The boundary control strategy for a fractional wave equation with external disturbances *

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This paper is concerned with the boundary control strategies of the fractional order wave equation with the boundary subject to persistent external disturbances in Hilbert spaces. By extending the sliding mode control approach to fractional order infinite-dimensional systems, the fractional order boundary sliding mode control is designed for the infinite dimensional setting. And based on the globally asymptotic stability theorem, it's applied for addressing the asymptotical stability of the state for the fractional order wave equation with an uncertain boundary. Finally, numerical simulations are presented to verify the viability and efficiency of the proposed fractional order controllers.

Keywords: The boundary control strategies; Fractional order uncertain wave equation; External disturbance; Sliding mode control.

1. Introduction

In practical engineering, many systems are described by partial differential equations, and these syetems are often subject to a significant degree of uncertainty.[1] As a class of distributed parameter systems, wave equation has been studied sufficiently. Over the past decade, the modeling and control of several classes of wave process have been researched as a hot point, and there has been emerged a considerable amount of results.[2,3] In particular, when the external disturbance exists on boundary, the boundary control has received a special attention.[4–6] Therefore, plentiful control methods has been applied to deal with the uncertainties such as the internal model principle for output regulation, the adaptive control for systems with unknown parameters, and the active disturbance rejection control method, to name just a few. Smyshlyaev[7] introduced a new integral transformation for

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wave equations and used it to obtain explicit controllers and observers for a wave equation with negative damping at the boundary. Liu[8] investigated a cascade of ODE-wave systems with the control actuator matched disturbance at the boundary of the wave equation by use of the sliding mode control (SMC) technique and the active disturbance rejection control (ADRC) method to overcome the disturbance, respectively.

It's well-known that, when focused on the system with uncertainty and external disturbance, sliding mode control (SMC) approach[9–11] is a noted control technique. The main advantage of the SMC is to switch the control law to force the states of the system from the initial states onto some predefined sliding mode surface, and the system on the sliding mode surface has desirable properties such as fast response, low sensitivity to external noises, robustness to the system uncertainties, and effortless realization and so on. Therefore, the sliding mode control has been recognized as a powerful control method to exhibit strong properties of robustness against significant classes of disturbances and model uncertainties, and has been used to address the control problems of variable systems.[12] Recently, a growing interest has emerged in extending sliding mode control to infinite-dimensional systems, and the study of sliding mode control for the distributed parameter systems has been a hot spot topic.[13,14]

Due to the advantage and simplicity of implementation for the sliding mode control method, it has widely been used in various situations. And in recent years, it has been applied to deal with the boundary control of the wave process. Baccoli[15] designed the combined twisting/PD algorithm which are shown to be capable of regulating uncertain and perturbed wave and reaction-diffusion processes based on the second-order sliding mode boundary control techniques. Yury Orlov[16] designed a boundary controller in an infinite dimensional systems setting based on a second-order sliding mode control algorithm, and the controller has been shown to provide for the regulation of an uncertain and perturbed wave process.

With the development of viscous-elastic material, it was shown that the viscous-elastic damping can be described by fractional differential model, and many of the physical laws are necessary to be described in terms of fractional calculus, thus, much attention has been drawn to the study of fractional order damping,[17] and many systems and industrial processes in practical engineering are governed by partial differential equations with fractional order operator, one of which is the fractional order wave equation. The fractional order wave equations are such systems that are obtained from the classical wave equations by replacing the second-order time derivative with a fractional order. And the robust control problem of fractional order wave process has attracted the attention of scientists and engineers from many fields such as mathematics, physics and engineering,[18–20] in particular, the boundary control of the fractional order wave equations. Pisano et al.[21] extended the so-called twisting and supertwisting 2-SM control algorithms to globally asymptotically stabilize uncertain wave and, respectively, heat equations under Dirichlet and Neumann boundary conditions. Dai et al.[22] considered the wave equation with boundary source term and fractional boundary dissipation, and proved the exponential growth for sufficiently large initial data. Liang et al.[23] investigated the integer order and fractional order boundary control laws of a fractional order wave equation. Based on the delayed boundary measurements and the Smith predict, the boundary controller was designed. Besides, for the first time, the authors confirmed that small time delay in boundary control law could destabilize the controlled system through extensive hybrid symbolic and numerical simulation, combined with parameter optimization; In Ref.,[24] Liang et al. considered a boundary controller at the boundary for the one-dimensional fractional diffusion-wave equation and confirmed the existing schemes that were focused on the boundary stabilization and disturbance rejection for integer order wave equations, which were still valid for fractional order diffusion-wave equations via hybrid symbolic and numerical simulation studies. Mbodje et al.[25] studied the wave equation with fractional derivative feedback at the boundary and proved the asymptotic decay of the solution. However, up to now, there are few achievements involving the boundary control of the fractional order wave equation via sliding mode approach when the external disturbance on boundary exists. Motivated by the boundary control of integer order wave equations and the challenge in the design of control strategies for distributed parameter systems,[16] this paper mainly focuses on the boundary control of wave equation with fractional order derivative.

