ARTICLE TEMPLATE

Existence of the solution and stability for a class of variable fractional order differential systems †

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ABSTRACT
In this paper, the existence results of the solution and stability are focused for the variable fractional order differential equation. In view of the definitions of three kinds of Caputo variable fractional order operator, the existence of the solution for the variable fractional order differential system is obtained by use of the Arzela-Ascoli theorem. Moreover, some criterions for the Mittag-Leffler stability and asymptotical stability of the variable fractional order differential system are proposed according to the Fractional Comparison Principle.

KEYWORDS
Variable fractional order differential equation; Existence of the solution; Arzela-Ascoli theorem; Variable fractional order type Mittag-Leffler stability

1. Introduction

Fractional calculus has been acknowledged extensively as a powerful tool to describe the natural behavior and complex phenomena of practical problems in many research fields, such as chemistry, engineering, mathematics, physics, material, finance, and so on [1–7]. However, the constant fractional order calculus is not the ultimate tool to model the phenomena in nature, the variable fractional order calculus is proposed as a natural candidate to describe the complex dynamics problems. Lorenzo and Hartley [8] gave the concept of variable fractional order operator in 1998. Whereafter, the definition of variable order integral and differential as well as its applications in physical and engineering were deeply discussed in [9].

Roughly speaking, the main advantage of variable fractional order calculus operator in dynamic system modeling is that it has adaptive memory for the past phenomena,

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and the variable fractional order derivatives and integrals are very useful for simulating temporal or spatial correlation phenomena [10]. Different authors have proposed different definitions for variable fractional order differential operators, each of which has a special meaning to meet the desired objectives [11]. Coimbra et al. [12] proposed a new concept of variable order calculus by describing a simple problem in mechanics, and gave a mathematical definition of variable fractional order differential operator suitable for mechanical modeling. In [13], two definitions of VO fractional derivative were introduced and compared in the analysis of abnormal relaxation process. Lynnette et al. [14] have compared the definitions of Riemann- Liouville type, Caputo type, Coimbra type and Marchaud type of variable fractional order operators. By using the selected variable fractional order operators, the significance of the order of functions is related to the dynamic properties of viscoelastic oscillators. Recently, due to its profound applications to many fields of science and technology, a series of research hotspots such as the existence, uniqueness, stability and control of solutions of differential operators with variable fractional order are presented [15–21].

Since the kernel of the variable fractional order operator contains a variable exponent, the variable fractional order operator is not invertible and cannot be simply transformed into Volterra integral equations. Therefore, the research on the solution of variable fractional order differential equations needs more precise and unique technical methods. Razminia et al. [15] studied the existence of solutions for a class of generalized fractional differential equations with non-autonomous fractional order operators, and the sufficient conditions for the existence of the solution of the equation were obtained, but the conditions for the uniqueness of the solution of the equation were not given. In the Ref [16], the existence and uniqueness of solutions were studied for variable fractional order differential equations defined by Caputo-type fractional operator, nevertheless, the order of the variable fractional order operator of Caputo type was only related to the integral variable, and the irreversibility of the operator with variable fractional order was neglected. For this reason, Zhang [17] transformed the initial value problem of the variable fractional order differential equation into the fixed point problem of the integral equation under the definition of Riemann- Liouville type, and the existence of the solution for the integral equation was studied by means of nonlinear functional analysis. Malesza et al. [18] studied the existence of solutions of linear fractional order differential equations under the definition of Grunvald- Letnikov type variable fractional order operator. Ref [19] was devoted to tackle the existence of solutions for linear fractional differential equations with variable fractional order operator defined by three different Grunvald-Letnikov operators. However, based on the present research results, there has not been a systematic study on the existence and uniqueness of solutions for three kinds of variable fractional order differential equations defined by Caputo-type variable fractional order operators.

The stability results of the constant order fractional differential systems provide an important theoretical basis for the stability of the variable fractional order differential systems [22–25]. But, it is more complex to study of the stability for the variable fractional order differential systems based on the fact that complexity of variable fractional order operators and the variety of definitions. To our best knowledge, there hardly exists a systematic study on the stability for Caputo-type variable fractional differential equations with the following form:

$$\begin{cases} {}_0^C D_t^{q(t,x(t))} x(t) = f(t, x(t)), t \in (0, T], \\ x(0) = u_0. \end{cases} \tag{1}$$

where $0 < q_1 \leq q(t, x(t)) \leq q_2 < 1$.

This paper is organized as follows. The rest of this section is used to recall some basic definitions of variable order fractional calculus and some lemmas which are important to derive the main result of this paper. Section 3 is devoted to study the existence of the solution for the variable fractional order differential equation. In Section 4, the stability of the variable order fractional order system are analyzed.

2. Preliminaries

To begin with, the definitions of variable order fractional derivative and fractional integral are introduced.

