# TOPOLOGICAL ENTROPY OF CONTINUOUS SELF–MAPS ON CLOSED SURFACES

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Abstract. The objective of the present work is to present sufficient conditions for having positive topological entropy for continuous self–maps defined on a closed surface by using the action of this map on the homological groups of the closed surface.

## 1. Introduction

Along this work by a *closed surface* we denote a connected compact surface with or without boundary, orientable or not. More precisely, *an orientable connected compact surface without boundary of genus* $g \geq 0$, $\mathbb{M}_g$, is homeomorphic to the sphere if $g = 0$, to the torus if $g = 1$, or to the connected sum of $g$ copies of the torus if $g \geq 2$. *An orientable connected compact surface with boundary of genus* $g \geq 0$, $\mathbb{M}_{g,b}$, is homeomorphic to $\mathbb{M}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows $\mathbb{M}_{g,0} = \mathbb{M}_g$.

*A non–orientable connected compact surface without boundary of genus* $g \geq 1$, $\mathbb{N}_g$, is homeomorphic to the real projective plane if $g = 1$, or to the connected sum of $g$ copies of the real projective plane if $g > 1$. *A non–orientable connected compact surface with boundary of genus* $g \geq 1$, $\mathbb{N}_{g,b}$, is homeomorphic to $\mathbb{N}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows $\mathbb{N}_{g,0} = \mathbb{N}_g$.

Let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map on a closed surface $\mathbb{X}$. A point $x \in \mathbb{X}$ is periodic of period $n$ if $f^n(x) = x$ and $f^k(x) \neq x$ for $k = 1, \ldots, n-1$.

The *topological entropy* of a continuous map $f : \mathbb{X} \to \mathbb{X}$ denoted by $h(f)$ is a non–negative real number (possibly infinite) which measures how much $f$ mixes up the phase space of $\mathbb{X}$. When $h(f)$ is positive the dynamics of the system is said to be *complicated* and the positivity of $h(f)$ is used as a measure of the so called *topological chaos*.

Here we introduce the topological entropy using the definition of Bowen [4].

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Since it is possible to embedded any surface orientable or not in $\mathbb{R}^4$ by the Whitney immersion theorem see [11], we consider the distance between two points of $\mathbb{X}$ as the distance of these two points in $\mathbb{R}^4$. Now, we define the distance $d_n$ on $G$ by

$$d_n(x,y) = \max_{0\le i\le n} d(f^i(x), f^i(y)), \quad \forall x,y \in G.$$

A finite set $S$ is called $(n,\varepsilon)$*–separated with respect to* $f$ if for different points $x,y\in S$ we have $d_n(x,y)>\varepsilon$. We denote by $S_n$ the maximal cardinality of an $(n,\varepsilon)$–separated set. Define

$$h(f,\varepsilon) = \limsup_{n\to\infty} \frac{1}{n}\log S_n.$$

Then

$$h(f) = \lim_{\varepsilon\to 0} h(f,\varepsilon)$$

is the *topological entropy* of $f$.

We have chosen the definition by Bowen because, probably it is the shorter one. The classical definition was due to Adler, Konheim and McAndrew [1]. See for instance the book of Hasselblatt and Katok [7] and [3] for other equivalent definitions and properties of the topological entropy. See [1, 2, 12] for more details on the topological entropy.

Our main results are the following.

**Theorem 1.** *Let $\mathbb{M}_g$ be an orientable connected compact surface without boundary of genus $g$. Then the following statements hold.*

(a) *If the degree $d \notin \{-1,0,1\}$, then the topological entropy of $f$ is positive.*
(b) *If the degree $d \in \{-1,0,1\}$, $2g$ is even and the number of roots of the characteristic polynomial of $f_{*1}$ equal to $\pm1$ or $0$ taking into account their multiplicities is not even, then the topological entropy of $f$ is positive.*
(c) *If the degree $d \in \{-1,0,1\}$, $2g$ is odd and the number of roots of the characteristic polynomial of $f_{*1}$ equal to $\pm1$ or $0$ taking into account their multiplicities is not odd, then the topological entropy of $f$ is positive.*