The rest of the paper is outlined as follows. In Section 2, basic definitions and preliminaries for the boundary control of the wave process are provided. Section 3 is devoted to introducing the boundary control of wave process, and showing that it can guarantees the global asymptotic stability of the system. In Section 4, simulation example is presented to illustrate the effectiveness of the proposed methods. Section 5 discusses the main features of the developed schemes and the promising direction of investigation for possible extensions of the obtained results. The last section concludes the obtained results of the present paper.

2. Basic definitions and preliminaries

The notation used in this paper is fairly standard. $H^l(0,1)$ with $l = 0,1,2,...,$ denotes the Sobolev space of defined on $(0,1)$, and $H^0(0,1) = L^2(0,1) = \left\{ z : ||z(\cdot)||_2 = \sqrt{\int_0^1 z^2(\tau)d\tau} \right\}$ stands for the square integrable functions space and $L_\infty(0,1)$ is a subspace of $L^2(0,1)$ with the norm $||z(\cdot)||_\infty = \max_{0\le t\le 1} z(t)$, for the detail, see Ref.[26]

Consider a class of uncertain infinite dimensional systems which is governed by a perturbed version of the wave equation:

$$ {}^C D^\alpha y(x,t) = \theta y_{xx}(x,t) \tag{2.1} $$

$$ y(x,0) = y^0(x) \in H^2(0,1),\ \ y_t(x,0) = y_t^0(x) \in H^2(0,1) \tag{2.2} $$

$$y_x(0,t) = c_0 D_t^r y(0,t),\ y_x(1,t) = u(t) + \psi(t) \tag{2.3}$$

Here (2.2) is the initial condition, (2.3) is the controlled and perturbed Neumann boundary condition. $u(t) \in L_2(0,1)$ is the boundary control input and $\psi(t) \in L_2(0,1)$ represents an uncertain disturbance source term. And c_0 is a positive constant, ${}^C D_t^{\alpha},\ 1 < \alpha \leq 2,\ r = \alpha/2$ is the Caputo-type fractional order derivative, $x \in [0,1]$ is the one dimensional space variable, $t > 0$ is the time variable. The coefficient $\theta \in \mathbb{R}$ stands for the elasticity. Let $H = L^2(0,1) \times L^2(0,1)$ equipped with the norm $||(y, {}^C D_t{}^{\alpha/2} y)||_H = ||y(\cdot,t)||_2 + ||{}^C D_t{}^{\alpha/2} y(\cdot,t)||_2$, where $(y, {}^C D_t{}^r y) \in H$.

Recent years, considerable interest has been shown in the so-called fractional calculus, such interest has been stimulated by the applications that this calculus finds in different areas of physics and engineering, possibly including fractal phenomena. And the calculus generalizes some basic topics of classical mathematical physics, which are treated by simple, linear, ordinary or partial differential equations. The fractional order wave equation is an evolution equation of order $1 < \alpha < 2$ which continues to the wave equation when $\alpha \to 2$.

The fractional derivative and fractional integral[27,28] adopted in this paper are the Caputo-type which are defined as following

$$I_t^q x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} x(s) ds, 0 < q < 1, \tag{2.4}$$

and

$${}^C D_t^q x(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} x'(s) ds, 0 < q < 1. \tag{2.5}$$

The following Lemmas which provides the basis for demonstrating the stability of many fractional order systems was obtained by Jiang, Cao and Chen[20]

Lemma 2.1 *Let $x(t),\ y(t) \in L_2(0,1)$ be continuous and derivable functions. Then, for any time instant $t \geq t_0$, the following inequality holds*

$${}^C D^{\gamma}(x(t)y(t)) \leq x(t) {}^C D^{\gamma} y(t) + y(t) {}^C D^{\gamma} x(t). \tag{2.6}$$

When the space variable x is fixed, the stability of the system (2.1) can be dealt with by the following theorem which also assures the globally asymptotic stability of the equilibrium of the fractional order system (2.1) in a finite time.

Theorem 2.1 *Let $x = 0$ be an equilibrium point for the non-autonomous fractional order system (2.1). Assume that there exists a Lyapunov function $V(t,x)$ satisfying the following conditions:*
1) $V(t,x)$ is positive definite;
2) ${}^C D^{\gamma} V(t,x)$ is negative definite;
3) $V(t,x)$ has an infinite upper-bounded;
4) $V(t,x)$ is radially unbounded;
where $\gamma \in (0,1)$. Then, the system (2.1) has global asymptotic stability;

3. The boundary control for the wave equation

Assume that the state vector $(y, {}^C D_t^r y)$ is available for measurements, then, $y, {}^C D_t^r y$ are also available for feedback. The following is an assumption about the disturbance.

Assumption 1 There exists a priori known constant $M > 0$ such that the unknown disturbance satisfies the following inequality,

$$|\psi(t)| \leq M. \tag{3.1}$$

The initial functions and admissible disturbances are specified by the following assumption.

Assumption 2 The initial functions in the IC's (2.2) are compatible to the boundary condition in the BC's (2.3)

$$y_x^0(0) = c_0 D_t^r y(0,t),\ y_x^0(1) = \psi(0).$$

In order to stabilize the dynamics (2.1), the discontinuous controller is designed as follows:

$$u(t) = -\lambda_1 sign(y(1,t)) - w_1 y(1,t) - \lambda_2 sign(D_t^r y(1,t)) - w_2 D_t^r y(1,t) \tag{3.2}$$

where $u(0) = 0, r = \alpha/2, w_1 > 0, w_2 > 0, \lambda_1,\ \lambda_2 \geq 0$ are constant control coefficients.