Definition 2.1. [9] Three kinds of definition of variable order fractional integration are defined as follows.
I.

$$I_t^{q(t,x(t))}x(t) = \frac{1}{\Gamma(q(t,x(t)))}\int_0^t (t-s)^{q(t,x(t))-1}x(s)ds, 0 < q(t,x(t)) < 1, \qquad (2)$$

II.

$$I_t^{q(t,x(t))}x(t) = \int_0^t \frac{(t-s)^{q(t-s,x(t-s))-1}}{\Gamma(q(t-s,x(t-s)))}x(s)ds, 0 < q(t,x(t)) < 1,\ t \in [0,T], \qquad (3)$$

III.

$$I_t^{q(t,x(t))}x(t) = \int_0^t \frac{(t-s)^{q(s,x(s))-1}}{\Gamma(q(s,x(s)))}x(s)ds, 0 < q(t,x(t)) < 1,\ t \in [0,T], \qquad (4)$$

provided the integration is point-wise defined, where $\Gamma(\cdot)$ is the Gamma function, $t \in [0,T]$.

Definition 2.2. [9] The following is the definition of variable order fractional derivatives in three different forms.
I.

$${}_0^C D_t^{q(t,x(t))}x(t) = \frac{1}{\Gamma(1-q(t,x(t)))}\int_0^t (t-s)^{-q(t,x(t))}x'(s)ds, 0 < q(t,x(t)) < 1, \qquad (5)$$

II.

$${}_0^C D_t^{q(t,x(t))}x(t) = \int_0^t \frac{(t-s)^{-q(t-s,x(t-s))}}{\Gamma(1-q(t-s,x(t-s)))}x'(s)ds, 0 < q(t,x(t)) < 1,\ t \in [0,T], \qquad (6)$$

III.

$${}_0^C D_t^{q(t,x(t))}x(t) = \int_0^t \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1-q(s,x(s)))}x'(s)ds, 0 < q(t,x(t)) < 1,\ t \in [0,T], \qquad (7)$$

provided the integration is point-wise defined, where $\Gamma(\cdot)$ is the Gamma function, $t \in [0,T]$.

When $q(t,x(t)) = q$, which q is a constant, the above variable fractional order operator presented in Definition 2.1 and Definition 2.2 is the usual Caputo constant fractional order operators.

The following definitions and lemma are important to obtain the main results given in section 3 and section 4.

Definition 2.3. The constant u_0 is an equilibrium point of variable fractional order system (1) if $f(t,u_0) = 0$.

Definition 2.4. ([26]) A continuous function $\alpha : [0,t) \to [0,\infty)$ belongs to class-$\mathcal{K}$ if it is strict increasing and $\alpha(0) = 0$.

The next is the famous Arzela-Ascoli Theorem which is employed to prove the existence of solution for system (1).

Lemma 2.5. *([27])(Arzela-Ascoli) If a sequence $\{u_n(t)\} \in C[0,T]$ is uniformly bounded and equicontinuous, then it has a uniformly convergent subsequence.*

The following presents one of the characteristics of Caputo variable fractional order calculus.

Unlike the Caputo constant fractional order calculus, the Caputo variable fractional order calculus ${}_0^CD^{q(t,u(t))}$, $I^{q(t,u(t))}$ does not pervasively satisfy

$$I^{p(t,x(t))}I^{q(t,x(t))} = I^{p(t,x(t))+q(t,x(t))},\ {}_0^CD^{p(t,x(t))}I^{p(t,x(t))}f(t) = f(t), \tag{8}$$

which means the variable fractional order differential equation can not be transformed into the equivalent integral equation. In order to overcome this obstacle, we propose the following transforming sequence

$$\begin{cases} x_n(t) = x_{n-1}(t) + \int_0^{t-\frac{T}{n}} \frac{(t-s)^{-q(t,x(t))}}{\Gamma(1-q(t,x(t)))} x_{n-1}(s)ds - f[t, \int_0^{t-\frac{T}{n}} x_{n-1}(s)ds + u_0], & t \in (\frac{T}{n}, T], \\ x_n(t) = 0, & t \in [0, \frac{T}{n}]. \end{cases} \tag{9}$$

3. The existence of the solution for the variable order differential system

Before giving the main result about the existence of the solution for the variable order differential system (1), the following assumptions are made.

Assumption

(I): f, x are continuous functions from $[0,T] \times \mathbb{R}$ to $\mathbb{R}$;

(II): $q : [0,T] \times R \to (0,1)$ is a continuous function;

The main result of this section is given in Theorem 3.1.

