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A toughness condition for fractional (k, m)–deleted graphs revisited

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Abstract In computer networks, toughness is an important parameter which is used to measure the vulnerability of the network. Zhou et al. obtains a toughness condition for a graph to be fractional (k, m)–deleted and presents an example to show the sharpness of the toughness bound. In this paper we remark that the previous example does not work and inspired by this fact, we present a new toughness condition for fractional (k, m)–deleted graphs improving the existing one. Finally, we state an open problem.

Keywords Graph, fractional factor, fractional (k, m)–deleted graph, toughness.

MR(2010) Subject Classification 05C70

1 Introduction

The problem of fractional factor can be discussed as a relaxation of the famous cardinality matching problem which is a crucial problem in operation research. It has wide applications in different areas such as combinatorial polyhedron, scheduling and network design. For instance, certain large data packets are sent to different destinations via several channels in a data transmission network. To finish this work efficiently, it is necessary to divide the large data packets into small ones, and the available assignments of data packets are equal to the problem of fractional flow which can be converted to a problem of fractional factor in a network graph. Some development methods on graph based networks design can refer to Ashwin and Postlethwaite [1], Crouzeilles et al. [5], de Araujo et al. [6], Fardad et al. [7], Haenggi et al. [15], Lanzeni [17], Pishvaee and Rabbani [19], Possani et al. [20], Rahimi and Haghighi [21] and Rizzelli et al. [22] for instance.

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All graphs considered in our article are simple (finite, loopless, and without multiple edges). Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $x \in V(G)$, we denote $d_G(x)$ and $N_G(x)$ by the degree and the neighborhood of x in G, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. For two vertex–disjoint subsets S and T of G, we set $e_G(S,T) = |\{e = xy | x \in S, y \in T\}|$ where $|\cdot|$ means cardinality. We denote the minimum degree of G by $\delta(G)$. For more information on the notations and terminologies used in this article, see [3].

Let $k \geq 1$ be an integer and $h : E(G) \to [0,1]$ be a function. If $\sum_{x \sim e} h(e) = k$ for any $x \in V(G)$, then we call $G[F_h]$ a *fractional* k*–factor* of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. Zhou [23] introduces the concept of fractional (k,m)–deleted graph, that is, a graph G is called a *fractional* (k,m)*–deleted graph* if removing any m edges from G, the resulting graph still has a fractional k–factor. A fractional (k,m)–deleted graph is simply called a fractional k–deleted graph if $m = 1$.

The notion of *toughness* was originally introduced by Chvátal [4] in the sense: If G is a complete graph, $t(G) := \infty$. If G is not complete,

$$t(G) := \min\{\frac{|S|}{\omega(G-S)} \Big| \omega(G-S) \geq 2\}$$

and where $\omega(G-S)$ is the number of connected components of $G - S$.

In what follows, we always assume that n is the order of G, i.e., $n = |V(G)|$. Many works have contributed to the analysis of the existence and characterization of fractional factors and fractional (k,m)–deleted graphs. For instance, Liu and Zhang [18] proves that a graph has a fractional k–factor if $t(G) \geq k - \frac{1}{k}$ where $k \geq 2$. Let $\sigma_2(G) := \min\{d_G(u) + d_G(v)\}$ for each pair of non–adjacent vertices u and v in G. Gao and Wang [12] and [13] determine the following sufficient conditions for a graph to be fractional (k,m)–deleted where $k \geq 2$ and $m \geq 0$:

- If $n \geq 4k + 4m - 5$ and $\delta(G) \geq \frac{n}{2}$, then G is a fractional (k,m)–deleted graph.
- If $n \geq 4k+4m-3$, $\delta(G) \geq k+m$ and $\max\{d_G(u), d_G(v)\} \geq \frac{n}{2}$ for each pair of non–adjacent vertices u and v of G, then G is a fractional (k,m)–deleted graph.
- If $n \geq \begin{cases} 4k+4m-5, & \text{if } (k,m) \neq (3,0) \\ 8, & \text{if } (k,m) = (3,0) \end{cases}$, $\delta(G) \geq k+m$ and $\sigma_2(G) \geq n$, then G is a fractional (k,m)–deleted graph.
- If $n \geq 8k+4m-7$, $\delta(G) \geq k+m$ and $|N_G(x) \cup N_G(y)| \geq \frac{n}{2}$ for each pair of non–adjacent vertices x, y of G, then G is a fractional (k,m)–deleted graph.

