On Vertex H-Irregularity Strengths of Some Graphs

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Abstract

For a graph G a *vertex- covering* of G is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each vertex of $V(G)$ belongs to at least one of the subgraphs H_i, $i = 1, 2, \ldots, t$. In this case we say that G admits an $(H_1, H_2, \ldots, H_t)$*-(vertex) covering*. An H*- covering* of graph G is an ($H_1, H_2, \ldots, H_t$)*-(vertex) covering* in which every subgraph H_i is isomorphic to a given graph H.

Let G be a graph admitting H- covering. A vertex k- labeling $\alpha : E(G) \rightarrow \{1, 2, \ldots, k\}$ is called an H*-irregular vertex* k*-labeling* of the graph G if for every two different subgraphs H' and H'' isomorphic to H their weights $wt_\alpha(H')$ and $wt_\alpha(H'')$ are distinct. The weight of a subgraph H under a vertex k-labeling α is the sum of labels of vertices belonging to H. The *vertex* H*-irregularity strength* of a graph G, denoted by $\mathrm{vhs}(G, H)$, is the smallest integer k such that G has an H-irregular vertex k-labeling. In this paper we determine the exact values of $\mathrm{vhs}(G, H)$ for prisms, antiprisms, triangular ladders and diagonal ladders.

Keywords: H-irregular vertex labeling, vertex H-irregularity strength, prism, antiprism, ladder graphs.
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1 Introduction

Consider a simple and finite graph $G = (V, E)$ of order at least 2. An edge k-labeling is a function $\alpha : E(G) \rightarrow \{1, 2, \ldots, k\}$, where k is a positive integer. Then the associated weight of a vertex $x \in V(G)$ is $w_\alpha(x) = \sum_{xy \in E(G)} \alpha(xy)$, where the sum is taken over all edges incident to x. Such a labeling α is called *irregular* if the obtained weights of all vertices are different. The smallest positive integer k for which there exists an irregular labeling of G is called the *irregularity strength* of G and is denoted by $\mathrm{s}(G)$. If it does not exist, then we write $\mathrm{s(G)} = \infty$.

The notion of the irregularity strength was firstly introduced by Chartrand et al. in [8]. Interesting and fascinating results on the irregularity strength can be found in [2, 3, 5, 7, 9, 10, 12, 13, 14].

A vertex k-labeling $\beta : V(G) \to \{1, 2, \dots, k\}$ is called an *edge irregular k-labeling* of the graph G if the weights $w_\beta(xy) \neq w_\beta(x'y')$ for every two distinct edges xy and $x'y'$, where the weight of an edge $xy \in E(G)$ is $w_\beta(xy) = \beta(x) + \beta(y)$. The minimum k for which a graph G admits an edge irregular k-labeling is called the *edge irregularity strength* of G, denoted by $\mathrm{es}(G)$. The notion of the edge irregularity strength was defined by Ahmad et al. in [1].

A family of subgraphs $H_1, H_2, \dots, H_t$ is said to be a *vertex- covering* of G if each vertex of $V(G)$ belongs to at least one of the sub graphs H_i, $i = 1, 2, \dots, t$. In this case we say that G admits an ($H_1, H_2, \dots, H_t$)-*(vertex) covering.* If every subgraph H_i, $i = 1, 2, \dots, t$, is isomorphic to a given graph H, then the graph G admits an *H-covering.*

Motivated by the irregularity strength and the edge irregularity strength of a graph G Ashraf et al. in [4] introduced a new parameter, vertex H-irregularity strength, as a natural extension of the parameters $\mathrm{s}(G)$ and $\mathrm{es}(G)$. Let G be a graph admitting H- covering. A vertex k-labeling α is called an *H-irregular vertex k-labeling* of the graph G if for every two different subgraphs H' and H'' isomorphic to H we have

$$wt_\alpha(H') = \sum_{v \in V(H')} \alpha(v) \neq \sum_{v \in V(H'')} \alpha(v) = wt_\alpha(H'').$$

The *vertex H-irregularity strength* of a graph G, denoted by $\mathrm{vhs}(G, H)$, is the smallest integer k for which G has a H-irregular vertex k-labeling.