Theorem 3.1. *Assume that assumption (I) and (II) hold, then the IVP (9) exists one solution $\tilde{x}(t) \in C[0,T]$.*

Proof. Let

$$T^{-q(t,x(t))} \begin{cases} \leq (\frac{1}{T})^{q_2}, & 0 < T < 1, \\ \leq 1, & 1 \leq T < +\infty, \end{cases} \tag{10}$$

and

$$\overline{T} = \max\{T^{-q_2}, 1\},$$

then $T^{-q(t,x(t))} \le \overline{T}$. By Minkowsk's inequality, the following holds

$$(\bar{a} + \bar{b})^r \le \bar{a}^r + \bar{b}^r. \tag{11}$$

where $\bar{a}, \bar{b}$ are non-negative constants in $\mathbb{R}$, $0 < r < 1$. Furthermore, when $0 < \bar{a} < \bar{b} < 1$, the function $\tilde{k}(t) := \bar{a}^t - \bar{b}^t$ is decreasing for $t \in (-1, 0)$.

Now, we claim that the sequence $\{x_n(t)\}$ defined by (9) is uniformly bounded on $[0, T]$.

Obviously, for $t \in [0, \frac{T}{n}]$, $\{x_n(t)\}$ defined by (9) is uniformly bounded. Thus, in the following, we need to verify that, for $t \in (\frac{T}{n}, T]$, the sequence $\{x_n(t)\}$ is uniformly bounded.

Let $G_1 = \max_{0 \le t \le T} |f(t, u_0)|$. For $t \in [0, T]$, $x_0 = 0$ is uniformly bounded, then,

$$|x_1(t)| = |f(t, u_0)| < G_1, t \in (\frac{T}{n}, T],$$

which implies that $x_1(t),\ t \in [\frac{T}{n}, T]$ is uniformly bounded.

In order to apply the method of the inductive hypothesis, we assume $x_{n-1}(t)$ is uniformly bounded on $[0, T]$ and let

$$|x_{n-1}(t)| \le G_{n-1},\ G_f = \max_{0 \le t \le T, |x_{n-1}(t)| \le G_{n-1}} \left| f(t, \int_0^t x_{n-1}(s)ds + u_0) \right|$$

on $t \in [0, T],\ \ n = 1, 2, ...$

According to the assumption (I), (II), then, $G_f < +\infty$. Then, the following inequalities hold

$$\begin{aligned}
|x_n(t)| &\le |x_{n-1}(t)| + \int_0^{t-\frac{T}{n}} \left| \frac{(t-s)^{-q(t,x(t))}}{\Gamma(1-q(t,x(t)))} \right| |x_{n-1}(s)| ds + \left| f(t, \int_0^{t-\frac{T}{n}} x_{n-1}(s)ds + u_0) \right| \\
&\le G_{n-1} + G_{n-1} \int_0^{t-\frac{T}{n}} \left| \frac{(t-s)^{-q(t,x(t))}}{\Gamma(1-q(t,x(t)))} \right| ds + G_f \\
&\le G_{n-1} + \frac{G_{n-1}}{\Gamma(1-q_1)} \int_0^{t-\frac{T}{n}} |(t-s)^{-q(t,x(t))}| ds + G_f \\
&\le G_{n-1} + \frac{G_{n-1}}{\Gamma(1-q_1)} \int_0^{t-\frac{T}{n}} |(\frac{t-s}{T})^{-q(t,x(t))}| T^{-q(t,x(t))} ds + G_f \\
&\le G_{n-1} + \frac{G_{n-1}}{\Gamma(1-q_1)} \overline{T} \int_0^{t-\frac{T}{n}} (\frac{t-s}{T})^{-q_2} ds + G_f \\
&\le G_{n-1} + \frac{G_{n-1}}{\Gamma(1-q_1)(1-q_2)} T\overline{T} + G_f = G_n,
\end{aligned}$$

where $G_n = G_{n-1} + \frac{G_{n-1}}{\Gamma(1-q_1)(1-q_2)} \overline{T} T + G_f$, which implies that $x_n(t)$ is uniformly bounded on $[\frac{T}{n}, T]$. Besides, $x_n(t) = 0$ for $t \in [0, \frac{T}{n})$, thus, it's obtained that $\{x_n(t)\}$ is uniformly bounded on $[0, T]$.

The following is to verify the equi-continuity of the sequence $\{x_n(t)\}, t \in [0,T]$.

It can be noticed that $x_0(t), t \in [0,T]$ is equi-continuous. Set $A = \{x_n(t); n \geq 1\}$, we consider the following three kinds of situations.

(I): if $0 \leq t_1 \leq t_2 \leq \frac{T}{n}$, we have

$$\lim_{t_1 \to t_2} |x_n(t_2) - x_n(t_1)| = 0$$

which is independent of x_n for each $x_n \in A$.