Due to the fractional derivative existed in the boundary control law in controller (3.2), compared with the integer order operator, the fractional order derivative is much more complicated due to the unique properties of fractional calculus, such as non-locality, memory-dependence and the power-law. Thus, the fractional order boundary control law has stronger adjustment ability.

From the representation of the controller(3.2), the controller is discontinuities on the two manifolds $y(1,t) = 0$ and $D_t^r y(1,t) = 0$. Thus, the meaning of the corresponding solutions has been given for the discontinuous system. Due to the non-smooth of the controller (3.2), the precise meaning of the solutions can be defined in the generalized sense listed as follows:

Definition 3.1 An absolutely continuous function $y^\delta(\cdot,t) \in L_2(0,1)$, defined on $[0,\tau)$, is said to be an approximate $\delta-$solution of the system(2.1) and (2.2)-(2.3) under the controller (3.2) if it is a strong solution of the corresponding boundary value problem with a continuous approximation $u^\delta(\cdot)$ substituted for the discontinuous control input (3.2) such that $||u^\delta - u||_2 \leq \delta$ for all $y_1, y_2 \in L_2(0,1)$ subject to $||y_1||_2 \geq \delta$ and $||y_2||_2 \geq \delta$, respectively, where $\delta > 0$.

Definition 3.2 An absolutely continuous function $y(\cdot,t) \in L_2(0,1)$, defined on $[0,\tau)$, is said to be a generalized solution of the system (2.1) and (2.2)-(2.3) under the controller (3.2) if there exists a family of approximate $\delta-$ solutions $y^\delta(\cdot,t)$ of the corresponding boundary value problem such that $\lim_{\delta\to 0} ||y^\delta(\cdot,t) - y(\cdot,t)||_2 = 0$, and $\lim_{\delta\to 0} ||{}^C D^\gamma y^\delta(\cdot,t) - {}^C D^\gamma y(\cdot,t)||_2 = 0$, uniformly in $t \in [0,\tau)$.

Similar to the finite dimensional case, a sliding mode is defined as a motion along the discontinuity manifold.

Remark 3.1 In the abstract framework of Hilbert space, the existence of generalized solutions has been established, whereas the uniqueness and well-posedness appear to follow from the fact that, no sliding mode occurs except in the origin $y_1 = y_2 = 0$.

Before giving the main results, we make the following assumptions about the system parameters.

Assumption 3. $\lambda_2 > M$.

Assumption 4. The parameters $\lambda_1, k_2, k_3, \theta$ satisfy the following inequalities

$$\begin{cases} \lambda_1 > \sqrt{\frac{Rk_2}{2\theta^2}}, \\ k_2 + k_3 < 1, \\ k_3 < \theta. \end{cases} \tag{3.3}$$

Assumption 5. Suppose the system parameters satisfy the next inequalities

$$\begin{cases} \lambda_2 > M + \frac{k_2\sqrt{2R}}{\theta}, \\ k_2 + (\frac{1}{c_0\theta} + c_0)k_3 < 2, \\ \lambda_1 > M + \lambda_2, \\ w_1 > \frac{c_0}{2}. \end{cases} \tag{3.4}$$

The following theorem claims the asymptotic stability of the generalized solutions of the wave equation (2.1) and (2.2)-(2.3) under the control strategy (3.2).

Theorem 3.1 *Consider the system* (2.1) *along with the ICs* (2.2) *and BCs* (2.3)*, and assume the parameters and external disturbance satisfy assumption 1, 2, 3, 4 and 5. Then, the control strategy* (3.2) *guarantees the exponential decay of* $||y_1(\cdot,t)||_2$ *and* $||y_2(\cdot,t)||_2$ *of the solutions* (2.1).

Proof. For simplifying the notation, the dependence of the system signals on the space and variables (x, t) is omitted. Firstly, we choose the following Lyapunov functional:

$$V_1(t) = \lambda_1\theta|y(1,t)| + \frac{1}{2}\theta w_1 y^2(1,t) + \frac{1}{2}\|D^r y(\cdot,t)\|^2 + \frac{1}{2}\theta\|y_x(\cdot,t)\|^2 \tag{3.5}$$

Then, taking its fractional derivative with respect to time along the solution of the system (2.1) under the controller (3.2) and by the Lemma 2.1, we have

$$\begin{aligned} D^r V_1(t) \le{}& \lambda_1\theta sign(y(1,t))D^r y(1,t) + \theta w_1 y(1,t)D^r y(1,t) + \int_0^1 D^r y(v,t)D^\alpha y(v,t)dv \\ &+\theta\int_0^1 y_x(v,t)D^r y_x(v,t)dv \\ ={}& \lambda_1\theta sign(y(1,t))D^r y(1,t) + \theta w_1 y(1,t)D^r y(1,t) + \theta\int_0^1 D^r y(v,t)y_{xx}(v,t)dv \end{aligned}$$