**Theorem 2.** *Let $\mathbb{M}_{g,b}$, $b>0$, be an orientable connected compact surface with boundary of genus $g$. Then the following statements hold.*

(a) *If $2g+b-1$ is even and the number of roots of the characteristic polynomial of $f_{*1}$ equal to $\pm1$ or $0$ taking into account their multiplicities is not even, then the topological entropy of $f$ is positive.*
(b) *If $2g+b-1$ is odd and the number of roots of the characteristic polynomial of $f_{*1}$ equal to $\pm1$ or $0$ taking into account their multiplicities is not odd, then the topological entropy of $f$ is positive.*

**Theorem 3.** *Let $\mathbb{N}_{g,b}$, $b \geq 0$, be a non–orientable connected compact surface with boundary of genus $g$. Then the following statements hold.*

(a) *If $g+b-1$ is even and the number of roots of the characteristic polynomial of $f_{*1}$ equal to $\pm 1$ or $0$ taking into account their multiplicities is not even, then the topological entropy of $f$ is positive.*
(b) *If $g+b-1$ is odd and the number of roots of the characteristic polynomial of $f_{*1}$ equal to $\pm 1$ or $0$ taking into account their multiplicities is not odd, then the topological entropy of $f$ is positive.*

## 2. Lefschetz zeta functions for surfaces

Let $f$ be a continuous self–map defined on $\mathbb{M}_{g,b}$ or $\mathbb{N}_{g,b}$, respectively. For a closed surface the homological groups with coefficients in $\mathbb{Q}$ are linear vector spaces over $\mathbb{Q}$. We recall the homological spaces of $\mathbb{M}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$, $n_1 = 2g + b - 1$ if $b > 0$, $n_2 = 1$ if $b = 0$, and $n_2 = 0$ if $b > 0$; and the induced linear maps $f_{*k} : H_k(\mathbb{M}_{g,b}, \mathbb{Q}) \to H_k(\mathbb{M}_{g,b}, \mathbb{Q})$ by $f$ on the homological group $H_k(\mathbb{M}_{g,b}, \mathbb{Q})$ are $f_{*0} = (1)$, $f_{*2} = (d)$ where $d$ is the *degree* of the map $f$ if $b = 0$, $f_{*2} = (0)$ if $b > 0$, and $f_{*1} = A$ where $A$ is an $n_1 \times n_1$ integral matrix (see for additional details [10, 13]).

We recall that the homological groups of $\mathbb{N}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{N}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = g + b - 1$ and $n_2 = 0$; and the induced linear maps are $f_{*0} = (1)$ and $f_{*1} = A$ where $A$ is an $n_1 \times n_1$ integral matrix (see again for additional details [10, 13]).

Let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map and let $\mathbb{X}$ be either $\mathbb{M}_{g,b}$ or $\mathbb{N}_{g,b}$. Then the *Lefschetz number* of $f$ is defined by

$$L(f) = \text{trace}(f_{*0}) - \text{trace}(f_{*1}) + \text{trace}(f_{*2}).$$

We shall use the Lefschetz numbers of the iterates of $f$, i.e. $L(f^n)$. In order to study the whole sequence $\{L(f^n)\}_{n\geq 1}$ it is defined the formal *Lefschetz zeta function* of $f$ as

$$Z_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n\right).$$

The Lefschetz zeta function is in fact a generating function for the sequence of the Lefschetz numbers $L(f^n)$.