(II): if $0 \leq t_1 < \frac{T}{n} < t_2 \leq T$, we have

$$\begin{aligned}
|x_n(t_2) - x_n(t_1)| &= |x_{n-1}(t_2) + \int_0^{t_2 - \frac{T}{n}} \frac{(t_2 - s)^{-q(t_2, x(t_2))}}{\Gamma(1 - q(t_2, x(t_2)))} x_{n-1}(s) ds \\
&\quad - f(t_2, \int_0^{t_2 - \frac{T}{n}} x_{n-1}(s) ds + u_0)| \\
&\leq |x_{n-1}(t_2)| + \frac{G_{n-1}}{\Gamma(1 - q_1)} \int_0^{t_2 - \frac{T}{n}} T^{-q(t_2, x(t_2))} (\frac{t_2 - s}{T})^{-q(t_2, x(t_2))} ds \\
&\quad + |f(t_2, \int_0^{t_2 - \frac{T}{n}} x_{n-1}(s) ds + u_0)| \\
&\leq |x_{n-1}(t_2)| + \frac{G_{n-1}\overline{T}}{\Gamma(1 - q_1)} \int_0^{t_2 - \frac{T}{n}} (\frac{t_2 - s}{T})^{-q_2} ds \\
&\quad + |f(t_2, \int_0^{t_2 - \frac{T}{n}} x_{n-1}(s) ds + u_0)| \\
&\leq |x_{n-1}(t_2)| - \frac{G_{n-1}\overline{T} T^{q_2}}{\Gamma(1 - q_1)(1 - q_2)} (\frac{T}{n})^{(1 - q_2)} \\
&\quad + \frac{G_{n-1}\overline{T} T^{q_2} t_2^{1 - q_2}}{\Gamma(1 - q_1)(1 - q_2)} + |f(t_2, \int_0^{t_2 - \frac{T}{n}} x_{n-1}(s) ds + u_0)|,
\end{aligned}$$

for each $x_n \in A$, which together with $|x_1(t_2) - x_1(t_1)| = |f(t_2, u_0)|$ implies

$$\lim_{t_2 \to t_1} |x_1(t_2) - x_1(t_1)| = |f(0, u_0)| = 0.$$

Based on the method of inductive hypothesis, we assume that

$$\lim_{t_2 \to t_1} |x_{n-1}(t_2) - x_{n-1}(t_1)| = 0,$$

then,

$$\begin{aligned}
\lim_{t_2 \to t_1} |x_n(t_2) - x_n(t_1)| &\leq \lim_{t_2 \to t_1} |x_{n-1}(t_2) - x_{n-1}(t_1)| \\
&\quad + \lim_{t_2 \to t_1} \left[- \frac{G_{n-1}\overline{T} T^{q_2}}{\Gamma(1 - q_1)(1 - q_2)} (\frac{T}{n})^{(1 - q_2)} + \frac{G_{n-1}\overline{T} T^{q_2} t_2^{1 - q_2}}{\Gamma(1 - q_1)(1 - q_2)} \right] \\
&\quad + \lim_{t_2 \to t_1} |f(t_2, \int_0^{t_2 - \frac{T}{n}} x_{n-1}(s) ds + u_0)| = 0
\end{aligned}$$

which is independent of $x_n \in A$.

According to the assumption (I), $f(t,x)$ is continuous with respect to t and x, moreover, $\lim_{t\to\frac{T}{n}}\int_0^{t-\frac{T}{n}}x(s)ds=0$ and $f(\frac{T}{n},u_0)=0$. Then

$$\lim_{t\to\frac{T}{n}} f(t,\int_0^{t-\frac{T}{n}}x(s)ds+u_0)=0.$$

(III): for $\frac{T}{n}\le t_1<t_2\le T$, the following equality holds:

$$|x_1(t_2)-x_1(t_1)|=|f(t_2,u_0)-f(t_1,u_0)|.$$

Based on assumption (I), $f(t,x)$ is a continuous function, then

$$\lim_{t_2\to t_1}|x_1(t_2)-x_1(t_1)|=0,$$

which is independent of $x_1\in A$. Supposing $\lim_{t_2\to t_1}|x_{n-1}(t_2)-x_{n-1}(t_1)|=0$, according to (11) and the following inequalities, it can be obtained