More results on the topic with fractional factor, fractional deleted graphs and other applications can refer to Zhou et al. [25] and [26], Jin [16], Basavanagoud et al. [2], Guirao and Luo [14], Gao et al. [8], and Gao and Wang [9], [10] and [11].

In Zhou et al. [24], the authors proved that a graph G is a fractional (k,m)–deleted graph if $\delta(G) \geq k+2m$ and $t(G) \geq k + \frac{2m-1}{k}$ for $k \geq 2$. Moreover, they presented an example to show that this toughness bound is "sharp". We remark that this example does not work to explain the sharpness of the toughness bound determined by Zhou et al. [24] as we show in section 2.

Inspired by this fact we consider that it is still meaningful to discuss the toughness condition for a graph to be fractional (k, m)–deleted, and it is natural to ask whether it would be possible to improve the toughness condition characterization of fractional (k, m)–deleted graph. Thus, in the present paper we shall study the relations between $t(G)$ and the fractional (k, m)–deleted graph. The statement of our main result which improves the Zhou's et al. previous bound is the following.

Theorem 1.1 *Let $k \geq 2$ be an integer. A graph G with $\delta(G) \geq k + m$ is a fractional (k, m)-deleted graph if $t(G) \geq k + \frac{m-1}{k}$.*

The structure of this paper is the following. In section 2 we analyze the Zhou's et al. example and we shall state that it does not work. In section 3 we present the auxiliary results that we shall need for the proof of Theorem 1.1 which shall be proved in section 4. Finally, in section 5 we shall state an open problem to be study in the future.

2 On the Zhou's et al. example

Now, we shall analyze the sharpness example which is stated in Zhou et al. [24]. Let $k \geq 2$, $n \geq 1$ and $m \geq 0$ be integers. Let K_i be the complete graph of order i and $(j)K_i$ denotes a subgraph of K_i containing j isolated vertices and no edge. The graph of the example is constructed from $K_{n(k-1)}$, $(nk + 1)K_{k-1}$ and $K_{(nk+1)(k-1)+2mn}$ as follows:

Let $V((nk+1)K_{k-1}) = \{x_1, x_2, \ldots, x_{(nk+1)(k-1)}\}$ and consider the subset $\{y_1, y_2, \ldots, y_{(nk+1)(k-1)}\} \subseteq V(K_{(nk+1)(k-1)+2mn})$. $V(G) = V(K_{n(k-1)}) \cup V((nk + 1)K_{k-1}) \cup V(K_{(nk+1)(k-1)+2mn})$ and $E(G) = E(K_{n(k-1)}) \cup E((nk+1)K_{k-1}) \cup E(K_{(nk+1)(k-1)+2mn}) \cup \{x_iy_i : i = 1, 2, \cdots, (nk+1)(k-1)\} \cup \{ux_i : u \in V(K_{n(k-1)}), i = 1, 2, \cdots, (nk + 1)(k - 1)\}$. It takes $U = (K_{(nk+1)(k-1)+2mn} - \{y_1\}) \cup \{x_1\} \cup K_{n(k-1)}$, and thus $|U| = (nk + n + 1)(k - 1) + 2mn$ and $\omega(G - U) = nk + 2$.

Thus, the authors deduced that $t(G) = \frac{(nk+n+1)(k-1)+2mn}{nk+2}$ which tends to $k + \frac{2m-1}{k}$ when $n \to \infty$. However, if we consider for the same construction $U := \{y_1, y_2, \cdots, y_{(nk+1)(k-1)}\} \cup K_{n(k-1)}$, then we infer $|U| = (nk + n + 1)(k - 1)$ and $\omega(G - U) = nk + 2$. Also, we can check that $\frac{(nk+n+1)(k-1)}{nk+2}$ tends to $k - \frac{1}{k}$ when $n \to \infty$. It implies that $t(G) < k - \frac{1}{k}$ by the definition of toughness. Therefore, we confirm that this example is not valid to explain the sharpness of the toughness bound determined by Zhou et al. [24].

