty,$$

from which we can deduce that if $c\neq 1$ then $Q^{-1}\in l^1(\mathbb{N})$, and if $Q^{-1}\notin l^1(\mathbb{N})$ the unique possibility for $\mathbf{c}=\{c\}_{n\in\mathbb{N}}\in\mathcal{L}_Q$ is $\log c=0$, i.e, $\mathbf{c}=\mathbf{u}$. □

Finally, we shall prove that the subspaces $\mathcal{L}_Q$ form decreasing chains of spaces being their intersection the set $\mathcal{L}_{\mathbb{N}}$ as it follows from the next proposition.

Proposition 16. *Let* $Q = \{q_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$, $\widetilde{Q}$ *subsequence from* Q. *Then* $\mathcal{L}_Q \subseteq \mathcal{L}_{\widetilde{Q}}$.

Proof. Let $\lambda \in \mathcal{L}_Q$. Let $\widetilde{Q} = \{\widetilde{q}_n\}_{n\in\mathbb{N}}$ in Q, then for every $n \in \mathbb{N}$ we have that $\widetilde{q}_n \geq q_n$. Therefore

$$\sum_{n\in\mathbb{N}} \frac{|\log \lambda_n|}{\widetilde{q}_n} \leq \sum_{n\in\mathbb{N}} \frac{|\log \lambda_n|}{q_n} < \infty$$

and consequently $\lambda \in \mathcal{L}_{\widetilde{Q}}$. □

So, from the previous proposition we can claim that

$$\bigcap_{Q\subseteq\mathbb{N}} \mathcal{L}_Q = \mathcal{L}_{\mathbb{N}}.$$

Concerning the union, taking account that for any sequence λ we always can find a Q such that the series $\sum_{n\in\mathbb{N}} \frac{|\log \lambda_n|}{q_n}$ be convergent, we can claim that the union of all spaces $\mathcal{L}_Q$ covers the space of all sequences of positive real terms. Moreover in the next results we present a constructive proof for a Q valid.

Proposition 17. *Let* $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ *a sequence of positive real terms. Then for every* $p > 0$ *there exists* $Q = \{q_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ *such that* $Q^{-1} \in l^p(\mathbb{N})$ *and* $\lambda \in \mathcal{L}_Q$.

Proof. Let $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ and $p > 0$. Let $Q = \{q_n\}_{n\in\mathbb{N}}$ be given by $q_n = \left[\max_{1\leq r\leq n} \left\{2^r |\log \lambda_r|, 2^{\frac{n}{p}}, q_{n-1}\right\}\right] + 1$, where $[\cdot]$ denotes the integer part. Then $Q^{-1} \in l^p(\mathbb{N})$ because of

$$\sum_{n\in\mathbb{N}} \left(\frac{1}{q_n}\right)^p \leq \sum_{n\in\mathbb{N}} \left(\frac{1}{2^{\frac{n}{p}}}\right)^p = \sum_{n\in\mathbb{N}} \frac{1}{2^n}.$$

Moreover, we have that

$$\sum_{n\in\mathbb{N}} \frac{|\log \lambda_n|}{q_n} \leq \sum_{n\in\mathbb{N}} \frac{|\log \lambda_n|}{2^n\,|\log \lambda_n|} = \sum_{n\in\mathbb{N}} \frac{1}{2^n}$$

and therefore $\lambda \in \mathcal{L}_Q$. □

Remark 18. *In short, for every* $p > 0$, *the union of the spaces* $\mathcal{L}_Q$ *such that* Q^{-1} *belongs to* $l^p(\mathbb{N})$ *covers the space of the sequences of positive real numbers, i.e.,*

$$\bigcup_{\substack{Q\subseteq\mathbb{N}\\ Q^{-1}\in l^p(\mathbb{N})}} \mathcal{L}_Q = \left\{\lambda : \mathbb{N} \longrightarrow \mathbb{R}^+\right\}.$$

Obviously, in the convergence study of the series

$$\sum_{n\in\mathbb{N}} \frac{|\log \lambda_n|}{q_n} < \infty,$$

we could have considered the contrary relation to $\mathcal{L}_Q$. More precisely, fixed a sequence $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ of positive real terms different sets $Q = \{q_n\}_{n\in\mathbb{N}}$ could be considered over them the previous series be convergent, i.e, working with sets of the form

$$\mathcal{Q}_\lambda = \left\{ Q = \{q_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}; \sum_{n\in\mathbb{N}} \frac{|\log \lambda_n|}{q_n} < \infty \right\}.$$

The interpretation and application of the sets $\mathcal{L}_Q$ and $\mathcal{Q}_\lambda$ is symmetric. And their study can be performance in a parallel way obtaining similar results.

Note that for each sequence $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ the set $\mathcal{Q}_\lambda$ is non empty and is an expansive set in the sense that for each sequence $Q \in \mathcal{Q}_\lambda$ then every sequence $P \geq Q$ belongs to the set $\mathcal{Q}_\lambda$. (Given two sequences $P = \{p_n\}_{n\in\mathbb{N}}$ y $Q = \{q_n\}_{n\in\mathbb{N}}$, $P \geq Q$ if and only if $p_n \geq q_n$ for every $n \in \mathbb{N}$).

Corollary 19. *The sum and the product element by element are internal operations in $\mathcal{Q}_\lambda$.*

Proof. Simple note that $P + Q > P$ and $PQ > P$. □

Proposition 20. *Let $\lambda = \{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$.*

a) *If $\lambda \in \mathcal{L}_\mathbb{N}$ then $\mathbb{N}$ is the minimum element in $\mathcal{Q}_\lambda$.*
b) *If $\lambda \notin \mathcal{L}_\mathbb{N}$ then $\mathcal{Q}_\lambda$ has no minimum element.*

Proof. If $\lambda \in \mathcal{L}_\mathbb{N}$ then we have that $\displaystyle\sum_{n\in\mathbb{N}} \frac{|\log \lambda_n|}{n} < \infty$. Thus $\mathbb{N} \in \mathcal{Q}_\lambda$. Moreover, it is a minimum in $\mathcal{Q}_\lambda$ due to for every Q we have $Q \geq \mathbb{N}$.
Assume now that $\lambda \notin \mathcal{L}_\mathbb{N}$ and we shall see that for every sequence $Q \in \mathcal{Q}_\lambda$ is possible to find another sequence $P \in \mathcal{Q}_\lambda$ such that $P < Q$.
Indeed, since $\lambda \notin \mathcal{L}_\mathbb{N}$ the series $\displaystyle\sum_{n\in\mathbb{N}} \frac{|\log \lambda_n|}{n}$ does not converge, i.e. $\mathbb{N} \notin \mathcal{Q}_\lambda$. Thus, given every sequence $Q \in \mathcal{Q}_\lambda$ we have that Q is a proper subset of $\mathbb{N}$ and therefore the set $\{k \in \mathbb{N}; q_k > k\}$ is non empty. Let n_0 the minimum element of such set.

Let $P = \{p_n\}_{n\in\mathbb{N}}$ such that $p_n = q_n$ for every $n \neq n_0$ and $p_{n_0} = q_{n_0} - 1$. Then we have that $P < Q$ and, obviously, $P \in \mathcal{Q}_\lambda$ since the series

$$\sum_{n\in\mathbb{N}} \frac{|\log \lambda_n|}{p_n} = \sum_{\substack{n\in\mathbb{N}\\ n\neq n_0}} \frac{|\log \lambda_n|}{q_n} + \frac{|\log \lambda_{n_0}|}{q_{n_0} - 1}$$

is convergent because $Q \in \mathcal{Q}_\lambda$, ending the proof. □

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