$$\begin{aligned}
&|x_n(t_2)-x_n(t_1)|\\
&\quad\le|x_{n-1}(t_2)-x_{n-1}(t_1)|+\Big|\int_0^{t_2-\frac{T}{n}}\frac{(t_2-s)^{-q(t_2,x_{n-1}(t_2))}}{\Gamma(1-q(t_2,x_{n-1}(t_2)))}x_{n-1}(s)ds\\
&\quad-\int_0^{t_1-\frac{T}{n}}\frac{(t_1-s)^{-q(t_1,x_{n-1}(t_1))}}{\Gamma(1-q(t_1,x_{n-1}(t_1)))}x_{n-1}(s)ds\Big|+\Big|f(t_2,\int_0^{t_2-\frac{T}{n}}x_{n-1}(s)ds+u_0)\\
&\quad-f(t_1,\int_0^{t_1-\frac{T}{n}}x_{n-1}(s)ds+u_0)\Big|\\
&\quad\le|x_{n-1}(t_2)-x_{n-1}(t_1)|+\Big|\int_0^{t_1-\frac{T}{n}}\Big[\frac{(t_2-s)^{-q(t_2)}}{\Gamma(1-q(t_2,x_{n-1}(t_2)))}-\frac{(t_1-s)^{-q(t_1,x_{n-1}(t_1))}}{\Gamma(1-q(t_1,x_{n-1}(t_1)))}\Big]\\
&\quad\cdot x_{n-1}(s)ds\Big|+\Big|f(t_2,\int_0^{t_2-\frac{T}{n}}x_{n-1}(s)ds+u_0)-f(t_1,\int_0^{t_1-\frac{T}{n}}x_{n-1}(s)ds+u_0)\Big|\\
&\quad+\Big|\int_{t_1-\frac{T}{n}}^{t_2-\frac{T}{n}}\frac{(t_2-s)^{-q(t_2,x_{n-1}(t_2))}}{\Gamma(1-q(t_2,x_{n-1}(t_2)))}x_{n-1}(s)ds\Big|\\
&\quad\le|x_{n-1}(t_2)-x_{n-1}(t_1)|+\frac{G_{n-1}}{\Gamma(1-q(t_2,x_{n-1}(t_2)))}\int_0^{t_1-\frac{T}{n}}\Big|(\frac{t_2-s}{T})^{-q(t_2,x_{n-1}(t_2))}-\\
&\quad(\frac{t_1-s}{T})^{-q(t_2,x_{n-1}(t_2))}\Big|T^{-q(t_2,x_{n-1}(t_2))}ds+G_{n-1}\int_0^{t_1-\frac{T}{n}}\Big|\frac{((t_1-s)/T)^{-q(t_2,x_{n-1}(t_2))}}{\Gamma(1-q(t_2,x_{n-1}(t_2)))}\\
&\quad-\frac{((t_1-s)/T)^{-q(t_2,x_{n-1}(t_2))}}{\Gamma(1-q(t_1,x_{n-1}(t_1)))}\Big|T^{-q(t_2,x_{n-1}(t_2))}ds+G_{n-1}\int_0^{t_1-\frac{T}{n}}\Big|\frac{(t_1-s)^{-q(t_2,x_{n-1}(t_2))}}{\Gamma(1-q(t_1,x_{n-1}(t_1)))}\\
&\quad-\frac{(t_1-s)^{-q(t_1,x_{n-1}(t_1))}}{\Gamma(1-q(t_1,x_{n-1}(t_1)))}\Big|ds+\Big|f(t_2,\int_0^{t_2-\frac{T}{n}}x_{n-1}(s)ds+u_0)-f(t_1,\int_0^{t_1-\frac{T}{n}}x_{n-1}(s)ds\\
&\quad+u_0)\Big|+\frac{G_{n-1}}{\Gamma(1-q(t_2,x_{n-1}(t_2)))}\Big|\int_{t_1-\frac{T}{n}}^{t_2-\frac{T}{n}}(\frac{t_2-s}{T})^{-q(t_2,x_{n-1}(t_2))}T^{-q(t_2,x_{n-1}(t_2))}ds\Big|\\
&\quad\le|x_{n-1}(t_2)-x_{n-1}(t_1)|+\frac{\overline{T}T^{q_2}G_{n-1}}{\Gamma(1-q_1)}\int_0^{t_1-\frac{T}{n}}\Big|(t_1-s)^{-q_2}-(t_2-s)^{-q_2}\Big|ds
\end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{\overline{T}T^{q_2}G_{n-1}}{\Gamma(1-q(t_2,x_{n-1}(t_2)))}-\frac{\overline{T}T^{q_2}G_{n-1}}{\Gamma(1-q(t_1,x_{n-1}(t_1)))}\right|\int_0^{t_1-\frac{T}{n}}(t_1-s)^{-q_2}ds\\
