The Extension Degree Conditions for Fractional Factor

Wei Gao[1], Weifan Wang[2], Juan L.G. Guirao[3*]

1. School of Information Science and Technology, Yunnan Normal University, Kunming 650500, China

2. Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

3. Departamento de Matemática Aplicada y Estadística, Universidad Politécnica de Cartagena, Hospital de Marina, 30203-Cartagena, Región de Murcia, Spain

Abstract: In Gao's previous work, the authors determined several graph degree conditions of a graph which admits fractional factor in particular settings. It was revealed that these degree conditions are tight if $b = f(x) = g(x) = a$ for all vertices x in G. In this paper, we continue to discuss these degree conditions for admitting fractional factor in the setting that several vertices and edges are removed and there is a difference Δ between $g(x)$ and $f(x)$ for every vertex x in G. These obtained new degree conditions reformulate Gao's previous conclusions, and show how Δ acts in the results. Furthermore, counterexamples are structured to reveal the sharpness of degree conditions in the setting $f(x) = g(x) + \Delta$.

Key words: fractional factor, degree condition, independent set

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1 Introduction

In many engineering applications, their mathematical models can be expressed as a (direct or undirect) graph. For example, we look upon the network as a graph. Some correspondences are given here: the site matches with a vertex and the channel matches with an edge in the graph. In conventional network, the mission of data transmission is based on the selection of the shortest way between vertices. However, the computation of network flow in software definition network determines the data transmission. It chooses the path that is least congested at present. In this view, the pattern of data transmission problem in SDN setting is just the existence of fractional factor in the corresponding graph.

Graph $G = (V, E)$ mentioned here are all simple graph with its edge set $E(G)$ and its vertex set $V(G)$. Throughout this paper, we set $n = |V(G)|$ as the order of graph. For a vertex x in G, $N_G(x)$ and $d_G(x)$ are used to denote the neighborhood and the degree of x in G, respectively. Let $N_G[x] = N_G(x) \cup \{x\}$. To simplify, we use $N(x)$,$d(x)$ and $N[x]$ to express $d_G(x)$, $N_G(x)$ and $N_G[x]$, respectively. Set $\delta(G)$ as the minimum degree of G. We set $G[S]$ as the sub-graph of G deduced from $S \subseteq V(G)$, and $G - S = G[V(G)\backslash S]$. Set $e_G(S_1, S_2) = |\{e = v_1v_2 | v_1 \in S_1, v_2 \in S_2\}|$ for any $S_1, S_2 \subseteq V(G)$ with $S_1 \cap S_2 = \emptyset$. Denote $\sigma_2(G) = \min\{d_G(u) + d_G(v) | uv \notin E(G)\}$. The other terms used without clear definitions here can be refered to classic graph theory book [1].

Functions f and g are two integer-valued defined on $V(G)$ satisfying $f(x) \geq g(x) \geq 0$ for all vertices x in G. A *fractional* (g, f)*-factor* is regarded as a score function h which maps to every element in $E(G)$ a real number belongs to [0,1]. As a result, for every vertex x we get $g(x) \leq d^h_G(x) \leq f(x)$, and $\sum_{e\in E(x)} h(e) = d^h_G(x)$ where $E(x) = \{y|yx \in E(G)\}$. Fractional f-factor is regarded as a special case of fractional (g, f)-factor if the values of two functions are equal for any vertex x in G. Fractional $[a, b]$-factor is another special case of fractional (g, f)-factor if $f(x) = b$, $g(x) = a$ for any

*Corresponding author: juan.garcia@upct.es

Table 1: Special cases of fractional (g, f, n', m)-critical deleted graph

setting (for any $v \in V(G)$)	name
$g(x) = f(x)$	fractional (f, n', m)-critical deleted graph
$f(x) = b$ and $g(x) = a$	fractional (a, b, n', m)-critical deleted graph
$f(x) = g(x) = k$	fractional (k, n', m)-critical deleted graph

vertex x in G. In addition, if the value of both f and g equal to $k \in \mathbb{N}$ for any vertex x in G, then it's a fractional k-factor.

A fractional (g, f, m)-deleted graph and a fractional (g, f, n')-critical graph imply the existence of fractional factor in special setting when delete m edges and n' vertices, respectively. As the combination of the above two concepts, Gao [2] introduced fractional (g, f, n', m)-critical deleted graph to denote a graph to be fractional (g, f, m)-deleted after removing any n' vertices. When functions g and f take special value for all vertices, the fractional (g, f, n', m)-critical deleted graph becomes various names which are presented in Table 1. Several recent contributions in this topic were presented in Zhou et al. [10], [11], [13], [15] and [16], and Gao et al. [3], [4], [5] and [7], Knor et al. [8], and Liu et al. [9].

In Zhou [12] and Zhou et al. [14], the setting was different from the previous situations in which there is a difference Δ between $g(x)$ and $f(x)$ for every vertex x in G, i.e., $b - \Delta \geq f(x) - \Delta \geq g(x) \geq a$ for every x in G. We observe that if $\Delta = 0$, then binding number (minimum $\frac{|N(X)|}{|X|}$ where $\emptyset \neq X \subset V(G)$) condition for ID-$(g, f)$-factor-critical graph (this concept will be explain later) is

$$bind(G) > \frac{(n-1)(b+2a-1)}{an-(b+a-2)}.$$

After adding the variable Δ (i.e., $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$), by the conclusion obtained by Zhou et al. [14], the binding number condition becomes

$$bind(G) > \frac{(n-1)(b+2a-1+\Delta)}{(a+\Delta)n-(b+a-2)}.$$

This fact reveals that if the setting changes, the lower bound of binding number for ID-(g, f)-factor-critical graph is changed as well, and the new binding number heavily depends on Δ. There is one thing we must emphasize here is that all the results in this paper are independent from the maximum degree of the graph, and Δ is only used to represent the difference between g and f throughout the article.

In our article, we consider the degree condition for the existence of fractional factors in the setting that $b - \Delta \geq f(x) - \Delta \geq g(x) \geq a$ for every vertex x. Intuitively, in our new setting, the new degree conditions should be relied on the variable Δ, or at least the new degree conditions are different from the previous ones. Thus, it inspired us to strictly study it theoretically.