$$
\begin{aligned}
&+\theta\int_0^1 y_x(v,t)D^r y_x(v,t)dv\\
&=\lambda_1\theta sign(y(1,t))D^r y(1,t)+\theta w_1 y(1,t)D^r y(1,t)+\theta y_x(1,t)D^r y(1,t)\\
&\quad-\theta y_x(0,t)D^r y(0,t)-\theta\int_0^1 y_x(v,t)D^r y_x(v,t)dv+\theta\int_0^1 y_x(v,t)D^r y_x(v,t)dv\\
&=\lambda_1\theta sign(y(1,t))D^r y(1,t)+\theta w_1 y(1,t)D^r y(1,t)+\theta D^r y(1,t)[u(t)+\psi(t)]\\
&\quad-\theta c_0[D^r y(0,t)]^2\\
&=\lambda_1\theta sign(y(1,t))D^r y(1,t)+\theta w_1 y(1,t)D^r y(1,t)+\theta D^r y(1,t)[-\lambda_1 sign(y(1,t))\\
&\quad-w_1 y(1,t)-\lambda_2 sign(D_t^r y(1,t))-w_2 D_t^r y(1,t)]+\theta D^r y(1,t)\psi(t)\\
&\quad-\theta c_0[D^r y(0,t)]^2\\
&=-\lambda_2\theta|D^r y(1,t)|-w_2\theta|D^r y(1,t)|^2+\theta D^r y(1,t)\psi(t)-\theta c_0|D^r y(0,t)|^2\\
&\le-\theta\left[(\lambda_2-M)|D_t^r y(1,t)|+c_0|D^r y(0,t)|^2+w_2|D_t^r y(1,t)|^2\right]\\
&\le 0. \qquad (3.6)
\end{aligned}
$$

Based on the assumption 3, it implies that the Lyapunov functional $V_1(t)$ is a non-increasing function with time, i,e

$$V_1(t_2)\le V_1(t_1),\forall\ t_2\ge t_1\ge 0.$$

Denote $D_R=\{(y_1,y_2)\in H:V_1(y_1,y_2)\le R\}$. Obviously, when an arbitrary $R\ge V_1(0)$ is fixed, the obtained domain D_R is proved to be invariant. For the purposes of the analysis below, we will consider that the states (y_1,y_2) belong to the domain D_R starting from the initial time $t=0$.

It's clear that the following inequalities hold

$$|y(1,t)|\le\frac{R}{\lambda_1\theta},\ \|D^r y(\cdot,t)\|^2\le 2R,\ \|y_x(\cdot,t)\|^2\le\frac{2R}{\theta}, \qquad (3.7)$$

Now we take into account the augmented functional

$$
\begin{aligned}
V_2(t)=V_1(t)+\frac{1}{2}k_2\theta w_2 y^2(1,t)+k_2\int_0^1 y(1,t)D^r y(v,t)dv\\
+k_3\int_0^1(v-1)y_x(v,t)D^r y(v,t)dv,
\end{aligned}
$$

where the parameters $k_2>0,\ k_3>0$ are to be specified below and $v\in[0,1]$.

Then, it's obtained that

$$
\begin{aligned}
\left|k_2\int_0^1 y(1,t)D^r y(v,t)dv\right|&\le k_2\int_0^1|y(1,t)D^r y(v,t)|dv\\
&\le\frac{k_2}{2}\int_0^1\left[y^2(1,t)+(D^r y(v,t))^2\right]dv\\
&=\frac{k_2}{2}y^2(1,t)+\frac{k_2}{2}\|D^r y(\cdot,t)\|^2\\
&\le\frac{Rk_2}{2\lambda_1\theta}|y(1,t)|+\frac{k_2}{2}\|D^r y(\cdot,t)\|^2. \qquad (3.8)
\end{aligned}
$$

Due to $\max_{v\in[0,1]}|v-1|=1$, the following equalities hold

$$\begin{aligned}|k_3\int_0^1(v-1)y_x(v,t)D^ry(v,t)dv| &= k_3\Big|\int_0^1(v-1)y_x(v,t)D^ry(v,t)dv\Big| \quad (3.9)\\ &\le k_3|\int_0^1 y_x(v,t)D^ry(v,t)dv|\\ &\le \frac{k_3}{2}||y_x(\cdot,t)||^2+\frac{k_3}{2}||D^ry(\cdot,t)||^2.\end{aligned}$$

Thus, the augmented functional $V_2(t)$ has the following inequality

$$\begin{aligned}V_2(t) &\ge V_1(t)+\frac{1}{2}k_2\theta w_2y^2(1,t)-\frac{Rk_2}{2\lambda_1\theta}|y(1,t)|-\frac{k_2}{2}||D^ry(\cdot,t)||^2\\ &-\frac{k_3}{2}||y_x(\cdot,t)||^2-\frac{k_3}{2}||D^ry(\cdot,t)||^2\\ &=(\lambda_1\theta-\frac{Rk_2}{2\lambda_1\theta})|y(1,t)|+(\frac{\theta w_1}{2}+\frac{k_2\theta w_2}{2})y^2(1,t)\\ &+(\frac{1}{2}-\frac{k_2}{2}-\frac{k_3}{2})||D^ry(\cdot,t)||^2+(\frac{\theta}{2}-\frac{k_3}{2})||y_x(\cdot,t)||^2.\end{aligned}$$