&+\frac{G_{n-1}}{\Gamma(1-q(t_1,x_{n-1}(t_1)))}\left|\int_0^{t_1-\frac{T}{n}}\left[(t_1-s)^{-q(t_2,x_{n-1}(t_2))}-(t_1-s)^{-q(t_1,x_{n-1}(t_1))}\right]ds\right|\\
&+\left|f(t_2,\int_0^{t_2-\frac{T}{n}}x_{n-1}(s)ds+u_0)-f(t_1,\int_0^{t_1-\frac{T}{n}}x_{n-1}(s)ds+u_0)\right|+\frac{\overline{T}T^{q_2}G_{n-1}}{\Gamma(1-q_1)(1-q_2)}\cdot\\
&\left|(t_2-t_1+\frac{T}{n})^{1-q_2}-(\frac{T}{n})^{1-q_2}\right|\\
=&\,|x_{n-1}(t_2)-x_{n-1}(t_1)|+\frac{\overline{T}T^{q_2}G_{n-1}}{\Gamma(1-q_1)(1-q_2)}\left|t_1^{1-q_2}-t_2^{1-q_2}+(t_2-t_1+\frac{T}{n})^{1-q_2}\right.\\
&\left.-(\frac{T}{n})^{1-q_2}\right|+\left|\frac{\overline{T}T^{q_2}G_{n-1}}{\Gamma(1-q(t_2,x_{n-1}(t_2)))}-\frac{\overline{T}T^{q_2}G_{n-1}}{\Gamma(1-q(t_1,x_{n-1}(t_1)))}\right|\left|\frac{{t_1}^{1-q_2}}{1-q_2}-\frac{(\frac{T}{n})^{1-q_2}}{1-q_2}\right|\\
&+\frac{G_{n-1}}{\Gamma(1-q(t_1,x_{n-1}(t_1)))}\left|\frac{(\frac{T}{n})^{1-q(t_1,x_{n-1}(t_1))}}{1-q(t_1,x_{n-1}(t_1))}-\frac{(\frac{T}{n})^{1-q(t_2,x_{n-1}(t_2))}}{1-q(t_2,x_{n-1}(t_2))}+\frac{{t_1}^{1-q(t_2,x_{n-1}(t_2))}}{1-q(t_2,x_{n-1}(t_2))}\right.\\
&\left.-\frac{{t_1}^{1-q(t_1,x_{n-1}(t_1))}}{1-q(t_1,x_{n-1}(t_1))}\right|+\left|f(t_2,\int_0^{t_2-\frac{T}{n}}x_{n-1}(s)ds+u_0)-f(t_1,\int_0^{t_1-\frac{T}{n}}x_{n-1}(s)ds+u_0)\right|\\
&+\frac{\overline{T}T^{q_2}G_{n-1}}{\Gamma(1-q_1)(1-q_2)}\left|(t_2-t_1+\frac{T}{n})^{1-q_2}-(\frac{T}{n})^{1-q_2}\right|\\
\le&\,|x_{n-1}(t_2)-x_{n-1}(t_1)|+\frac{\overline{T}T^{q_2}G_{n-1}}{\Gamma(1-q_1)(1-q_2)}(t_2-t_1)^{1-q_2}+\frac{\overline{T}T^{q_2}G_{n-1}}{\Gamma(1-q_1)(1-q_2)}\left|t_1^{1-q_2}\right.\\
&\left.-t_2^{1-q_2}+(t_2-t_1)^{1-q_2}\right|+\left|\frac{\overline{T}T^{q_2}G_{n-1}}{\Gamma(1-q(t_2,x_{n-1}(t_2)))}-\frac{\overline{T}T^{q_2}G_{n-1}}{\Gamma(1-q(t_1,x_{n-1}(t_1)))}\right|\left|\frac{{t_1}^{1-q_2}}{1-q_2}\right.\\
&\left.-\frac{(\frac{T}{n})^{1-q_2}}{1-q_2}\right|+\frac{G_{n-1}}{\Gamma(1-q(t_1,x_{n-1}(t_1)))}\left|\frac{(\frac{T}{n})^{1-q(t_1,x_{n-1}(t_1))}}{1-q(t_1,x_{n-1}(t_1))}-\frac{(\frac{T}{n})^{1-q(t_2,x_{n-1}(t_2))}}{1-q(t_2,x_{n-1}(t_2))}\right.\\
&\left.+\frac{{t_1}^{1-q(t_2,x_{n-1}(t_2))}}{1-q(t_2,x_{n-1}(t_2))}-\frac{{t_1}^{1-q(t_1,x_{n-1}(t_1))}}{1-q(t_1,x_{n-1}(t_1))}\right|+\left|f(t_2,\int_0^{t_2-\frac{T}{n}}x_{n-1}(s)ds+u_0)\right.\\
&\left.-f(t_1,\int_0^{t_1-\frac{T}{n}}x_{n-1}(s)ds+u_0)\right|.
\end{aligned}
$$