In the following context, we first present the major results of part one in fractional (g, f, n', m)-critical deleted setting and prove it in details which extended Theorem 1-3 raised in Gao et. al. [6], perspectively.

Theorem 1 *Assume G is a graph with n vertices, and set b, a, n', m, and Δ as non-negative integers meeting $b - \Delta \geq a \geq 2$ and $n > \frac{(b+a+2m-2)(b+a)}{\Delta+a} + n'$. Functions g, f are integer-valued on its vertex set and $b - \Delta \geq f(x) - \Delta \geq g(x) \geq a$ for every vertex x. Then G is fractional (g, f, n', m)-critical deleted if $\delta(G) \geq \frac{(b-\Delta)n+(\Delta+a)n'}{b+a}$.*

Theorem 2 *Assume G is a graph with n vertices, and set b, a, n', m, and Δ as non-negative integers meeting $b - \Delta \geq a \geq 2$, $\delta(G) \geq m + n' + \frac{b(b-\Delta)}{\Delta+a}$ and $n > \frac{(b+a+2m-1)(a+b)}{\Delta+a} + n'$. Functions g, f as integer-valued on its vertex set and meet $b - \Delta \geq f(x) - \Delta \geq g(x) \geq a$ for every vertex x in G. Then G is fractional (g, f, n', m)-critical deleted if for any $xy \neq E(G)$, we have*

$$\max\{d_G(x), d_G(y)\} \geq \frac{(b-\Delta)n+(\Delta+a)n'}{b+a}.$$

Table 2: Three degree conditions of fractional (g, f, m)-deleted graph by setting $n' = 0$ in above three theorems

order of graph	degree condition	additional condition
$n > \frac{(b+a+2m-2)(a+b)}{\Delta+a}$	$\delta(G) \geq \frac{(b-\Delta)n}{b+a}$	
$n > \frac{(b+a+2m-1)(b+a)}{\Delta+a}$	$\max\{d_G(x), d_G(y)\} \geq \frac{(b-\Delta)n}{b+a}$	$\delta(G) \geq \frac{b(b-\Delta)}{\Delta+a} + m$
$n > \frac{(b+a+2m-2)(b+a)}{\Delta+a}$	$\sigma_2(G) \geq \frac{2(b-\Delta)n}{b+a}$	$\delta(G) \geq \frac{b(b-\Delta)}{\Delta+a} + m$

Table 3: Special cases of fractional ID-(g, f, m)-deleted graphs

setting (for any $v \in V(G)$)	name
$g = f$	fractional ID-(f, m)-deleted graph
$f(x) = b$ and $g(x) = a$	fractional ID-(a, b, m)-deleted graph
$m = 0$	fractional ID-(g, f)-factor-critical graph

Theorem 3 *Assume G is a graph with n vertices, and set $b, a, n', m,$ and Δ as non-negative integers meeting $b - \Delta \geq a \geq 2$, $\delta(G) \geq \frac{b(b-\Delta)}{\Delta+a} + m + n'$ and $n > \frac{(b+a+2m-2)(a+b)}{\Delta+a} + n'$. Functions g, f are integer-valued defined on the vertex set so that $b - \Delta \geq f(x) - \Delta \geq g(x) \geq a$ for every vertex x in G. Then G is fractional (g, f, n', m)-critical deleted if $\sigma_2(G) \geq \frac{2(n(b-\Delta)+n'(\Delta+a))}{b+a}$.*

The above three theorems manifest conditions for a graph to be fractional (g, f, n', m)-critical deleted from different aspects. The corollaries on fractional (g, f, m)-deleted graphs can be stated in Table 2.

The data in Table 2 can be regarded as the extension of Corollary 1, Corollary 2 and Corollary 3 in Gao et al. [6], respectively. Furthermore, we will further to discuss the relevant conditions in setting both f and g are constant functions in subsection 2.4.

Set $d_H(T) = \sum_{x \in T} d_H(x)$ and $f(S) = \sum_{x \in S} f(x)$. The lemma below will be used in the demonstration process of our Theorem 1-3.

Lemma 1 (Gao [2]) *Assume G is a graph, functions f and g are integer-valued on its vertex set meeting $f(x) \geq g(x)$ for every x in G. Set n', $m \in \mathbb{N}^{+} \cup \{0\}$. Then G is fractional (g, f, n', m)-critical deleted iff*

$$\begin{aligned} & f(S) - g(T) + d_{G-S}(T) \\ \geq & \max_{U \subseteq S, H \subseteq E(G-U), |U|=n', |H|=m} \{f(U) + \sum_{x \in T} d_H(x) - e_H(T, S)\} \end{aligned} \quad (1)$$

for any subsets S, T of $V(G)$ with $S \cap T = \emptyset$ and $|S| \geq n'$.

In very special circumstances, n' vertices consist an independent set, then it comes to *fractional ID-(g, f, m)-deleted graph*. Analogously, when functions g and f take special value for all vertices, it becomes different names which are presented in Table 3.

The following results in fractional ID-(g, f, m)-deleted setting as second part of main conclusions which are the extension of Theorem 4, Theorem 5 and Theorem 6 showed in Gao et al. [6], respectively.