3 Auxiliary results

In this section we present the auxiliary results that we shall need in the proof of our main result.

Lemma 3.1 (Zhou [23]) *Let $k \geq 1$ and $m \geq 0$ be two integers, let G be a graph and H be a subgraph of G with m edges. Then G is a fractional (k, m)–deleted graph if and only if for any subset S of $V(G)$,*

$$k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| \geq \sum_{x \in T} d_H(x) - e_H(S, T),$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) - d_H(x) + e_H(x, S) \leq k - 1\}$.

We note that Lemma 3.1 can be easily reformulated in the following equivalent form:

Lemma 3.2 *Let $k \geq 1$ and $m \geq 0$ be two integers, let G be a graph and H be a subgraph of*

G with m edges. Then G is a fractional (k, m)–deleted graph if and only if

$$k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| \geq \sum_{x \in T} d_H(x) - e_H(S, T) \tag{3.1}$$

for all disjoint subsets S and T of $V(G)$.

The relationship between minimum degree and toughness of graph can be stated as follows.

Lemma 3.3 (Chvátal [4]) *If a graph G is not complete, then $t(G) \leq \frac{\delta(G)}{2}$.*

The following two lemmas are presented in Liu and Zhang [18] and show the characters of *independent set* (i.e. subsets of the vertex set of G in which any two of them are not adjacent because they reflect the sparsity and stability of the graph system in somehow) and *covering set* (i.e. subsets of the vertex set of G in which each edge has at least one vertex in this set).

Lemma 3.4 (Liu and Zhang [18]) *Let G be a graph and let $H = G[T]$ such that $\delta(H) \geq 1$ and $1 \leq d_G(x) \leq k - 1$ for every $x \in V(H)$ where $T \subseteq V(G)$ and $k \geq 2$. Let $T_1, \ldots, T_{k-1}$ be a partition of the vertices of H satisfying $d_G(x) = j$ for each $x \in T_j$ where we allow some T_j to be empty. If each component of H has a vertex of degree at most $k - 2$ in G, then H has a maximal independent set I and a covering set $C = V(H) - I$ such that*

$$\sum_{j=1}^{k-1} (k - j)c_j \leq \sum_{j=1}^{k-1} (k - 2)(k - j)i_j,$$

where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for $j = 1, \ldots, k - 1$.

Clearly, Lemma 3.4 also works for $\delta(H) \geq 0$. By the proving process of Lemma 2.2 in [18], we obtain the following Lemma that will be play an important role in the proof of our main result.

Lemma 3.5 (Liu and Zhang [18]) *Let G be a graph and let $H = G[T]$ such that $d_G(x) = k-1$ for every $x \in V(H)$ and no component of H is isomorphic to K_k where $T \subseteq V(G)$ and $k \geq 2$. Then there exists an independent set I and a covering set $C = V(H) - I$ of H satisfying*

$$|V(H)| \leq \sum_{i=1}^{k} (k - i + 1)|I^{(i)}| - \frac{|I^{(1)}|}{2}$$

and

$$|C| \leq \sum_{i=1}^{k} (k - i)|I^{(i)}| - \frac{|I^{(1)}|}{2}$$

where $I^{(i)} = \{x \in I, d_H = k - i\}$ for $1 \leq i \leq k$ and $\sum_{i=1}^{k} |I^{(i)}| = |I|$.

Note that for each vertex $x \in I$, $d_H(x) = k - 1$ and there exists a vertex $y \in I - \{x\}$ such that $N_H(x) \cap N_H(y) \neq \emptyset$.