From the work of Franks in [6] we have for a continuous self–map of a closed surface that its Lefschetz zeta function is the rational function

$$Z_f(t) = \frac{\det(I - tf_{*1})}{\det(I - tf_{*0})\det(I - tf_{*2})}, \tag{1}$$

where in $I - tf_{*k}$ the $I$ denotes the $n_k \times n_k$ identity matrix, and $\det(I - tf_{*2}) = 1$ if $f_{*2} = (0)$. Then for a continuous map $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ we have

$$Z_f(t) = \begin{cases} \dfrac{\det(I - tA)}{(1-t)(1-dt)} & \text{if } b = 0, \\ \dfrac{\det(I - tA)}{1-t} & \text{if } b > 0, \end{cases}$$

and for a continuous map $f : \mathbb{N}_{g,b} \to \mathbb{N}_{g,b}$ we have

$$Z_f(t) = \frac{\det(I - tA)}{1-t}.$$

## 3. Basic results

In this section we present the main result stated in Theorem 7 for proving Theorems 1, 2 and 3. Since its proof is short and important for this work we provide it here.

For a polynomial $H(t)$ we define $H^*(t)$ by

$$H(t) = (1-t)^\alpha (1+t)^\beta t^\gamma H^*(t),$$

where $\alpha$, $\beta$ and $\gamma$ are non-negative integers such that $1-t$, $1+t$ and $t$ do not divide $H^*(t)$.

The *spectral radii* of the maps $f_{*k}$ are denoted $\mathrm{sp}(f_{*k})$, and they are equal to the largest modulus of all the eigenvalues of the linear map $f_{*k}$. The *spectral radius of* $f_*$ is

$$\mathrm{sp}(f_*) = \max_{k=0,\dots,m} \mathrm{sp}(f_{*k}).$$

The next result is due to Manning [9].

**Theorem 4.** *Let $f : X \to X$ be a continuous map on a closed surface $\mathbb{X}$. Then $\log \max\{1, \mathrm{sp}(f_{*1})\} \le h(f)$.*

**Lemma 5.** *Let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map and let $\mathbb{X}$ be a closed surface. If the topological entropy of $f$ is zero, then all the eigenvalues of the induced homomorphism $f_{*1}$ are zero or root of unity.*

*Proof.* Since the topological entropy is zero, by Theorem 4 we have $\mathrm{sp}(f_{*1}) = 1$. So, all the eigenvalues of $f_{*1}$ have modulus in the interval $[0, 1]$ and at least one of them is 1. Then the characteristic polynomial of $f_{*1}$ is of the form $t^m p(t)$, where $m$ is a non–negative integer, positive if the zero is an eigenvalue. And $p(t)$ is a polynomial with integer coefficients and whose independent term $a_0$ is non–zero. Since the product of all non-zeros eigenvalues of $f_{*1}$ is the integer $a_0$ and, these eigenvalues have modulus in $(0, 1]$, we have that any of these eigenvalues can have modulus smaller than one, otherwise we are in contradiction with the fact $a_0$ is an integer. In short, all the non–zero eigenvalues have modulus one, and consequently $a_0 = 1$.

Since if a polynomial has integer coefficients, constant term 1 and all of whose roots have modulus 1, then all of its roots are roots of unity, see [14], the lemma follows. ☐

The $n$–th cyclotomic polynomial is defined by

$$c_n(t) = \prod_k (w_k - t),$$

being $w_k = e^{2\pi i k/n}$ a primitive $n$–th root of unity and where $k$ runs over all the relative primes $\leq n$. See [8] for the properties of these polynomials.

For a positive integer $n$ the *Euler function* is $\varphi(n) = n \prod_{p|n, p \text{ prime}} \left(1 - \frac{1}{p}\right)$. It is known that the degree of the polynomial $c_n(t)$ is $\varphi(n)$. Note that $\varphi(n)$ is even for $n > 2$.

A proof of the next result can be found in [8].

**Proposition 6.** *Let $\xi$ be a primitive $n$–th root of the unity and $P(t)$ a polynomial with rational coefficients. If $P(\xi) = 0$ then $c_n(t)|P(t)$.*

The proofs of our results are strongly based in the next theorem.