Theorem 4 *Assume G is a graph with n vertices, and b, a, m, Δ are non-negative integers meeting $b - \Delta \geq a \geq 2$ and $n > \frac{(b+\Delta+2a)(b+2m+a-2)}{\Delta+a}$. Functions g, f are integer-valued on its vertex set satisfy $b - \Delta \geq f(x) - \Delta \geq g(x) \geq a$ for every vertex x. Then G is fractional ID-(g, f, m)-deleted if $\delta(G) \geq \frac{(b+a)n}{b+2a+\Delta}$.*

Theorem 5 *Assume G is a graph with n vertices, and b, a, m, Δ are non-negative integers meeting $b - \Delta \geq a \geq 2$, $\delta(G) \geq \frac{(\Delta+a)n}{b+2a+\Delta} + \frac{b(b-\Delta)}{\Delta+a} + m$ and $n > \frac{(b+a+2m-1)(b+2a+\Delta)}{\Delta+a}$. Functions g, f as* *integer-valued on its vertex set satisfy $b-\Delta \geq f(x)-\Delta \geq g(x) \geq a$ for every vertex x in G. Then G is fractional ID-(g,f,m)-deleted if for any $xy \neq E(G)$, we have*

$$\max\{d_G(y), d_G(x)\} \geq \frac{(b+a)n}{b+2a+\Delta}.$$

Theorem 6 *Assume G is a graph with n vertices, and b, Δ, a, m as non-negative integers meeting $b-\Delta \geq a \geq 2$, $\delta(G) \geq \frac{(\Delta+a)n}{b+2a+\Delta} + \frac{b(b-\Delta)}{a+\Delta} + m$ and $n > \frac{(b+2m+a-2)(b+\Delta+2a)}{\Delta+a}$. Functions g, f are integer-valued on its vertex set satisfy $b-\Delta \geq f(x)-\Delta \geq g(x) \geq a$ for every vertex x in G. Then G is fractional ID-(g,f,m)-deleted if $\sigma_2(G) \geq \frac{2(b+a)n}{b+2a+\Delta}$.*

2 Proof of first part results: Theorem 1-3

By observing, we find that $\delta(G) \geq \frac{(b-\Delta)n+(a+\Delta)n'}{a+b}$ in Theorem 1 implies $\sigma_2(G) \geq \frac{2((b-\Delta)n+(a+\Delta)n')}{a+b}$ and $\delta(G) \geq n'+m+\frac{(b-\Delta)b}{\Delta+a}$ in Theorem 3. Hence, it's sufficient to make Theorem 2 and 3 proved.

We deduce the conclusion on graph without non-adjacent vertices below.

Lemma 2 *Assume G is a complete graph with n vertices, and b, Δ, a, n', m as non-negative integers meeting $b-\Delta \geq a \geq 2$ and $n > \frac{(b+a+2m-2)(b+a)}{a+\Delta} + n'$. Functions g, f as integer-valued on its vertex set satisfy $b-\Delta \geq f(x)-\Delta \geq g(x) \geq a$ for every vertex x in G. Then G is fractional (g,f,n',m)-critical deleted.*

Proof. Assume G meets the conditions of Lemma 2 without being fractional (g,f,n',m)-critical deleted. Clearly, $T \neq \emptyset$. According to Lemma 1 and the fact that $\sum_{x\in T} d_H(x) - e_H(T,S)$ at most $2m$, subsets T and S of $V(G)$ with $T \cap S\emptyset$ exist to satisfy

$$f(S) + d_{G-S}(T) - g(T) \leq \max_{U\subseteq S, |U|=n'} f(U) - 1 + 2m$$

or

$$f(S-U) - g(T) + d_{G-S}(T) \leq 2m-1, \tag{2}$$

in which $|S| \geq n'$. Choose T and S with minimum $|T|$. Hence, for every $x \in T$, we derive $b-1-\Delta \geq g(x)-1 \geq d_{G-S}(x)$.

Note that $G-S$ is also complete for each vertex subset S. Thus, for disjoint subsets $T, S \subseteq V(G)$, we deduce

$$\begin{aligned}
& f(S-U) - g(T) - 2m + d_{G-S}(T) \\
\geq\ & (|S|-n')(\Delta+a) + \sum_{x\in T} d_{G-S}(x) - (b-\Delta)|T| - 2m \\
\geq\ & (|S|-n')(\Delta+a) - (n-|S|)(b-n-\Delta+1+|S|) - 2m \\
=\ & (b+a-2n+1)|S| + |S|^2 - (b-\Delta)n + n^2 - n - (a+\Delta)n' - 2m.
\end{aligned}$$

Regarding it as the function of $|S|$, we look into the following cases in view of the fact that $|S|$ is an integer.

Case 1. $a+b$ is even. Since $n > \frac{(b+a+2m-2)(b+a)}{\Delta+a} + n'$ and $b+a \geq 4$, we have

$$\begin{aligned}
& (b+a+1-2n)|S| + |S|^2 - (b-\Delta)n + n^2 - 2m - n - (a+\Delta)n' \\
\geq\ & (n - \frac{a+b}{2})(b+a+1-2n) + (n - \frac{b+a}{2})^2 - n - (b-\Delta)n + n^2 - 2m - (a+\Delta)n' \\
=\ & (\Delta+a)n - \frac{b+a}{2} - (\frac{b+a}{2})^2 - 2m - (\Delta+a)n' \\
>\ & (\frac{(b+a+2m-2)(b+a)}{\Delta+a} + n')(\Delta+a) - 2m - \frac{b+a}{2} - (\frac{b+a}{2})^2 - (\Delta+a)n' \\
=\ & \frac{3}{4}(b+a)^2 + (b+a-1)2m - \frac{5}{2}(b+a) \\
\geq\ & \frac{3}{4}\cdot 16 - \frac{5}{2}\cdot 4 > 0,
\end{aligned}$$

which contradicts (2).

Case 2. $b-a \equiv 1 \pmod 2$. By $n > \frac{(b+a+2m-2)(b+a)}{\Delta+a} + n'$ and $b+a \geq 5$, we get

$$\begin{aligned}
& (b+a+1-2n)|S| + |S|^2 - (b-\Delta)n + n^2 - n - (\Delta+a)n' - 2m \\
\geq\ & (n - \frac{a+b+1}{2})(b+a-2n+1) + (n - \frac{b+a+1}{2})^2 \\
& -(b-\Delta)n + n^2 - 2m - n - (a+\Delta)n' \\
=\ & (\Delta+a)n - 2m - (\frac{b+a+1}{2})^2 - (a+\Delta)n' \\
>\ & (\Delta+a)(\frac{(b+a)(b+a+2m-2)}{\Delta+a} + n') - (\frac{b+a+1}{2})^2 - (\Delta+a)n' - 2m \\
=\ & (b+a-1)2m + \frac{3}{4}(b+a)^2 - \frac{1}{4} - \frac{5}{2}(b+a) \\
\geq\ & \frac{3}{4} \cdot 25 - \frac{5}{2} \cdot 5 - \frac{1}{4} > 0,
\end{aligned}$$

a contradiction.