4 Proof of Theorem 1.1

If G is complete, clearly G is a fractional (k, m)–deleted graph by means of $\delta(G) \geq k + m$. In the sequel, we assume that G is not complete. Suppose that G satisfies the hypothesis of Theorem 1.1, but is not a fractional (k, m)–deleted graph. By Lemma 3.2 and the fact that

$\sum_{x\in T} d_H(x) - e_H(T, S) \leq 2m$ for all S, T disjoint subsets of $V(G)$ and $H \subseteq E(G)$, there exist disjoint subsets S and T of $V(G)$ such that

$$k|S| + d_{G-S}(T) - k|T| \leq \sum_{x\in T} d_H(x) - e_H(T, S) - 1 \leq 2m - 1. \quad (4.1)$$

We select S and T such that $|T|$ is minimum. If $T = \emptyset$, then $\sum_{x\in T} d_H(x) - e_H(T, S) = 0$ in view of its definition. Thus, (4.1) becomes $k|S| \leq -1$, a contradiction. Hence, we have $T \neq \emptyset$.

If there exists $x \in T$ satisfying $d_{G-S}(x) \geq k$, then the subsets S and $T \setminus \{x\}$ satisfy (4.1) as well. This contradicts the selection rule of S and T. It implies that $d_{G-S}(x) \leq k-1$ for any $x \in T$. Furthermore, we infer that $S \neq \emptyset$.

Let l be the number of the components of $H' = G[T]$ which are isomorphic to K_k and let $T_0 = \{x \in V(H')|d_{G-S} = 0\}$. Let H be the subgraph obtained from $H' - T_0$ by deleting those l components isomorphic to K_k.

If $|V(H)| = 0$, then from (4.1) we obtain $1 \leq |S| \leq |T_0| + l + \frac{2m-1}{k}$. Clearly, $|T_0| + l \geq 1$ since $|T| \neq 0$. If $|T_0| + l = 1$, then $|S| \leq 1 + \frac{2m-1}{k}$ and $2k + \frac{2m-2}{k} \leq 2t(G) \leq \delta(G) \leq k - 1 + |S| \leq k + \frac{2m-1}{k}$ which contradicts to $k \geq 2$. Hence $\omega(G - S) \geq |T_0| + l \geq 2$ and $t(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{|S|}{|T_0|+l} \leq \frac{1}{2} + \frac{2m-1}{2k}$. This contradicts $t(G) \geq k + \frac{m-1}{k}$ and $k \geq 2$. Therefore, we have $|V(H)| > 0$.

Let $H = H_1 \cup H_2$ where H_1 is the union of components of H which satisfies that $d_{G-S}(x) = k - 1$ for every vertex $x \in V(H_1)$ and $H_2 = H - H_1$. By Lemma 3.5, H_1 has a maximum independent set I_1 and the covering set $C_1 = V(H_1) - I_1$ such that

$$|V(H_1)| \leq \sum_{i=1}^{k} (k - i + 1)|I^{(i)}| - \frac{|I^{(1)}|}{2}, \quad (4.2)$$

and

$$|C_1| \leq \sum_{i=1}^{k} (k - i)|I^{(i)}| - \frac{|I^{(1)}|}{2}, \quad (4.3)$$

where $I^{(i)} = \{x \in I_1, d_{H_1}(x) = k - i\}$ for $1 \leq i \leq k$ and $\sum\limits_{i=1}^{k} |I^{(i)}| = |I_1|$. On the other hand, let $T_j = \{x \in V(H_2)|d_{G-S}(x) = j\}$ for $1 \leq j \leq k - 1$. By the definition of H and H_2 we can also see that each component of H_2 has a vertex of degree at most $k - 2$ in $G - S$. According to Lemma 3.4, H_2 has a maximal independent set I_2 and the covering set $C_2 = V(H_2) - I_2$ such that

$$\sum_{j=1}^{k-1} (k - j)c_j \leq \sum_{j=1}^{k-1} (k - 2)(k - j)i_j, \quad (4.4)$$

where $c_j = |C_2 \cap T_j|$ and $i_j = |I_2 \cap T_j|$ for every $j = 1, \ldots, k - 1$. Set $W = V(G) - S - T$ and $U = S \cup C_1 \cup (N_G(I_1) \cap W)) \cup C_2 \cup (N_G(I_2) \cap W)$. In light of computation, we infer