According to assumption 4, in the invariant domain D_R, the augmented functional $V_2(t)$ is positive definite and furthermore, since the parameters R can be selected arbitrarily large, then, the augmented functional becomes radially unbounded as $R\to+\infty$. Besides,

$$\begin{aligned}V_2(t) &\le V_1(t)+\frac{1}{2}k_2\theta w_2y^2(1,t)+\frac{k_2}{2}y^2(1,t)+\frac{k_2}{2}||D^ry(\cdot,t)||^2\\ &+\frac{k_3}{2}||y_x(\cdot,t)||_0^2+\frac{k_3}{2}||D^ry(\cdot,t)||_0^2\\ &\le\lambda_1\theta|y(1,t)|+(\frac{k_2}{2}+\frac{1}{2}\theta w_1+\frac{1}{2}k_2\theta w_2)y^2(1,t)+(\frac{1}{2}+\frac{k_2}{2}+\frac{k_3}{2})||D^ry(\cdot,t)||_0^2\\ &+(\frac{k_3}{2}+\frac{\theta}{2})||y_x(\cdot,t)||_0^2.\end{aligned}$$

Then, it implies that the augmented functional $V_2(t)$ has an infinite upper-bounded.

Taking $V_2(t)$ with r fractional derivative with respect to time along the solution of the system, then

$$\begin{aligned}D^rV_2(t) &\le D^rV_1(t)+k_2\theta w_2y(1,t)D^ry(1,t)+k_2D^ry(1,t)\int_0^1D^ry(v,t)dv \quad (3.10)\\ &+k_2\int_0^1y(1,t)D^{2r}y(v,t)dv+k_3\int_0^1(v-1)D^ry_x(v,t)D^ry(v,t)dv\\ &+k_3\int_0^1(v-1)y_x(v,t)D^{2r}y(v,t)dv,\\ &\equiv D^rV_1(t)+k_2\theta w_2y(1,t)D^ry(1,t)+K_1+K_2+K_3,\end{aligned}$$

which the third term can be reduced to the following corresponding inequality

$$|K_1|=\Big|k_2D^ry(1,t)\int_0^1D^ry(v,t)dv\Big|\le k_2|D^ry(1,t)|\int_0^1|D^ry(v,t)|dv \quad (3.11)$$

$$\le k_2|D^r y(1,t)|||D^r y(\cdot,t)dv||_0$$
$$\le k_2\sqrt{2R}|D^r y(1,t)|,$$

and the fourth term can be simplified into the following inequality

$$\begin{aligned}
K_2 = k_2\int_0^1 y(1,t)D^{2r}y(v,t)dv &= k_2\int_0^1 y(1,t)\theta y_{xx}dv \\
&= k_2\theta y(1,t)y_x(1,t) - k_2\theta y(1,t)y_x(0,t) \\
&= k_2\theta y(1,t)[u(t)+\psi(t)] - k_2\theta c_0 y(1,t)D^r y(0,t) \\
&= k_2\theta y(1,t)\psi(t) - k_2\theta\lambda_1 y(1,t)sign(y(1,t)) - k_2\theta w_1 y^2(1,t) \\
&\quad -k_2\theta\lambda_2 y(1,t)sign(D^r y(1,t)) - k_2\theta w_2 y(1,t)D^r y(1,t) \\
&\quad -k_2\theta c_0 y(1,t)D^r y(0,t),
\end{aligned}$$

then, because of the following inequality holding

$$|k_2\theta y(1,t)\psi(t)| \le k_2\theta M|y(1,t)|$$

$$|k_2\theta y(1,t)sign(D^r y(1,t))| \le k_2\theta|y(1,t)|$$

$$|k_2\theta c_0 y(1,t)D^r y(0,t)| \le \frac{1}{2}k_2\theta c_0|D^r y(0,t)|^2 + \frac{1}{2}k_2\theta c_0|y(1,t)|^2.$$

Thus, the fourth term of inequality (3.10) can be finally reduced to the following inequality

$$\begin{aligned}
|K_2| = |k_2\int_0^1 y(1,t)D^{2r}y(v,t)dv| &\le k_2\theta M|y(1,t)| - k_2\theta\lambda_1|y(1,t)| - k_2\theta w_1 y^2(1,t) \\
&\quad +k_2\theta\lambda_2|y(1,t)| - k_2\theta w_2 y(1,t)D^r y(1,t) + \frac{1}{2}k_2\theta c_0 y^2(1,t) \\
&\quad +\frac{1}{2}k_2\theta c_0|D^r y(0,t)|^2.
\end{aligned} \tag{3.12}$$

For the fifth and sixth term of inequality (3.10), it can be obtained that

$$\begin{aligned}
K_3 &= k_3\int_0^1 (v-1)D^r y_x(v,t)D^r y(v,t)dv + k_3\int_0^1 (v-1)y_x(v,t)D^{2r}y(v,t)dv \\
&= k_3\int_0^1 (v-1)D^r y_x(v,t)D^r y(v,t)dv + k_3\theta\int_0^1 (v-1)y_x(v,t)y_{xx}(v,t)dv \\
&= k_3\int_0^1 vD^r y_x(v,t)D^r y(v,t)dv - k_3\int_0^1 D^r y_x(v,t)D^r y(v,t)dv \\
&\quad +k_3\theta\int_0^1 vy_x(v,t)y_{xx}(v,t)dv - k_3\theta\int_0^1 y_x(v,t)y_{xx}(v,t)dv, \\
&\equiv K_{31} - K_{32} + K_{33} - K_{34}.
\end{aligned} \tag{3.13}$$

which each of the terms can be simplified into the following equalities respectively:

The first term of inequality (3.13) is reduced into the following form

$$K_{31} = k_3\int_0^1 vD^r y_x(v,t)D^r y(v,t)dv = \frac{k_3}{2}\int_0^1 vd[D^r y(v,t)]^2$$

$$= \frac{k_3}{2} v[D^r y(v,t)]^2|_0^1 - \frac{k_3}{2}||D^r y(\cdot,t)||^2$$
$$= \frac{k_3}{2}[D^r y(1,t)]^2 - \frac{k_3}{2}||D^r y(\cdot,t)||^2.$$

The second term of inequality (3.13) is reduced into the following form

$$K_{32} = k_3 \int_0^1 D^r y_x(v,t) D^r y(v,t) dv = \frac{k_3}{2}[D^r y(v,t)]^2|_0^1 = \frac{k_3}{2}[D^r y(1,t)]^2 - \frac{k_3}{2}[D^r y(0,t)]^2.$$

The third term of inequality (3.13) is reduced into the following form

$$\begin{aligned} K_{33} &= k_3\theta \int_0^1 v y_x(v,t) y_{xx}(v,t) dv = \frac{k_3\theta}{2}\int_0^1 v d[y_x(v,t)]^2 \\ &= \frac{k_3\theta}{2} v[y_x(v,t)]^2|_0^1 - \frac{k_3\theta}{2}||y_x(\cdot,t)||^2 \\ &= \frac{k_3\theta}{2} y_x^2(1,t) - \frac{k_3\theta}{2}||y_x(\cdot,t)||^2, \end{aligned}$$

and the last term of inequality (3.13) is reduced into the following form

$$K_{34} = k_3\theta \int_0^1 y_x(v,t) y_{xx}(v,t) dv = \frac{\theta k_3}{2} y_x^2(1,t) - \frac{\theta k_3}{2} y_x^2(0,t).$$

Then, we get

$$K_3 = \frac{k_3}{2}[D^r y(0,t)]^2 - \frac{k_3}{2}||D^r y(\cdot,t)||^2 - \frac{k_3\theta}{2}||y_x(\cdot,t)||^2 + \frac{\theta k_3}{2} y_x^2(0,t). \quad (3.14)$$

Thus, together with (3.10), (3.11), (3.12) and (3.14), the functional $V_2(t)$ can be simplified that

$$\begin{aligned} D^r V_2(t) \le\ & D^r V_1(t) + k_2\theta w_2 y(1,t) D^r y(1,t) + k_2\sqrt{2R}|D^r y(1,t)| + k_2\theta M|y(1,t)| \\ & - k_2\theta\lambda_1|y(1,t)| - k_2\theta w_1 y^2(1,t) + k_2\theta\lambda_2|y(1,t)| - k_2\theta w_2 y(1,t) D^r y(1,t) \\ & + \frac{k_2\theta c_0}{2} y^2(1,t) + \frac{k_2\theta c_0}{2}|D^r y(0,t)|^2 + \frac{k_3}{2}[D^r y(1,t)]^2 - \frac{k_3}{2}||D^r y(\cdot,t)||^2 \\ & - \frac{k_3}{2}[D^r y(1,t)]^2 + \frac{k_3}{2}[D^r y(0,t)]^2 + \frac{k_3\theta}{2} y_x^2(1,t) - \frac{k_3\theta}{2}||y_x(\cdot,t)||^2 \\ & - \frac{k_3\theta}{2} y_x^2(1,t) + \frac{k_3\theta}{2} y_x^2(0,t) \\ \le\ & -\theta[(\lambda_2 - M)|D^r y(1,t)| + c_0|D^r y(0,t)|^2 + w_2|D^r y(1,t)|^2] \\ & + k_2\sqrt{2R}|D^r y(1,t)| + (k_2\theta M - k_2\theta\lambda_1 + k_2\theta\lambda_2)|y(1,t)| - k_2\theta w_1 y^2(1,t) \\ & + \frac{k_2\theta c_0}{2} y^2(1,t) + (\frac{k_2\theta c_0}{2} + \frac{k_3}{2})|D^r y(0,t)|^2 \\ & - \frac{k_3}{2}||D^r y(\cdot,t)||^2 - \frac{k_3\theta}{2}||y_x(\cdot,t)||^2 + \frac{k_3\theta}{2} y_x^2(0,t) \\ =\ & -[\theta(\lambda_2 - M) - k_2\sqrt{2R}]|D^r y(1,t)| - [c_0\theta - \frac{k_2\theta c_0}{2} - \frac{k_3}{2} - \frac{k_3\theta c_0^2}{2}]|D^r y(0,t)|^2 \\ & - \theta w_2|D^r y(1,t)|^2 - [k_2\theta\lambda_1 - k_2\theta M - k_2\theta\lambda_2]|y(1,t)| \end{aligned}$$

$$-(k_2\theta w_1 - \frac{k_2\theta c_0}{2})y^2(1,t) - \frac{k_3}{2}||D^r y(\cdot,t)||^2 - \frac{\theta k_3}{2}||y_x(\cdot,t)||^2.$$