The proof of complete graph setting is done. □

Clearly, Lemma 2 is the extension of previous conclusion on the complete graph marked in Lemma 2 of Gao et al. [6]. By setting $n' = 0$ in Lemma 2, the corollary present below will be employed in Section 3.

Corollary 1 *Assume G is a complete graph having n vertices, and b, Δ, a, m as non-negative integers meeting $b - \Delta \geq a \geq 2$ and $n > \frac{(b+a)(b+a+2m-2)}{a+\Delta}$. Functions g, f are integer-valued on its vertex set satisfy $b - \Delta \geq f(x) - \Delta \geq g(x) \geq a$ for every vertex x in G. Therefore, G is fractional (g, f, m)-deleted.*

Graph is supposed to be non-complete in what follows. From this point of view, the degree condition $\max\{d_G(y), d_G(x)\} \geq \frac{(b-\Delta)n+(a+\Delta)n'}{a+b}$ for every $xy \neq E(G)$ in Theorem 2 and $\sigma_2(G) \geq \frac{2((b-\Delta)n+(a+\Delta)n')}{a+b}$ in Theorem 3 are meaningful.

2.1 Correctness of Theorem 2

Assume G meets all the assumptions of Theorem 2 without being fractional (g, f, n', m)-critical deleted. It can be inferred $|T| \geq 1$, and there exist disjoint subsets $T, S \subseteq V(G)$ satisfies (2) with $|S| \geq n'$. We have $b - 1 - \Delta \geq g(x) - 1 \geq d_{G-S}(x)$ for all vertex x in T by means of selecting S and T with minimum $|T|$.

Let $d_1 = \min\{d_{G-S}(x) : x \in T\}$. We deduce $b - 1 - \Delta \geq d_1 \geq 0$ and

$$f(S-U) + d_{G-S}(T) - g(T) \geq d_1|T| + (\Delta+a)(|S| - n') - (b-\Delta)|T|.$$

This implies

$$2m - 1 \geq (|S| - n')(\Delta + a) - (b - \Delta - d_1)|T|. \tag{3}$$

We choose vertex x_1 in T to meet $d_{G-S}(x_1) = d_1$.

If $|T| \leq b - \Delta$, in terms of (3) and $|S| + d_1 \geq d_G(x_1) \geq \delta(G) \geq \frac{b(b-\Delta)}{a+\Delta} + n' + m$, we verify

$$\begin{aligned}
& 2m - 1 \\
\geq\ & (|S| - n')(\Delta + a) + |T|(\Delta + d_1 - b) \\
\geq\ & (a+\Delta)(\frac{b(b-\Delta)}{a+\Delta} + n' + m - d_1 - n') + (\Delta + d_1 - b)(b - \Delta) \\
=\ & (b - a - \Delta)d_1 + am + \Delta(m - d_1 - \Delta + b) \\
\geq\ & 2m,
\end{aligned}$$

which gets contradicted. Thus, $|T| \geq b + 1 - \Delta \geq 1 + a$.

On the condition that $T - N_T[x_1] \neq \emptyset$, set $d_2 = \min\{d_{G-S}(x) : x \in T - N_T[x_1]\}$ and take vertex x_2 belongs to $T - N_T[x_1]$ such that $d_{G-S}(x_2) = d_2$. Hence, $d_1 \leq d_2 \leq b - \Delta - 1$. Since $b - \Delta - 1 \geq d_{G-S}(x)$ for any vertex x in T and $|T| \geq b - \Delta + 1$, $T - N_T[x_1] \neq \emptyset$, thus x_1, x_2 must be existed. Considering the non-adjacent vertices assumption, we deduce

$$\frac{n'(a+\Delta) + n(b-\Delta)}{b+a} \leq \max\{d_G(x_1), d_G(x_2))\} \leq |S| + d_2,$$

which reveals

$$|S| \geq \frac{n'(a+\Delta) + n(b-\Delta)}{b+a} - d_2. \tag{4}$$

In view of $b - \Delta - d_2 > 0$ and $n - |S| - |T| \geq 0$, we infer

$$\begin{aligned}
& (n - |T| - |S|)(b - \Delta - d_2) \\
\geq\ & (\Delta + a)(|S| - n') + \sum_{x \in T}(d_{G-S}(x) - b + \Delta) + 1 - 2m \\
\geq\ & (d_1 + \Delta - b)|N_T[x_1]| - 2m + (\Delta + a)|S| + 1 \\
& +(d_2 - b + \Delta)(|T| - |N_T[x_1]|) - (a + \Delta)n' \\
=\ & (a + \Delta)|S| + (d_1 - d_2)|N_T[x_1]| + (d_2 + \Delta - b)|T| - (a + \Delta)n' - 2m + 1 \\
\geq\ & (d_1 - d_2)(1 + d_1) + (a + \Delta)|S| + (\Delta + d_2 - b)|T| - (a + \Delta)n' - 2m + 1.
\end{aligned}$$

It follows that

$$0 \leq n(b - d_2 - \Delta) - (b + a - d_2)|S| + 2m + (1 + d_1)(d_2 - d_1) + (a + \Delta)n' - 1. \tag{5}$$

Using (4), (5), $n > \frac{(a+b)(a+b+2m-1)}{a+\Delta} + n'$ and $d_1 \leq d_2 \leq b - 1 - \Delta$, we obtain

$$\begin{aligned}
0\ \leq\ & (b - d_2 - \Delta)n - (b + a - d_2)(\frac{(a+\Delta)n' + (b-\Delta)n}{b+a} - d_2) + (d_1 + 1)(d_2 - d_1) \\
& +(a + \Delta)n' - 1 + 2m \\
=\ & -nd_2\frac{a+\Delta}{b+a} + d_2\frac{(a+\Delta)n'}{a+b} + (b + a)d_2 - d_1^2 - d_2^2 + d_1 d_2 + d_2 - d_1 + 2m - 1 \\
<\ & -d_1^2 - d_2^2 + d_1 d_2 + 2d_2 - d_1 + 2m(1 - d_2) - 1.
\end{aligned}$$