$$\begin{aligned}
& |C_2| + |N_G(I_2) \cap W| = |V(H_2)| - |I_2| + |N_{G-S-T}(I_2)| \\
= \quad & |V(H_2)| - |I_2| + |N_{G-S}(I_2)| - |N_T(I_2)| \\
= \quad & (|V(H_2)| - |I_2| - |N_T(I_2)|) + |N_{G-S}(I_2)| \\
\leq \quad & (|V(H_2)| - |I_2| - |N_{H_2}(I_2)|) + |N_{G-S}(I_2)|
\end{aligned}$$

$$\leq \quad 0+\sum_{j=1}^{k-1} j i_j=\sum_{j=1}^{k-1} j i_j.$$

Furthermore, we get

$$|U| \leq |S|+|C_1|+\sum_{j=1}^{k-1} j i_j+\sum_{i=1}^{k}(i-1)|I^{(i)}| \tag{4.5}$$

and

$$\omega(G-U) \geq t_0+l+|I_1|+\sum_{j=1}^{k-1} i_j, \tag{4.6}$$

where $t_0=|T_0|$. Let $t(G)=t$. Then when $\omega(G-U)>1$, we have

$$|U| \geq t\omega(G-U), \tag{4.7}$$

and it is also held when $\omega(G-U)=1$ (in this case, $t\omega(G-U)=t \leq \frac{\delta(G)}{2} \leq \frac{|S|+d_{G-S}(x)}{2} \leq \frac{|U|}{2}$ where $x \in T$).

By (4.5)-(4.7), we yield

$$|S|+|C_1| \geq \sum_{j=1}^{k-1}(t-j) i_j+t(t_0+l)+t|I_1|-\sum_{i=1}^{k}(i-1)|I^{(i)}|. \tag{4.8}$$

Note that $|T|=t_0+lk+|V(H_1)|+|V(H_2)|$, and

$$\begin{aligned}
& k|T|-d_{G-S}(T)=kt_0+lk^2+k|V(H_1)|+k|V(H_2)|-d_{G-S}(T) \\
= \quad & kt_0+lk^2+k|V(H_1)|+k|V(H_2)|-lk(k-1)-(k-1)|V(H_1)| \\
& -\sum_{j=1}^{k-1} j(i_j+c_j) \\
= \quad & kt_0+lk+|V(H_1)|+\sum_{j=1}^{k-1}(k-j) i_j+\sum_{j=1}^{k-1}(k-j) c_j.
\end{aligned}$$

Thus, from (4.1) we have

$$kt_0+kl+|V(H_1)|+\sum_{j=1}^{k-1}(k-j) i_j+\sum_{j=1}^{k-1}(k-j) c_j \geq k|S|-2m+1.$$

Combining with (4.8) we have

$$\begin{aligned}
& kt_0+kl+|V(H_1)|+\sum_{j=1}^{k-1}(k-j) i_j+\sum_{j=1}^{k-1}(k-j) c_j+k|C_1| \\
\geq \quad & \sum_{j=1}^{k-1}(kt-kj) i_j+kt(t_0+l)+kt|I_1|-k\sum_{i=1}^{k}(i-1)|I^{(i)}|-2m+1.
\end{aligned}$$

Thus

$$\begin{aligned}
& |V(H_1)|+\sum_{j=1}^{k-1}(k-j) c_j+k|C_1| \qquad (4.9)\\
\geq \quad & \sum_{j=1}^{k-1}(kt-kj-k+j) i_j+k(t-1)(t_0+l)+kt|I_1|
\end{aligned}$$

$$-k\sum_{i=1}^{k}(i-1)|I^{(i)}|-2m+1. \tag{4.10}$$

By (4.2) and (4.3),

$$|V(H_1)|+k|C_1|\le\sum_{i=1}^{k}(k^2-(i-1)k-(i-1))|I^{(i)}|-\frac{(k+1)|I^{(1)}|}{2}. \tag{4.11}$$

Using (4.4),(4.9) and (4.11), we have

$$\begin{aligned}&\sum_{j=1}^{k-1}(k-2)(k-j)i_j+\sum_{i=1}^{k}(k^2-(i-1)k-(i-1))|I^{(i)}| \qquad (4.12)\\ \ge\ &\sum_{j=1}^{k-1}(kt-kj-k+j)i_j+k(t-1)(t_0+l)+kt|I_1|+\frac{(k+1)|I^{(1)}|}{2}\\ &-k\sum_{i=1}^{k}(i-1)|I^{(i)}|-2m+1.\end{aligned}$$

Now, we shall discuss the following cases according to the value of t_0+l.