Combing with the assumption 5, $D^r V_2(t)$ is negative definite. Besides, the Lyapunov functional $V_2(t)$ is radially unbounded as $R \to +\infty$, then, according to Theorem 2.1, $V(t)$ can be employed as a radially unbounded Lyapunov functional, and thus, the theorem is completed. □

4. Numerical simulation

In this part, the L_2 approach[29] is implemented to discrete the Caputo-derivative operator. For the perturbed equation (2.1), the BCs and ICs are set as $y^0(x) = 10 + 5cos(4\pi x)$, $y_t^0(x) = 2cos(2\pi x)$, $\theta = 5$, $c_0 = 1$. The disturbance is chosen as $\psi(t) = 1 + 2\sin(t)$ which meets the BCs, and the magnitude M of the disturbance is estimated as $M = 3$. Then, for the controller (3.2), the parameters are set in accordance with Assumption 2, 3, 4, 5 as $\lambda_1 = 20$, $\lambda_2 = 10$, $w_1 = 10$, $w_2 = 10$, $R = 2, k_2 = 0.5, k_3 = 0.4$. Here, the steps of space and time are taken as 0.02 and 0.002, respectively.

Figure 1 a) depicts the spatiotemporal profiles of $y(x,t)$ with no boundary control, which implies the destabilizing of the disturbance effect. While, under the controller (3.2), the state variables $y(x,t)$ of the system converge to the origin with the fractional order $\alpha = 1.8$, which is described by Figure 1(b).

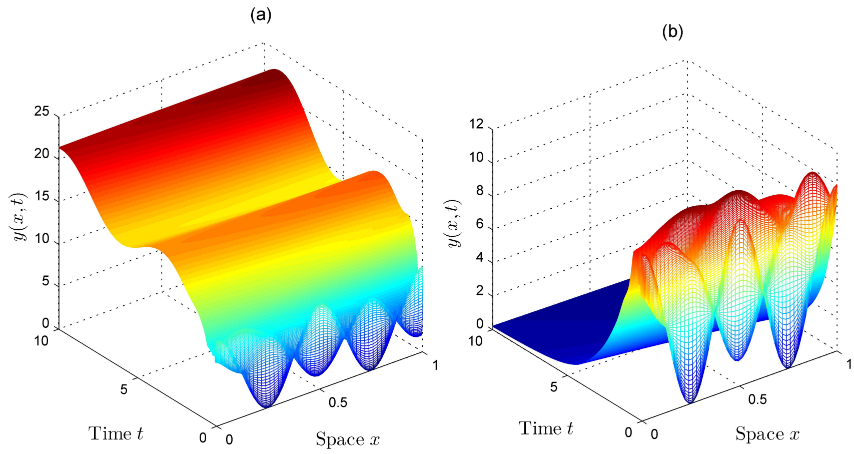

Fig. 1. (a): the solution $y(x,t)$ of the wave equation with on boundary control at $\alpha = 1.8$. (b): the solution $y(x,t)$ of the controlled wave equation under the controller (3.2) with $\alpha = 1.8$.

Due to the controller (3.2) involved with the fractional order operator, thus, the behavior of the asymptotic stability for the fractional order wave equation under the controller (3.2) is changed with the order of the fractional order derivative. Figure 2 depicts the corresponding plots of the solution $y(x,t)$ and the solution $||y(\cdot,t)||$ with $\alpha = 1.6,\ \alpha = 1.8,\ \alpha = 2$. From the figure, we can see that the solution of the controlled wave equation under the controller (3.2) converges to the origin as confirmed in Theorem 3.1, and the boundary controller is verified viability and efficiency. In addition, from Figure 2 (a), Figure 2 (b) and Figure 2 (c), we can find that the state variable $y(x,t)$ of the system converges to the origin more fast with the increase of the fractional order α.