If $d_2 = 0$, then we have $d_1 = d_2 = 0$. According to (4), we have $|S| \geq \frac{(b-\Delta)n+(a+\Delta)n'}{a+b}$ and $|T| \leq n - |S| \leq \frac{(a+\Delta)n-(a+\Delta)n'}{a+b}$. By $\sum_{x \in T} d_H(x) - e_G(T, S) \leq d_{G-S}(T)$, we yield

$$\begin{aligned}
& f(S - U) - g(T) + d_{G-S}(T) - (\sum_{x \in T} d_H(x) - e_G(T, S)) \\
\geq\ & (a + \Delta) \cdot (\frac{(b-\Delta)n + (\Delta+a)n'}{b+a} - n') - (b - \Delta) \cdot \frac{(\Delta+a)n - n'(\Delta+a)}{b+a} \\
& +e_G(T, S) + d_{G-S}(T) - \sum_{x \in T} d_H(x) \\
\geq\ & 0,
\end{aligned}$$

a contradiction.

If $d_2 \geq 1$, we infer

$$0 < -d_1^2 - d_2^2 + d_1 d_2 + 2d_2 - d_1 + 2m(1 - d_2) - 1 \leq -d_2^2 + (d_1 + 2)d_2 - d_1^2 - d_1 - 1.$$

Let

$$h_1(d_2) = -d_2^2 + (d_1 + 2)d_2 - d_1^2 - d_1 - 1.$$

This implies,

$$\max\{h_1(d_2)\} = h_1(\frac{d_1+2}{2}) = -\frac{3}{4}d_1^2 \leq 0,$$

which is a contradiction. Thus, we complete the derivation for the correctness. □

2.2 Correctness of Theorem 3

Assume G meets all the assumptions of Theorem 3 without being fractional (g, f, n', m)-critical deleted. Apparently, $|T| \geq 1$ and there exist $T, S \subseteq V(G)$ with $T \cap S = \emptyset$ satisfies (2) with $|S| \geq n'$. Selecting T and S with minimum $|T|$, we obtain $b - 1 - \Delta \geq g(x) - 1 - \Delta \geq d_{G-S}(x)$ for any vertex x in T.

Set d_1, d_2, x_1 and x_2 as defined before. Similarly as discussed in Section 2.1, we yield $d_1 \leq d_2 \leq b - \Delta - 1$, $|T| \geq b - \Delta + 1$ and x_1, x_2 must be existed.

By means of degree assumption, we arrive

$$\frac{2(n(b-\Delta) + n'(\Delta + a))}{b+a} \leq \sigma_2(G) \leq 2|S| + d_2 + d_1,$$

which reveals

$$|S| \geq \frac{(a+\Delta)n' + (b-\Delta)n}{a+b} - \frac{d_2 + d_1}{2}. \tag{6}$$

Using the consideration in Subsection 2.1, (5) holds as well. In light of (5), (6), $n > \frac{(b+a-2+2m)(b+a)}{\Delta+a} + n'$ and $d_1 \leq d_2 \leq b - 1 - \Delta$, we derive

$$\begin{aligned} 0 &\leq (d_1+1)(d_2-d_1) + n(b-d_2-\Delta) - (b+a-d_2)(\frac{n'(\Delta+a)+n(b-\Delta)}{b+a} - \frac{d_1+d_2}{2}) \\ &\quad +(\Delta+a)n' - 1 + 2m \\ &= d_2\frac{(\Delta+a)n'}{b+a} - nd_2\frac{\Delta+a}{b+a} + (b+a)\frac{d_1+d_2}{2} - d_1^2 - \frac{d_2^2}{2} + \frac{d_1 d_2}{2} + d_2 - d_1 + 2m - 1 \\ &< -d_2(a+b-3) + \frac{a+b}{2}(d_1+d_2) - d_1^2 - \frac{d_2^2}{2} + \frac{d_1 d_2}{2} - d_1 + 2m(1-d_2) - 1. \end{aligned}$$

The case for $d_2 = 0$ can be proved in the similar way as Subsection 2.1.

If $d_2 \geq 1$, then we verify

$$\begin{aligned} 0 &< -d_2(a+b-3) + \frac{a+b}{2}(d_1+d_2) - d_1^2 - \frac{d_2^2}{2} + \frac{d_1 d_2}{2} - d_1 + 2m(1-d_2) - 1 \\ &\leq -\frac{d_2^2}{2} - d_2(\frac{a+b}{2} - 3 - \frac{d_1}{2}) - d_1^2 + (\frac{a+b}{2} - 1)d_1 - 1. \end{aligned}$$

Let

$$h_2(d_2) = -\frac{d_2^2}{2} - d_2(\frac{a+b}{2} - 3 - \frac{d_1}{2}) - d_1^2 + (\frac{a+b}{2} - 1)d_1 - 1.$$

If d_2 can reach to $3 + \frac{d_1}{2} - \frac{a+b}{2}$ (i.e., $3 + \frac{d_1}{2} - \frac{a+b}{2} \geq 1$), then

$$\max\{h_2(d_2)\} = h_2(3 + \frac{d_1}{2} - \frac{a+b}{2}),$$

and $d_2 \leq 1$ in terms of $b \geq a \geq 2$ and $d_1 \leq b - 1$. Hence, $(d_1, d_2) = (0, 1)$ or $d_1 = d_2 = 1$. By $b \geq a \geq 2$, we get $h_2(d_2) \leq 0$ for both $(d_1, d_2) = (0, 1)$ and $(d_1, d_2) = (1, 1)$, a contradiction.