Case 1. $t_0+l\ge 1$**.** In this case, by (4.12) and $k(t-1)(t_0+l)-2m+1\ge k(k+\frac{m-1}{k}-1)-2m+1=k^2-m-1$,

$$\begin{aligned}&\sum_{j=1}^{k-1}(k-2)(k-j)i_j+\sum_{i=1}^{k}(k^2-(i-1)k-(i-1))|I^{(i)}| \qquad (4.13)\\ \ge\ &\sum_{j=1}^{k-1}(kt-kj-k+j)i_j+kt|I_1|+\frac{(k+1)|I^{(1)}|}{2}-k\sum_{i=1}^{k}(i-1)|I^{(i)}|\\ &+k^2-m-1.\end{aligned}$$

Claim 4.1 If $t_0+l\ge 1$, then $|I_2|\ne 0$.

Proof Suppose $|I_2|=0$. Then $|I_1|\ne 0$ by $|V(H)|>0$ and (4.13) becomes

$$(|I_1|-1)k^2-(t|I_1|+\frac{|I^{(1)}|}{2})k-\sum_{i=1}^{k}(i-1)|I^{(i)}|-\frac{|I^{(1)}|}{2}+m+1\ge 0. \tag{4.14}$$

In light of (4.14) and $t\ge k+\frac{m-1}{k}$ we have

$$\begin{aligned}0\ &\le\ (|I_1|-1)k^2-(t|I_1|+\frac{|I^{(1)}|}{2})k-\sum_{i=1}^{k}(i-1)|I^{(i)}|-\frac{|I^{(1)}|}{2}+m+1\\ &\le\ (|I_1|-1)k^2-((k+\frac{m-1}{k})|I_1|+\frac{|I^{(1)}|}{2})k-\sum_{i=1}^{k}(i-1)|I^{(i)}|\\ &\quad -\frac{|I^{(1)}|}{2}+m+1\\ &=\ -k^2-\frac{|I^{(1)}|}{2}k-(m-1)|I_1|-\sum_{i=1}^{k}(i-1)|I^{(i)}|-\frac{|I^{(1)}|}{2}+m+1<0,\end{aligned}$$

a contradiction. The last step follows from $|I_1|\ge 1$ and $k\ge 2$.

Claim 4.2 If $t_0+l\ge 1$, then $|I_1|\ne 0$.

Proof Suppose $|I_1|=0$. We yield $|I_2|\neq 0$ by $|V(H)|>0$ and (4.13) becomes

$$\sum_{j=1}^{k-1}(k-2)(k-j)i_j \geq \sum_{j=1}^{k-1}(kt-kj-k+j)i_j+k^2-m-1.$$

Note that

$$\begin{aligned} & (k-2)(k-j)-kt+kj+k-j \\ \leq\ & (k-2)(k-j)-k(k+\frac{m-1}{k})+kj+k-j \\ =\ & -k+j-m+1\leq -m. \end{aligned}$$

We have

$$\sum_{j=1}^{k-1}(k-2)(k-j)i_j-\sum_{j=1}^{k-1}(kt-kj-k+j)i_j-k^2+m+1<0$$

in view of $k\geq 2$.