**Theorem 7** (Theorem 3.2 of [5])**.** *Let $\mathbb{X}$ be a closed surface, $f : \mathbb{X} \to \mathbb{X}$ be a continuous self–map, and let $\mathcal{Z}_f(t) = P(t)/Q(t)$ be its Lefschetz zeta function. If $P^*(t)$ or $Q^*(t)$ has odd degree, then the topological entropy of $f$ is positive.*

*Proof.* From the definitions of a polynomial $H^*$ and of the Lefschetz zeta function we have

$$\mathcal{Z}_f(t) = \frac{P(t)}{Q(t)} = (1-t)^a(1+t)^b t^c \frac{P^*(t)}{Q^*(t)},$$

where $a, b$ and $c$ are integers.

Assume now that the topological entropy $h(f) = 0$. Then by Lemma 5 all the eigenvalues of the induced homomorphisms $f_{*1}$'s are zero or roots of unity. Therefore, by (1) all the roots of the polynomials $P^*(t)$ and $Q^*(t)$ are roots of the unity different from $\pm 1$ and zero. Hence, by Proposition 6 the polynomials $P^*(t)$ and $Q^*(t)$ are product of cyclotomic polynomials different from $c_1(t) = 1 - t$ and $c_2(t) = 1 + t$. Consequently $P^*(t)$ and $Q^*(t)$ have even degree because all the cyclotomic polynomials which appear in them have even degree due to the fact that the Euler function $\varphi(n)$ for $n > 2$ only takes even values. But this is a contradiction with the assumption that $P^*(t)$ or $Q^*(t)$ has odd degree. ☐

## 4. Proof of Theorems 1, 2 and 3

*Proof of Theorem 1.* Since $\mathbb{M}_g$ is an orientable connected compact surface without boundary of genus $g$, then the Lefschetz zeta function of $f$ is equal to

$$Z_f(t) = \frac{\det(I - tA)}{(1-t)(1-dt)},$$

where $d$ is the degree of $f$ and $2g$ is the dimension of the characteristic polynomial $\det(I - tA)$ of $f_{*1} = A$. Note here that if $d \notin \{-1, 0, 1\}$, then $Q^*(t) = 1 - dt$ and therefore by Theorem 7 statement (a) of Theorem 1 is proved.

Assume now that $d \in \{-1, 0, 1\}$. Note that in this case $Q(t) = (1-t)(1-dt)$ and $Q^*(t) = 1$. So, by Theorem 7 the main role will be play by the $2g$ degree polynomial $P(t) = \det(I - tA)$ where $f_{*1} = A$. If $2g$ is even and the number of roots of the characteristic polynomial of $f_{*1}$ equal to $\pm 1$ or 0 taking into account their multiplicities is not even, then $P^*(t)$ has odd degree. Therefore, statement (b) of Theorem 1 follows by the application of Theorem 7.

On the other hand, if $2g$ is odd and the number of roots of the characteristic polynomial of $f_{*1}$ equal to $\pm 1$ or 0 taking into account their multiplicities is not odd then, $P^*(t)$ has odd degree and as before the proof of statement (c) of Theorem 1 follows. □

*Proof of Theorem 2.* Note now, since $\mathbb{M}_{g,b}$ is an orientable connected compact surface with boundary $(b > 0)$ of genus $g$, then the Lefschetz zeta function of $f$ is equal to

$$Z_f(t) = \frac{\det(I - tA)}{1 - t}$$

being $2g + b - 1$ the degree of the characteristic polynomial $\det(I - tA)$ of $f_{*1} = A$. Now the proof is similar to the statements (b) and (c) of Theorem 1. □

*Proof of Theorem 3.* For a non–orientable connected compact surface with or without boundary $(b \geq 0)$ of genus $g \geq 1$, the Lefschetz zeta function of $f$ is equal to

$$Z_f(t) = \frac{\det(I - tA)}{1 - t}$$

being $g + b - 1$ the degree of the characteristic polynomial $\det(I - tA)$ of $f_{*1} = A$. Then the proof if this theorems follows in a similar way to the proof of statements (b) and (c) of Theorem 1. □

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