If d_2 can't take $3 + \frac{d_1}{2} - \frac{a+b}{2} - \frac{1}{a+b}$ as its value, then we have

$$\begin{aligned} 0 &< -\frac{d_2^2}{2} - d_2(\frac{a+b}{2} - 3 - \frac{d_1}{2}) - d_1^2 + (\frac{a+b}{2} - 1)d_1 - 1 \\ &\leq -\frac{d_1^2}{2} - d_1(\frac{a+b}{2} - 3 - \frac{d_1}{2}) - d_1^2 + (\frac{a+b}{2} - 1)d_1 - 1 \\ &= -d_1^2 + 2d_1 - 1 \leq 0, \end{aligned}$$

which also gets contradicted. In result, Theorem 3 is proven. □

2.3 Sharpness

In this part, we give an example to prove the sharpness of the degree conditions in Theorem 1-3 in some sense. That is to say, the minimal condition $\delta(G) \geq \frac{(b-\Delta)n+(a+\Delta)n'}{a+b}$ can't be changed to $\delta(G) \geq \frac{(b-\Delta)n+(a+\Delta)n'}{a+b} - 1$; we can't replace $\max\{d_G(y), d_G(x)\} \geq \frac{n(b-\Delta)+n'(a+\Delta)}{a+b}$ by $\max\{d_G(y), d_G(x)\} \geq \frac{n(b-\Delta)+n'(a+\Delta)}{a+b} - 1$ in Theorem 2; and the degree sum condition $\sigma_2(G) \geq \frac{2((b-\Delta)n+(a+\Delta)n')}{a+b}$ in Theorem 3 can't be transferred to $\sigma_2(G) \geq \frac{2((b-\Delta)n+(a+\Delta)n')}{a+b} - 1$.

Let $b = a + \Delta$, $G_1 = K_{at+n'}$ be a complete graph, $G_2 = (bt + 1)K_1$, and $G = G_1 \vee G_2$, where $t \in \mathbb{N}$ is a large number which ensures the graph to meet $\delta(G) \geq m + n' + \frac{b(b-\Delta)}{\Delta+a}$ and $n > \frac{(b+a-2+2m)(b+a)}{\Delta+a} + n')$, so $n = |G_1| + |G_2| = (a+b)t + 1 + n'$. Let $a = g(x)$ and $b = a + \Delta = f(x)$ for every vertex x in G. We have

$$\frac{n'(a+\Delta)+n(b-\Delta)}{b+a} > \delta(G) = n' + at > \frac{n'(a+\Delta)+n(b-\Delta)}{b+a} - 1,$$

$$\frac{n'(a+\Delta)+n(b-\Delta)}{b+a} > \max\{d_G(y), d_G(x)\} = n' + at > \frac{n'(a+\Delta)+n(b-\Delta)}{b+a} - 1,$$

$$\frac{2(n'(a+\Delta)+n(b-\Delta))}{b+a} > \sigma_2(G) = 2(at+n') \geq \frac{2(n'(a+\Delta)+n(b-\Delta))}{b+a} - 1.$$

Let $T = V(G_2)$ and $S = V(G_1)$, we get

$$\begin{aligned} & f(S) - g(T) + d_{G-S}(T) - \max_{U\subseteq S, |U|=n', H\subseteq E(G-U), |H|=m}\{f(U) - e_H(T,S) + \sum_{x\in T} d_H(x)\} \\ = \; & b|S| - a|T| - (a+\Delta)n' = -a < 0. \end{aligned}$$

According to Lemma 1, G isn't fractional (g, f, n', m)-critical deleted.

2.4 Specific case in setting $(g, f) = (a, b)$

According to the techniques in the proof of Lemma 2, we infer a likely conclusion for a graph without non-adjacent vertices.

Lemma 3 *Assume G is a complete graph having n vertices, and b, n', a, m, Δ are non-negative integers meeting $n > \frac{(b+a-2+2m)(b+a)}{\Delta+a} + n'$ where $b - \Delta \geq a \geq 2$. Then G is fractional (a, b, n', m)-critical deleted.*

We arrive the corollary below by setting $n' = 0$ in Lemma 3, which is a sufficient condition for a fractional (a, b, m)-deleted complete graph.

Corollary 2 *Assume complete graph G having n vertices, and b, a, m, Δ are non-negative integers meeting $n > \frac{(b+a-2+2m)(b+a)}{\Delta+a}$ where $b - \Delta \geq a \geq 2$. Then G is fractional (a, b, m)-deleted.*

Note that Lemma 3 and Corollary 2 here are the extension results for the corresponding conclusions in Gao et al. [6].

Set $f(x) = b$, $g(x) = a$ for arbitrary vertex x in G. The necessary and sufficient condition is achieved from Lemma 1.

Lemma 4 *Assume G is a graph, b, a, n', and m are non-negative integers meeting $b \geq a$. Therefore, G is fractional (a, b, n', m)-critical deleted iff for any disjoint subsets $T, S \subseteq V(G)$ with $|S| \geq n'$, we have*

$$b|S| + d_{G-S}(T) - a|T| \geq \max_{|H|=m}\{(a+\Delta)n' + \sum_{x\in T} d_H(x) - e_H(T,S)\}. \tag{7}$$

Table 4: Degree conditions in fractional (a, b, n', m)-critical deleted setting

order of graph	degree condition	additional condition
$n > \frac{(b+a-2+2m)(b+a)}{\Delta+a} + n'$	$\delta(G) \geq \frac{(a+\Delta)n'+(b-\Delta)n}{b+a}$	
$n > \frac{(b+a-2+2m)(b+a)}{\Delta+a} + n'$	$\max\{d_G(x), d_G(y)\} \geq \frac{(\Delta+a)n'+(b-\Delta)n}{b+a}$	$\frac{b(b-\Delta)}{\Delta+a} + m + n' \leq \delta(G)$
$n > \frac{(b+a-2+2m)(b+a)}{\Delta+a} + n'$	$\sigma_2(G) \geq \frac{2(n'(\Delta+a)+n(b-\Delta))}{b+a}$	$\frac{b(b-\Delta)}{\Delta+a} + m + n' \leq \delta(G)$