From Claim 4.1 and Claim 4.2, we can see that $|I_1|>0$ and $|I_2|>0$. According to $|I_2|\geq 1$ and

$$\sum_{j=1}^{k-1}(k-2)(k-j)i_j-\sum_{j=1}^{k-1}(kt-kj-k+j)i_j+m\leq 0,$$

we have

$$\begin{aligned} & \sum_{i=1}^{k}(k^2-(i-1)k-(i-1))|I^{(i)}| \\ \geq\ & kt|I_1|+\frac{(k+1)|I^{(1)}|}{2}-k\sum_{i=1}^{k}(i-1)|I^{(i)}|+k^2-1. \end{aligned}$$

That is to say,

$$(|I_1|-1)k^2-(t|I_1|+\frac{|I^{(1)}|}{2})k-\sum_{i=1}^{k}(i-1)|I^{(i)}|-\frac{|I^{(1)}|}{2}+1\geq 0. \tag{4.15}$$

In light of (4.15) and $t\geq k+\frac{m-1}{k}$ we have

$$\begin{aligned} 0\ \leq\ & (|I_1|-1)k^2-(t|I_1|+\frac{|I^{(1)}|}{2})k-\sum_{i=1}^{k}(i-1)|I^{(i)}|-\frac{|I^{(1)}|}{2}+1 \\ \leq\ & (|I_1|-1)k^2-((k+\frac{m-1}{k})|I_1|+\frac{|I^{(1)}|}{2})k-\sum_{i=1}^{k}(i-1)|I^{(i)}| \\ & -\frac{|I^{(1)}|}{2}+1 \\ =\ & -k^2-\frac{|I^{(1)}|}{2}k-(m-1)|I_1|-\sum_{i=1}^{k}(i-1)|I^{(i)}|-\frac{|I^{(1)}|}{2}+1<0, \end{aligned}$$

a contradiction.

Case 2. $t_0+l=0$**.** In this case, by (4.12) we have,

$$\sum_{j=1}^{k-1}(k-2)(k-j)i_j+\sum_{i=1}^{k}(k^2-(i-1)k-(i-1))|I^{(i)}| \tag{4.16}$$

$$\geq \sum_{j=1}^{k-1}(kt-kj-k+j)i_j+kt|I_1|+\frac{(k+1)|I^{(1)}|}{2}-k\sum_{i=1}^{k}(i-1)|I^{(i)}| -2m+1.$$

Claim 4.3 If $t_0+l=0$, then $|I_2|\neq 0$.

Proof Suppose $|I_2|=0$. Then $|I_1|\neq 0$ by $|V(H)|>0$, $|V(T)|=|V(H_1)|$ and $k|S|\leq k|T|-d_{G-S}(T)+2m-1=|T|+2m-1$. If $|I_1|\leq k$, then $|T|\leq k^2$ and $|S|\leq \frac{|T|+2m-1}{k}\leq k+\frac{2m-1}{k}$. Then $2k+\frac{2m-2}{k}\leq 2t\leq \delta(G)\leq |S|+(k-1)\leq 2k-1+\frac{2m-1}{k}$, which contradicts to $k\geq 2$. Hence, we have $|I_1|\geq k+1$.

Using (4.16), we have

$$|I_1|k^2-(t|I_1|+\frac{|I^{(1)}|}{2})k-\sum_{i=1}^{k}(i-1)|I^{(i)}|-\frac{|I^{(1)}|}{2}+2m-1\geq 0. \tag{4.17}$$

In light of (4.17), $t\geq k+\frac{m-1}{k}$, $|I_1|\geq k+1$ and $k\geq 2$, we get

$$\begin{aligned}
0 &\leq |I_1|k^2-(t|I_1|+\frac{|I^{(1)}|}{2})k-\sum_{i=1}^{k}(i-1)|I^{(i)}|-\frac{|I^{(1)}|}{2}+2m-1\\
&\leq |I_1|k^2-((k+\frac{m-1}{k})|I_1|+\frac{|I^{(1)}|}{2})k-\sum_{i=1}^{k}(i-1)|I^{(i)}|-\frac{|I^{(1)}|}{2}\\
&\quad +2m-1\\
&= -\frac{|I^{(1)}|}{2}k-(m-1)|I_1|-\sum_{i=1}^{k}(i-1)|I^{(i)}|-\frac{|I^{(1)}|}{2}+2m-1<0,
\end{aligned}$$

a contradiction.