Next theorem proved in [4] give lower bound of the vertex H-irregularity strength.

Theorem 1. *Let G be a graph admitting an H-covering given by t subgraphs isomorphic to H. Then*

$$\mathrm{vhs}(G, H) \geq \left\lceil 1 + \frac{t-1}{|V(H)|} \right\rceil.$$

In this paper we determine exact values of the vertex H-irregularity strength for prisms, Möbius ladder, antiprisms, triangular ladders and diagonal ladders.

2 Results for Prism D_n and M$\ddot{o}$bius Ladder M_n

The prism D_n can be defined as the Cartesian product $C_n \Box P_2$ of a cycle on n vertices with a path on 2 vertices. Let $V(D_n) = \{a_i,\ b_j |\ 1 \leq i \leq n\}$ be the vertex setv and $E(D_n) = \{a_i a_{i+1} |\ 1 \leq i \leq n\} \cup \{b_i b_{i+1} |\ 1 \leq i \leq n\} \cup \{a_i b_i |\ 1 \leq i \leq n\}$ be the edge set, where the suffix“i” is taken modulo n. So, the graph D_n has $2n$ vertices and $3n$ edges.

Theorem 2. *Let $D_n = C_n \Box P_2$, $n \geq 3$, be a prism. Then*

$$vhs(D_n, C_4) = \left\lceil \frac{n+3}{4} \right\rceil.$$

Proof. Let $k = \lceil \frac{n+3}{4} \rceil$. The prism D_n, $n \geq 3$, admits a C_4- covering with exactly n cycles C_4. Every cycle C_4 has the vertex set $V(C_4^i) = \{a_i, a_{i+1}, b_i, b_{i+1}\}$ and the edge set $E(C_4^i) = \{a_ia_{i+1}, b_ib_{i+1}, a_ib_i, a_{i+1}b_{i+1}\}$. From Theorem 1 it follows that $\text{vhs}(D_n, C_4) \geq \lceil \frac{n+3}{4} \rceil$. To show that $\lceil \frac{n+3}{4} \rceil$ is an upper bound for the vertex C_4-irregularity strength of D_n we define a C_4-irregular vertex labeling $\alpha_1 : V(D_n) \to \{1, 2, \ldots, \lceil \frac{n+3}{4} \rceil\}$, in the following way:

For $n = 3$, $\alpha_1(a_1) = \alpha_1(a_2) = \alpha_1(b_2) = 1$, $\alpha_1(a_3) = \alpha_1(b_1) = \alpha_1(b_3) = 2$ and the weights are $wt_{\alpha_1}(C_4^1) = 5$, $wt_{\alpha_1}(C_4^2) = 6$, $wt_{\alpha_1}(C_4^3) = 7$.

Case I. When $n \equiv 0 \pmod 4$;

$$\alpha_1(a_i) = \begin{cases} \lceil \frac{i}{2} \rceil & \text{for } 1 \leq i \leq \frac{n}{2} + 1, \\ \lceil \frac{n-i+3}{2} \rceil & \text{for } \frac{n}{2} + 2 \leq i \leq n. \end{cases}$$

$$\alpha_1(b_i) = \begin{cases} 1 & \text{for } i = 1, 2, n, \\ \lceil \frac{i}{2} \rceil & \text{for } 3 \leq i \leq \frac{n}{2}, \\ k & \text{for } i = \frac{n}{2} + 1, \\ \lceil \frac{n-i+2}{2} \rceil & \text{for } \frac{n}{2} + 2 \leq i \leq n - 1, \end{cases}$$