Based on the assumption I), II), it's obtained that

$$\lim_{t_2\to t_1}f(t_2,\int_0^{t_2-\frac{T}{n}}x_{n-1}(s)ds+u_0)=f(t_1,\int_0^{t_1-\frac{T}{n}}x_{n-1}(s)ds+u_0).$$

Combined with the continuity of the Gamma function $\Gamma(t)$ on $(0,1]$, then

$$\lim_{t_2\to t_1}|x_{n-1}(t_2)-x_{n-1}(t_1)|=0.$$

Thus, the set A is equicontinuous. According to Arzela-Ascoli theorem, it's derived that, for $\forall t\in[0,T]$, there exists a subsequence still denoted by the sequence $\{x_n\}$ which uniformly converges to a continuous function x^*. Since the sequence $\{x_n\}$ satisfies the system (9), Let $n \to \infty$ in both side of the equation (9), we have

$$\begin{cases} x^*(t) = x^*(t) + \int_0^t \frac{(t-s)^{-q(t,x(t))}}{\Gamma(1-q(t,x(t)))} x^*(s)ds - f[t, \int_0^t x^*(s)ds + u_0], & t \in (0,T], \\ x^*(t) = 0, & t = 0. \end{cases} \tag{12}$$

Since $x^*(t) \in C[0,T]$, then, $\int_0^t x^*(t)$ is smooth, if necessary, set $x(t) = \int_0^t x^*(t) + u_0$, we have

$$\begin{cases} {}_0^C D_t^{q(t,x(t))} x(t) = f(t,x), t \in (0,T], \\ x(0) = u_0. \end{cases}$$

When $T = +\infty$, we can proceed the similar process, and the theorem also holds. □

Remark 1. For variable fractional order operator defined in the other form (see (3),(4)in Definition 2.1,(6) and (7) in Definition 2.2), the theorem also holds.

4. The stability of the variable fractional order system

In this section, we propose a Lyapunov stability theorem which plays an important role in stability analysis for nonlinear systems. Moreover, we develop the Mittag-Leffler asymptotical stability theorem of the constant fractional order systems to the variable fractional order situation.

Theorem 4.1. *Assume $x = 0$ is an equilibrium point of the variable fractional order system* (1), *and there exists a Lyapunov function $V(t, x(t))$ and class-$\mathcal{K}$ functions α_i $(i = 1, 2, 3)$ satisfying*

$$\alpha_1(||x||) \le V(t,x) \le \alpha_2(||x||), \ {}_0^C D_t^{q(t,x(t))} V(t,x(t)) \le -\alpha_3(||x||), \tag{13}$$

where $0 < q_1 \le q(t,x(t)) \le q_2 < 1,\ t \in [0,\infty)$. Then, the equilibrium point of system (1) *is asymptotically stable.*

Proof. Let the time interval $[0,\infty)$ be divided into a series of subintervals $[t_k, t_{k+1}],\ k = 1, \cdots$, and $\lim_{k\to\infty} t_k = \infty$. For $k = 1, 2, \cdots$, we set $T_k = t_{k+1} - t_k,\ \inf_k T_k > 0$, and $0 < \sup_k T_k < 1,\ k = 1, \cdots$.

Firstly, we give the following inequalities

$$T^{-q(t)} \le \begin{cases} (\frac{1}{T})^{q_2}, & 0 < T < 1, \\ (\frac{1}{T})^{q_1}, & 1 \le T, \end{cases} \tag{14}$$

with $\widehat{T} = max\{(\frac{1}{T})^{q_2}, (\frac{1}{T})^{q_1}\}$.

For $t \in (0,1]$, the Gamma function $\Gamma(t)$ is continuous and decreasing which is obtained that $\Gamma(1-q_1) \le \Gamma(1-q(t)) \le \Gamma(1-q_2)$.

Based on Definition 2.1 and 2.2, for $t \in [t_k, t_{k+1}),\ k = 1, \cdots$, then, we calculate

$$\begin{aligned} {}_{t_k}^C D_t^{q(t,x(t))} V(t,x(t)) &= \int_{t_k}^t \frac{(t-s)^{-q(t,x(t))}}{\Gamma(1-q(t,x(t)))} V'(s,x(s))ds \\ &\ge \frac{1}{\Gamma(1-q_1)} \int_{t_k}^t (t-s)^{-q(t,x(t))} V'(s,x(s))ds \end{aligned} \tag{15}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-q_1)} \int_{t_k}^{t} (\frac{t-s}{T_k})^{-q(t,x(t))} T_k^{-q(t,x(t))} V'(s,x(s))ds \\
&\geq \frac{\widehat{T_k}}{\Gamma(1-q_1)} \int_{t_k}^{t} (\frac{t-s}{T_k})^{-q_2} V'(s,x(s))ds \\
&= \frac{T_k^{q_2}\widehat{T_k}}{\Gamma(1-q_1)} \int_{t_k}^{t} (t-s)^{-q_2} V'(s,x(s))ds \\
&= \frac{1}{L} {}_{t_k}^{C}D_t^{q_2} V(t,x(t)),
\end{aligned}
$$

where $\widehat{T_k} = max\{(\frac{1}{T_k})^{q_2}, (\frac{1}{T_k})^{q_1}\}$, $L = \frac{\Gamma(1-q_1)}{T_k^{q_2}\widehat{T_k}\Gamma(1-q_2)} > 0$, which implies that

$$
{}_{t_k}^{C}D_t^{q_2} V(t,x(t)) \leq -L\ \alpha_3(\alpha_2^{-1}(V(t,x(t)))), \text{ for } t \in [t_k, t_{k+1}).
$$

Then,

$$
{}_{0}^{C}D_t^{q_2} V(t,x(t)) \leq -L\ \alpha_3(\alpha_2^{-1}(V(t,x(t)))), \quad \text{for } t \in [0,\infty).
$$