Claim 4.4 If $t_0+l=0$, then $|I_1|\neq 0$.

Proof Suppose $|I_1|=0$. Then $|I_2|\neq 0$ using $|V(H)|>0$. In terms of (4.16), we infer

$$\sum_{j=1}^{k-1}(k-2)(k-j)i_j\geq \sum_{j=1}^{k-1}(kt-kj-k+j)i_j-2m+1.$$

Note that

$$\max\{(k-2)(k-j)-kt+kj+k-j\}=-m.$$

Since $|C_2|\leq (k-2)+(|I_1|-1)(k-1-1)=|I_2|(k-2)$ and $|T|\leq |I_2|(k-1)$, we get $|S|\leq\frac{|T|+2m-1}{k}\leq |I_2|+\frac{2m-1-|I_2|}{k}$. If $|I_2|=1$, then $|S|\leq 1+\frac{2m-2}{k}$ and $2k+\frac{2m-2}{k}\leq 2t\leq \delta(G)\leq |S|+(k-1)=k+\frac{2m-2}{k}$, a contradiction. Hence, we get $|I_2|\geq 2$ and

$$\sum_{j=1}^{k-1}(k-2)(k-j)i_j-\sum_{j=1}^{k-1}(kt-kj-k+j)i_j+2m-1<0.$$

This implies that $|I_1|\neq 0$.

From Claim 4.3 and Claim 4.4, we can see that $|I_1|>0$ and $|I_2|>0$. In terms of $|I_2|\geq 1$, we deduce $\sum_{j=1}^{k-1}(kt-kj-k+j)i_j\geq\sum_{j=1}^{k-1}(k-2)(k-j)i_j+m$. Thus, we obtain

$$\sum_{i=1}^{k}(k^2-(i-1)k-(i-1))|I^{(i)}|$$

$$\geq \quad kt|I_1| + \frac{(k+1)|I^{(1)}|}{2} - k\sum_{i=1}^{k}(i-1)|I^{(i)}| - m + 1,$$

which reveals

$$|I_1|k^2 - (t|I_1| + \frac{|I^{(1)}|}{2})k - \sum_{i=1}^{k}(i-1)|I^{(i)}| - \frac{|I^{(1)}|}{2} + m - 1 \geq 0. \tag{4.18}$$

In light of (4.18) $|I_1| \geq 1$ and $t \geq k + \frac{m-1}{k}$, we have

$$\begin{aligned} 0 \quad &\leq \quad |I_1|k^2 - (t|I_1| + \frac{|I^{(1)}|}{2})k - \sum_{i=1}^{k}(i-1)|I^{(i)}| - \frac{|I^{(1)}|}{2} + m - 1 \\ &\leq \quad |I_1|k^2 - ((k + \frac{m-1}{k})|I_1| + \frac{|I^{(1)}|}{2})k - \sum_{i=1}^{k}(i-1)|I^{(i)}| - \frac{|I^{(1)}|}{2} \\ &\quad\quad +m - 1 \\ &= \quad -\frac{|I^{(1)}|}{2}k - (m-1)|I_1| - \sum_{i=1}^{k}(i-1)|I^{(i)}| - \frac{|I^{(1)}|}{2} + m - 1 < 0. \end{aligned}$$

Therefore the proof of Theorem 1.1 is over. ∎

5 Conclusions

From the detailed proof of Theorem 1.1 presented in the previous section, we see that the condition $\delta(G) \geq k+m$ is only used to ensure that G is a fractional (k,m)–deleted graph when G is complete. If we assume that G is not complete in our main result, then the condition $\delta(G) \geq k+m$ can be deleted.

We believe that the toughness bound presented in our paper is still not sharp, although it improves the previous existing result. Therefore, we give the following conjecture on the relationship between fractional (k,m)–deleted graph and toughness.

Conjecture 5.1 *If $t(G) \geq k - \frac{1}{k}$ and $k \geq \Theta(m)$ (here $\Theta(m)$ is a function of m and $\Theta(m) \sim O(m)$), then G is a fractional (k,m)–deleted graph.*

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