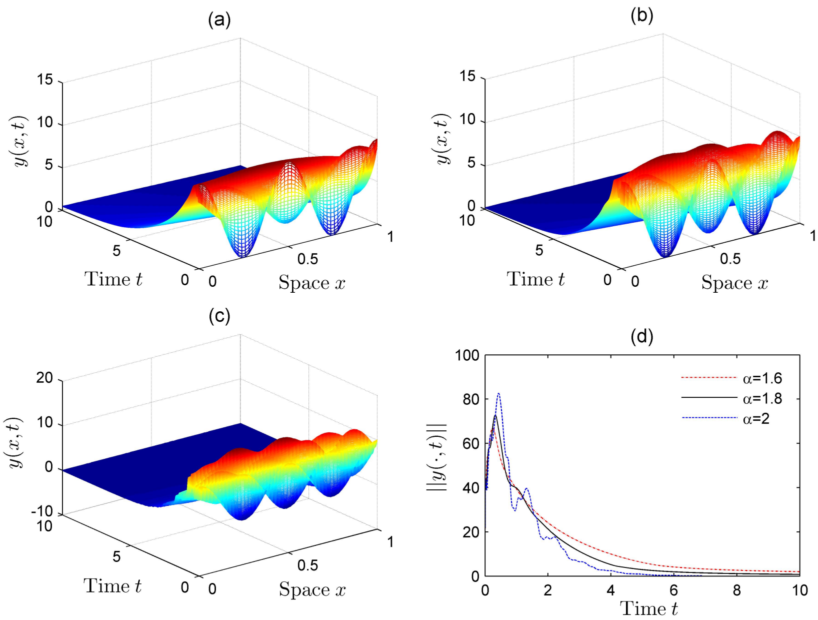

Fig. 2. (a), (b), (c): the solution $y(x,t)$ of the controlled wave equation under the controller (3.2), respectively with $\alpha = 1.6,\ \alpha = 1.8,\ \alpha = 2$. (d): the different view of the solution $||y(\cdot,t)||$ of the controlled wave equation (2.1).

Figure 3 demonstrates that the $D^r y(x,t)$ and $||D^r y(\cdot,t)||$ of the fractional wave equation under the controller (3.2) with $\alpha = 1.6,\ 1.8,\ 2$ respectively, which formulate the convergence of the solution of system (2.1) under control in the space $L^2(0,1) \times L^2(0,1)$.

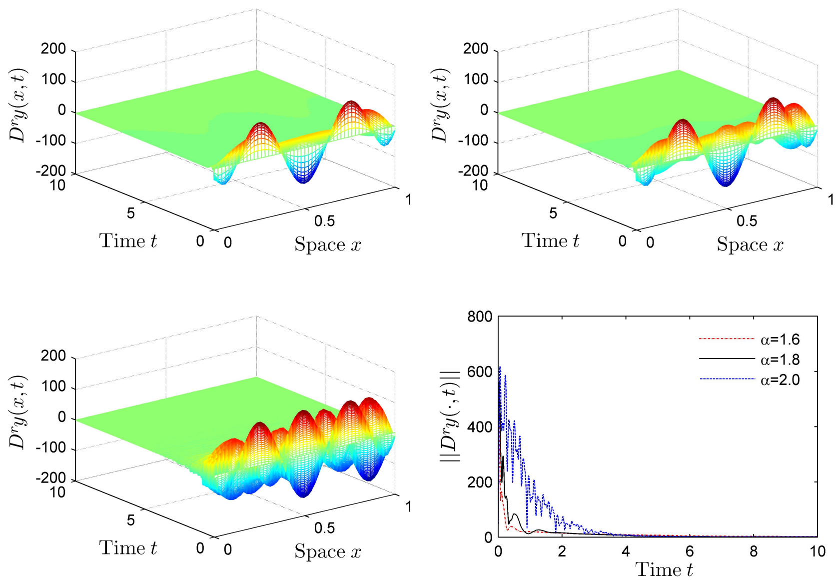

Fig. 3. (a), (b), (c): the solution $D^r y(x,t)$ of the controlled wave equation under the controller (3.2), respectively with $\alpha = 1.6,\ \alpha = 1.8,\ \alpha = 2$. (d): the different view of the solution $||D^r y(\cdot,t)||$ of the controlled wave equation (2.1).

5. Discussion

In the recent years, the controllability of fractional differential systems has become active. For the various real world problems in physical and engineering sciences when subject to abrupt changes at certain instants during the evolution process, the fractional order distributed parameter systems have been used for the system model. However, the controllability of such systems has not been extensively studied, in particular, the boundary control when an uncertainty exists on boundary. Hence, it is worthwhile to establish a robust control approach in order to provide a boundary control system with the desired dynamics in the presence of plant uncertainties on boundary. Furthermore, the sliding mode control theory for distributed parameter systems needs to be developed urgently in order to provide an efficient, robust and adaptive control approach for a wide class of distributed parameter systems.

This paper studies the boundary control strategies for the wave equation with fractional derivative. Our results can be extended to study a wide class of distributed parameter systems with fractional derivative. But, it should be noted that the discontinuous control requires that the state should be directly accessible for measurement, which is too strong for the wave equation with fractional derivative and make the proposed approaches of mainly theoretical interest. However, the developed methods open the way to further improvements, analogous to similar ones attained in the finite-dimensional setup, such as the boundary control and/or the pointwise output measurement feed-back implementation of the proposed feedback controllers, which will be addressed in future research activities.

6. Conclusions

In this paper, the boundary sliding mode control algorithm have been designed for a fractional order wave equation, under Neumann boundary conditions with an uncertain. The resulting schemes have been applied to solve the boundary control problem of the fractional order uncertain wave equation, and the control algorithms are extended to globally asymptotically stabilize the fractional order wave equation with an uncertain boundary conditions. At last, the numerical simulations are presented to verify the viability and efficiency of the proposed fractional order controllers.

The main contributions can be summarized as follows:
(1) As an appropriate extension of the infinite dimensional integer order system, the proposed control algorithm is focused on the infinite dimensional fractional order system. The controller is associated with the fractional order parameter, which influences the convergence rate of the proposed control algorithm.
(2) By means of appropriate Lyapunov functionals, the stability of the resulting controlled wave equation under the boundary controller is proven in the L_2-space based on the fractional order globally asymptotic stability theorem .
(3) At last, numerical simulation is presented to verify the efficiency of the proposed fractional controller.

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