Table 5: Degree conditions in fractional (a, b, m)-deleted setting

order of graph	degree condition	additional condition
$n > \frac{(b+a-2+2m)(b+a)}{\Delta+a}$	$\delta(G) \geq \frac{(b-\Delta)n}{b+a}$	
$n > \frac{(b+a-2+2m)(b+a)}{\Delta+a}$	$\max\{d_G(x), d_G(y)\} \geq \frac{(b-\Delta)n}{b+a}$	$\delta(G) \geq \frac{b(b-\Delta)}{\Delta+a} + m$
$n > \frac{(b+a-2+2m)(b+a)}{\Delta+a}$	$\sigma_2(G) \geq \frac{2n(b-\Delta)}{a+b}$	$\delta(G) \geq \frac{b(b-\Delta)}{\Delta+a} + m$

Using Lemma 3 and Lemma 4, in view of the approaches used in Subsection 2.1 and Subsection 2.2, we deduce the degree conditions depicted in Table 4 in fractional (a, b, n', m)-critical deleted setting, which are corresponding to Theorem 1-3. We omit the detailed proof.

Again, three theorems above present the new extension versions of Theorem 7-9 in Gao et al. [6], respectively. Moreover, the example in Subsection 2.3 shows that these degree conditions in Table 4 are tight.

In particular, by taking $n' = 0$ in Table 4, the corresponding degree conditions in fractional (a, b, m)-deleted setting are obtained in Table 5.

3 Proof of second part results: Theorem 4-6

Since $\delta(G) \geq \frac{n(a+b)}{2a+\Delta+b}$ in Theorem 4 implies $\delta(G) \geq \frac{(\Delta+a)n}{b+2a+\Delta} + m + \frac{(b-\Delta)b}{\Delta+a}$ and $\sigma_2(G) \geq \frac{2n(b+a)}{2a+\Delta+b}$ in Theorem 6, it is sufficient for the proof of Theorem 5-6.

3.1 Correctness of Theorem 5-6

Here, first let's prove Theorem 5. Let $G' = G - I$ for arbitrary independent set I. The conclusion is deduced by making sure that G' meets Table 2 or Corollary 1.

If every two vertices has an edge in G', we obtain

$$|G'| \geq \frac{n(b+a)}{b+2a+\Delta} > \frac{(b+a-1+2m)(b+a)}{a+\Delta} > \frac{(b+2m+a-2)(b+a)}{\Delta+a}.$$

The conclusion holds in light of Corollary 1.

If I only contain one vertex, we yield $|V(G')| > \frac{(b+2m+a-1)(b+2a+\Delta)-\Delta-a}{\Delta+a} > \frac{(b+a-1+2m)(b+a)}{\Delta+a}$. Hence, $\delta(G') \geq \frac{(b-\Delta)b}{a+\Delta} + m$ and

$$\max\{d_{G'}(u), d_{G'}(v)\} \geq \frac{|V(G')|(b-\Delta)}{b+a} = \frac{(n-1)(b-\Delta)}{b+a}$$

for any $uv \notin E(G')$. Hence, the result obtained in view of Table 2.

If $|I| \geq 2$ and G' isn't complete. Applying degree condition, we infer $|V(G')| \geq \frac{(b+a)n}{2a+b+\Delta} > \frac{(b+a)(b+2m+a-1)}{\Delta+a}$. If $\max\{d_{G'}(u), d_{G'}(v)\} < \frac{|V(G')|(b-\Delta)}{b+a}$ for some $uv \notin E(G')$, we arrive

$$\frac{(|V(G')| + |I|)(b+a)}{b+2a+\Delta} \leq \max\{d_G(v), d_G(u)\} < \frac{|V(G')|(b-\Delta)}{b+a} + |I|,$$

which implies

$$|V(G')| < \frac{(a+\Delta)(b+a)}{a^2+\Delta(2a+\Delta)}|I| \leq \frac{(b+a)(a+\Delta)}{a^2+\Delta(2a+\Delta)} \frac{n(\Delta+a)}{2a+\Delta+b} = \frac{(b+a)n}{2a+\Delta+b}.$$

Table 6: Degree conditions in fractional ID-(a, b, m)-deleted setting

order of graph	degree condition	additional condition
$n > \frac{(b+2a+\Delta)(b+a-2+2m)}{\Delta+a}$	$\delta(G) \geq \frac{n(b+a)}{2a+\Delta+b}$	
$n > \frac{(b+2a+\Delta)(b+a+2m-1)}{\Delta+a}$	$\max\{d_G(x), d_G(y)\} \geq \frac{(b+a)n}{b+2a+\Delta}$	$\delta(G) \geq \frac{(\Delta+a)n}{b+2a+\Delta} + \frac{b(b-\Delta)}{\Delta+a} + m$
$n > \frac{(b+a-2+2m)(b+2a+\Delta)}{\Delta+a}$	$\sigma_2(G) \geq \frac{2(b+a)n}{b+2a+\Delta}$	$\delta(G) \geq \frac{(\Delta+a)n}{b+2a+\Delta} + \frac{b(b-\Delta)}{a+\Delta} + m$

It contradicts $|I| \geq 2$ and $\max\{d_G(v), d_G(u)\} \geq \frac{(b+a)n}{b+2a+\Delta}$. Thus,

$$\max\{d_{G'}(v), d_{G'}(u)\} \geq \frac{|V(G')|(b-\Delta)}{b+a}$$

for any $uv \notin E(G')$. Further, we get $m + \frac{b(b-\Delta)}{\Delta+a} \leq \delta(G')$ in view of $|I| \leq \frac{(a+\Delta)n}{2a+b+\Delta}$ and $\frac{n(\Delta+a)}{b+2a+\Delta} + m + \frac{b(b-\Delta)}{a+\Delta} \leq \delta(G)$. Therefore, the result is obtained from Table 2.

Hence, we finish the proof of Theorem 5. By means of Table 2 and Corollary 1, Theorem 6 can be checked in the similar techniques. We omit the detailed procedure. □

3.2 Tight of results

To show the tight of Theorem 4, Theorem 5 and Theorem 6, we need the following lemma follows from the corollary of Lemma 1.