It is easy to see that under the labeling α_1 all vertices labels are at most $\lceil \frac{n+3}{4} \rceil$. The C_4- weights of the cycle C_4^i, $i = 1, 2, \ldots, n$, under the vertex labeling α_1, are given by;

$$\begin{aligned} wt_{\alpha_1}(C_4^i) &= \sum_{v \in V(C_4^i)} \alpha_1(v), \\ &= \alpha_1(a_i) + \alpha_1(a_{i+1}) + \alpha_1(b_i) + \alpha_1(b_{i+1}) \\ &= \begin{cases} 2(i+1) & \text{for } 1 \leq i \leq \frac{n}{2}, \\ 1 + 2(n-i+2) & \text{for } \frac{n}{2} + 1 \leq i \leq n. \end{cases} \end{aligned}$$

Case II. When $n \equiv 1 \pmod 4$;

$$\alpha_1(a_i) = \begin{cases} \lceil \frac{i}{2} \rceil & \text{for } 1 \leq i \leq \frac{n+1}{2} + 1, \\ \lceil \frac{n-i+3}{2} \rceil & \text{for } \frac{n+1}{2} + 2 \leq i \leq n \end{cases}$$

$$\alpha_1(b_i) = \begin{cases} 1 & \text{for } i = 1, 2, n, \\ \lceil \frac{i}{2} \rceil & \text{for } 3 \leq i \leq \frac{n+1}{2}, \\ \lceil \frac{n-i+2}{2} \rceil & \text{for } \frac{n+1}{2} + 2 \leq i \leq n - 1. \end{cases}$$

It is easy to see that under the labeling α_1 all vertices labels are at most $\lceil \frac{n+3}{4} \rceil$. The C_4- weights of the cycle C_4^i, $i = 1, 2, \ldots, n$, under the vertex labeling α_1, are given by;

$$\begin{aligned} wt_{\alpha_1}(C_4^i) &= \sum_{v \in V(C_4^i)} \alpha_1(v), \\ &= \alpha_1(a_i) + \alpha_1(a_{i+1}) + \alpha_1(b_i) + \alpha_1(b_{i+1}) \\ &= \begin{cases} 2(i+1) & \text{for } 1 \leq i \leq \frac{n+1}{2}, \\ 1 + 2(n-i+2) & \text{for } \frac{n+1}{2} + 1 \leq i \leq n. \end{cases} \end{aligned}$$

Case III. When $n \equiv 2 \pmod 4$;

$$\alpha_1(a_i) = \begin{cases} 1 & \text{for } i = 1, \\ i - k & \text{for } 2 \le i \le k, \\ k & \text{for } k+1 \le i \le n - \lceil \frac{n-k}{3} \rceil, \\ 2\lceil \frac{n-i+1}{2} \rceil + 1 & \text{for } \quad n - \lceil \frac{n-k}{3} \rceil + 1 \le i \le n, \end{cases}$$

$$\alpha_1(b_i) = \begin{cases} 1 & \text{for } 1 \le i \le k, \\ i - k & \text{for } k+1 \le i \le \frac{n}{2}, \\ k-1 & \text{for } i = \frac{n}{2} + 1, \\ k & \text{for } i = \frac{n}{2} + 2, \\ k-2 & \text{for } i = \frac{n}{2} + 3, \\ \max\{k - \left\lceil \frac{1+3-\lceil \frac{n}{2} \rceil}{2} \right\rceil, 1\} & \text{for } \quad \frac{n}{2} + 4 \le i \le n. \end{cases}$$

It is easy to see that under the labeling α_1 all vertices labels are at most $\lceil \frac{n+3}{4} \rceil$. The C_4- weights of the cycle C_4^i, $i = 1, 2, \dots, n$, under the vertex labeling α_1, are given by;

$$\begin{aligned} wt_{\alpha_1}(C_4^i) &= \sum_{v \in V(C_4^i)} \alpha_1(v), \\ &= \alpha_1(a_i) + \alpha_1(a_{i+1}) + \alpha_1(b_i) + \alpha_1(b_{i+1}) \\ &= \begin{cases} 4 & \text{for } \; i = 1, \\ 2i+1 & \text{for } 2 \le i \le \frac{n}{2} - 1, \\ 4k-4 & \text{for } i = \frac{n}{2}, \\ 4k-1 & \text{for } i = \frac{n}{2} + 1, \\ 4k-2 & \text{for } i = \frac{n}{2} + 2, \\ 2(n-i+3) & \text{for } \; \frac{n}{2} + 3 \le i \le n. \end{cases} \end{aligned}$$