Based on Fractional Comparison Principle of the constant fractional order type [28], $V(t,x(t))$ can be bounded by the following scalar differential equation,

$$
\begin{aligned}
&V(t,x(t)) \leq g(t),\ for\ \ t \in [0,\infty), \\
&{}_{0}^{C}D_t^{q_2} g(t) = -L\ \alpha_3(\alpha_2^{-1}(g(t))),\ g(0) = V(0,x(0)).
\end{aligned} \tag{16}
$$

Since $\alpha_3\alpha_2^{-1}$ is a class-$\mathcal{K}$ function, it follows from Definition 2.3 that

$$
\begin{cases} g(t) = 0, & if\ g(0) = 0,\ for\ 0 \leq t < \infty, \\ g(t) \geq 0, & otherwise,\ for\ 0 \leq t < \infty. \end{cases}
$$

Then, it can be obtained from (16) that ${}_{0}^{C}D_t^{q_2} g(t) \leq 0$, which implies

$$
g(t) \leq g(0),\ t \in [0,\infty). \tag{17}
$$

In the following, we conduct the proving of the asymptotic stability for (16) by contradiction.

Situation i): Suppose that a constant $t_1 \geq 0$ is existed to satisfy

$$
{}_{0}^{C}D_{t_1}^{q_2} g(t) = -L\ \alpha_3(\alpha_2^{-1}(g(t_1))) = 0,
$$

then, for any $t \geq t_1$, it's obtained that

$$
{}_{0}^{C}D_t^{q_2} g(t) = {}_{t_1}^{C}D_t^{q_2} g(t) = -L\ \alpha_3(\alpha_2^{-1}(g(t))).
$$

According to Definition 2.3, $x = 0$ is the equilibrium point of ${}_{0}^{C}D_t^{q_2} g(t) = -L\ \alpha_3(\alpha_2^{-1}(g(t)))$. Then $g(t) = 0$ for $t \geq t_1$ if $g(t_1) = 0$.

Situation ii): Suppose that a positive constant ε is existed with $g(t) \geq \varepsilon$ for $t \geq 0$. Thus, based on (16), it's obtained that

$$
\varepsilon \leq g(t) \leq g(0). \tag{18}
$$

Then, from (16), we can obtain that

$$\begin{aligned}-L\ \alpha_3(\alpha_2^{-1}(g(t))) &\leq -L\ \alpha_3(\alpha_2^{-1}(\varepsilon))\\ &= -\frac{L\ \alpha_3(\alpha_2^{-1}(g(t))}{g(0)}g(0) \leq -lg(t),\end{aligned}$$

where, $l = \frac{L\ \alpha_3(\alpha_2^{-1}(g(t)))}{g(0)}$. And, for $0 \leq t < \infty$, we have that

$${}_0^C D_t^{q_2} g(t) = -L\ \alpha_3(\alpha_2^{-1}(g(t)) \leq -lg(t).$$

Thus, it implies that $g(t) \leq g(0)E_{q_2}(-lt^{q_2}),\ t \in [0,\infty)$. It is a contradiction for the assumption that $g(t) \geq \varepsilon$.

Summing up the above results for Situation i) and Situation ii), we obtain that $g(t)$ tends to zero as $t \to \infty$. Thus, $V(t, x(t))$ tends to zero as $t \to \infty$. According to (13), $\lim_{t\to\infty} x(t) = 0$. The proof is completed. □

Besides, for the definition of the variable order fractional operator (3), (4), (6), (7), the Theorem 4.1 also holds.

Theorem 4.2. *Let $x = 0$ be an equilibrium point of the variable fractional order system* (1)*. Assume that there exists a Lyapunov function $V(t, x(t))$ and class-$\mathcal{K}$ functions α_4 satisfying*

$$\alpha_4(||x||) \leq V(t,x),\ {}_0^C D_t^{q(t,x(t))} V(t,x(t)) \leq 0, \tag{19}$$

where $0 < q_1 \leq q(t) \leq q_2 < 1,\ t \in [0,\infty)$. Then, the equilibrium point of system (1) *is globally stable.*

Proof. By the same method as in the proof of Theorem 4.1, we can get from (6) and (19) that

$${}_{t_k}^C D_t^{q_2} V(t,x(t)) \leq 0,$$

then,

$${}_0^C D_t^{q_2} V(t,x(t)) \leq 0,\ t \in [0,\infty). \tag{20}$$

Based on Fractional Comparison Principle, it's obtained that

$$V(t,x(t)) \leq V(0,x(0)). \tag{21}$$

According to (19), it implies

$$||x(t)|| \leq \alpha_4^{-1}(V(0,x(0))), \tag{22}$$

for $t \in [0,\infty)$. Thus, the equilibrium is stable. □

Remark 2. The above two stability theorems provide an important tool to guarantee the stability of the controlled system with variable fractional order operator, we will pursue this line in the further.

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