Lemma 5 *Assume G is a graph, functions g, f are integer-valued on its vertex set meeting $f(x) \geq g(x)$ for every vertex x in G. Set $m \in \mathbb{N}^+ \cup \{0\}$. Then G is fractional (g, f, m)-deleted iff for all disjoint subsets $T, S \subseteq V(G)$, we have*

$$f(S) + d_{G-S}(T) - g(T) \geq \max_{|H|=m}\{\sum_{x\in T} d_H(x) - e_H(T, S)\}. \quad (8)$$

Let $b = a+\Delta$. Take $G = (bt+1)K_1 \vee K_{at} \vee (bt+1)K_1$, where $t \in \mathbb{N}$ is a large number. Apparently, $n = 2 + (2b + a)t$. Set $f(x) = b$ and $g(x) = a$ for any vertex x in G. We have

$$\frac{(b+a)n}{b+\Delta+2a} > \delta(G) = (b+a)t + 1 > \frac{(b+a)n}{b+\Delta+2a} - 1,$$

$$\frac{(b+a)n}{b+\Delta+2a} > \max\{d_G(u), d_G(v)\} = (b+a)t + 1 > \frac{(b+a)n}{b+\Delta+2a} - 1,$$

$$\frac{2(b+a)n}{b+\Delta+2a} > \sigma_2(G) = 2 + 2(b+a)t > \frac{2(b+a)n}{b+\Delta+2a} - 1.$$

Let $I = (bt+1)K_1$. For $G' = K_{at} \vee (bt+1)K_1$, let $S = K_{at}$ and $T = (bt+1)K_1$. We confirm that $e_H(T, S) = \sum_{x\in T} d_H(x)$ for arbitrary $H \subseteq E(G')$ having m edges. As a result,

$$f(S) + d_{G-S}(T) - g(T) - (\sum_{x\in T} d_H(x) - e_H(T, S)) = b(at) - a(bt+1) = -a.$$

To sum up, G isn't fractional ID-(g, f, m)-deleted due to Lemma 5 and G' isn't fractional (g, f, m)-deleted.

3.3 Specific case in setting $(g, f) = (a, b)$

The below degree conditions in Table 6 in setting $g(x) = a$ and $f(x) = b$ are derived in terms of Corollary 2, Table 5, and the approaches in Subsection 2.4 and Subsection 3.1.

One important thing we emphasize here is that the results presented in Table 6 are also the extensions of Theorem 10-12 in Gao et. al. [6]. Moreover, in terms of the example presented in Subsection 3.2, we ensure that these degree conditions in Table 6 are also tight.

4 Conclusion

In our work, we mainly discuss the degree conditions for the existence of fractional factor in the setting that $b - \Delta \geq f(x) - \Delta \geq g(x) \geq a$ for each vertex x in G, and some elements of graph are forbidden. Our results reveal that Δ is a key factor in this setting, and it specifically points out how Δ plays a role in the conclusion.

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References

[1] J. A. Bondy, U. S. R. Murty, Graph theory, Springer Press, Berlin, 2008.

[2] W. Gao, Some results on fractional deleted graphs, PhD thesis, Soochow University, Suzhou, China, 2012.

[3] W. Gao, D. Dimitrov, H. Abdo, Tight independent set neighborhood union condition for fractional critical deleted graphs and ID deleted graphs, *Discrete Cont. Dyn. S.-S*, 2019, 12, 711–721.

[4] W. Gao, J. L. G. Guirao, M. Abdel-Aty, W. F. Xi, An independent set degree condition for fractional critical deleted graphs, *Discrete Cont. Dyn. S.-S*, 2019, 12, 877–886.

[5] W. Gao, J. L. G. Guirao, H. L. Wu, Two tight independent set conditions for fractional (g, f, m)-deleted graphs systems, *Qual. Theory Dyn. Syst.*, 2018, 17, 231–243.

[6] W. Gao, L. Liang, T. W. Xu, J. X. Zhou, Degree conditions for fractional (g, f, n', m)-critical deleted graphs and fractional ID-(g, f, m)-deleted graphs, *Bull. Malays. Math. Sci. Soc.*, 2016, 39, 315–330.

[7] W. Gao, Y. Q. Zhang, Y. J. Chen, Neighborhood condition for all fractional (g, f, n', m)-critical deleted graphs, *Open Phys.*, 2018, 16, 544–553.

[8] M. Knor, R. Škrekovski, A. Tepeh, Convexity result and trees with large Balaban index, *Appl. Math. Nonl. Sc.*, 2018, 3, 433–446.

[9] D. L. Liu, C. X. Wang, S. H. Wang, Hamilton-connectivity of interconnection networks modeled by a product of graphs, *Appl. Math. Nonl. Sc.*, 2018, 3, 419–426.

[10] S. Z. Zhou, Remarks on orthogonal factorizations of digraphs, *Int. J Comput. Math.*, 2014, 91, 2109–2117.

[11] S. Z. Zhou, Some results about component factors in graphs, *RAIRO-Oper. Res.*, https://doi.org/10.1051/ro/2017045.

[12] S. Z. Zhou, Q. X. Bian, An existence theorem on fractional deleted graphs, *Period. Math. Hungar.*, 2015, 71, 125–133.

[13] S. Z. Zhou, Z. R. Sun, Neighborhood conditions for fractional ID-k-factor-critical graphs, *Acta Math. Appl. Sin-E*, 2018, 34, 636–644.

[14] S. Z. Zhou, Z. R. Sun, Y. Xu, A theorem on fractional ID-(g, f)-factor-critical-graphs, *Contributions to Discrete Mathematics*, 2015, 10, 31–38.

[15] S. Z. Zhou, F. Yang, L. Xu, Two sufficient conditions for the existence of path factors in graphs, *Sci. Iran.*, DOI: 10.24200/SCI.2018.5151.1122.

[16] S. Z. Zhou, T. Zhang, Some existence theorems on all fractional (g, f)-factors with prescribed properties, *Acta Math. Appl. Sin-E.*, 2018, 34, 344–351.