Case IV. When $n \equiv 3 \pmod 4$;

$$\alpha_1(a_i) = \begin{cases} 1 & \text{for } i = 1, \\ i - 1 & \text{for } 2 \le i \le k, \\ k & \text{for } k+1 \le i \le n - \lceil \frac{n-k}{3} \rceil, \\ 2\lceil \frac{n-i+1}{2} \rceil + 1 & \text{for } \quad n - \lceil \frac{n-k}{3} \rceil + 1 \le i \le n, \end{cases}$$

$$\alpha_1(b_i) = \begin{cases} 1 & \text{for } 1 \le i \le k, \\ i - k & \text{for } k+1 \le i \le \frac{n+1}{2}, \\ k & \text{for } i = \frac{n+1}{2} + 1, \\ k-1 & \text{for } i = \frac{n+1}{2} + 2, \\ k-2 & \text{for } i = \frac{n+1}{2} + 3, \\ \max\{k - \left\lceil \frac{1+3-\lceil \frac{n}{2} \rceil}{2} \right\rceil, 1\} & \text{for } \quad \frac{n+1}{2} + 4 \le i \le n. \end{cases}$$

It is routine matter to check that under the labeling α_1 all vertices labels are at most $\lceil \frac{n+3}{4} \rceil$. The C_4- weights of the cycle C_4^i, $i = 1, 2, \dots, n$, under the vertex labeling α_1, are given by;

$$
\begin{aligned}
wt_{\alpha_1}(C_4^i) &= \sum_{v\in V(C_4^i)} \alpha_1(v),\\
&= \alpha_1(a_i)+\alpha_1(a_{i+1})+\alpha_1(b_i)+\alpha_1(b_{i+1})\\
&= \begin{cases} 4 & \text{for } \ i=1,\\ 2i+1 & \text{for } 2\le i\le \frac{n+1}{2}-1,\\ 4k-2 & \text{for } i=\frac{n+1}{2},\\ 4k-1 & \text{for } i=\frac{n+1}{2}+1,\\ 4k-3 & \text{for } i=\frac{n+1}{2}+2,\\ 2(n-i+3) & \text{for } \ \frac{n+1}{2}+3\le i\le n. \end{cases}
\end{aligned}
$$

One can see that the weights of cycles C_4^i, for $i=1,2,\dots,n$, in all the cases are distinct. This shows that vhs$(D_n, C_4) \le \lceil \frac{n+3}{4} \rceil$. Hence the proof is concluded. □

The Möbius ladder graph M_n is the graph obtained from the prism graph D_n with two twisted edges. Let $V(M_n)=\{a_i,\ b_j|\ 1\le i\le n\}$ be the vertex set and $E(M_n)=\{a_ia_{i+1}|\ 1\le i\le n-1\}\cup\{b_ib_{i+1}|\ 1\le i\le n-1\}\cup\{a_ib_i|\ 1\le i\le n\}\cup\{a_nb_1, b_na_1\}$ be the edge set, where the suffix"i" is taken modulo n. So, the graph M_n has $2n$ vertices and $3n$ edges.

Theorem 3. *Let M_n, $n\ge 3$, be a Möbius ladder. Then*

$$
vhs(M_n, C_4) = \left\lceil \frac{n+3}{4} \right\rceil .
$$

Proof. The prism M_n, $n\ge 3$, admits a C_4- covering with exactly n cycles C_4. The C_4 cycles has the vertex set $V(C_4^i)=\{a_i, a_{i+1}, b_i, b_{i+1}\}$ for $1\le i\le n-1$ and the edge set $E(C_4^i)=\{a_ia_{i+1}, b_ib_{i+1}, a_ib_i, a_{i+1}b_{i+1}\}$ for $1\le i\le n-1$ with n^{th} cycle $V(C_4^n)=\{a_n, a_1, b_n, b_1\}$ and $E(C_4^n)=\{a_nb_1, b_na_1, a_nb_n, a_1b_1\}$. Observe that the C_4 covering of M_n is the same as of D_n in Theorem 2 except for the *nth* cycle C_4^n which has the same vertices as the preceding theorem. Therefore the labeling α_1 of Theorem 2 works here. This is enough to conclude the result. □

3 Results for Triangular and Diagonal Ladder Graphs

Let $L_n \cong P_n\Box P_2$, $n\ge 2$, be a ladder with the vertex set $V(L_n)=\{a_i, b_i : i=1,2,\dots,n\}$ and the edge set $E(L_n)=\{a_ia_{i+1}, b_ib_{i+1} : i=1,2,\dots,n-1\}\cup\{a_ib_i : i=1,2,\dots,n\}$. The triangular ladder graph TL_n, $n\ge 2$, is obtained from a ladder L_n by adding the edges b_ia_{i+1} for $i=1,2,\dots,n-1$.

Theorem 4. *Let TL_n, $n\ge 2$, be a triangular ladder. Then*

$$
\text{vhs}(TL_n, C_3) = \left\lceil \frac{2n}{3} \right\rceil .
$$

Proof. The triangular ladder TL_n, $n\ge 2$, admits a C_3- covering with exactly $2(n-1)$ cycles C_3. There are two types of cycles C_3 that covers TL_n. The first type of cycles C_3^i have the vertex set $V(C_3^i)=\{a_i, a_{i+1}, b_i : 1\le i\le n-1\}$ and the edge set $E(C_3^i)=$

$\{a_ia_{i+1}, a_ib_i, b_ia_{i+1} : 1 \le i \le n-1\}$. The second type of cycles C_3^i have the vertex set $V(C_3^i) = \{b_i, b_{i+1}, a_{i+1} : 1 \le i \le n-1\}$ and the edge set $E(C_3^i) = \{b_ib_{i+1}, b_ia_{i+1}, b_{i+1}a_{i+1} : 1 \le i \le n-1\}$. According to Theorem 1 it follows that $\text{vhs}(TL_n, C_3) \ge \lceil \frac{2n}{3} \rceil$. To show that $\lceil 2n/3 \rceil$ is an upper bound for the edge C_3-irregularity strength of TL_n we define a C_3-irregular vertex labeling $\alpha_2 : V(TL_n) \to \{1, 2, \dots, \lceil \frac{2n}{3} \rceil\}$, in the following way:

Case 1. When $i \equiv 0 \pmod 3$;

$$\alpha_2(a_i) = \alpha_2(b_i) = \left\lceil \frac{2i}{3} \right\rceil, \text{for} \quad 1 \le i \le n,$$

Case 2. When $i \equiv 1 \pmod 3$;

$$\alpha_2(a_i) = \alpha_2(b_i) = \frac{2i+1}{3}, \text{for} \quad 1 \le i \le n,$$

Case 3. When $i \equiv 2 \pmod 3$;

$$\begin{aligned} \alpha_2(a_i) &= \frac{2i-1}{3}, && \text{for} \quad 1 \le i \le n, \\ \alpha_2(b_i) &= \frac{2i+2}{3} && \text{for} \quad 1 \le i \le n. \end{aligned}$$

It is a routine matter to verify that under the labeling α_2 all edge labels are at most $\lceil 2n/3 \rceil$. It is not difficult to see that under the edge labeling α_2 the weights of the first type of cycles C_3^i, $1 \le i \le n-1$, are of the form:

$$\begin{aligned} wt_{\alpha_2}(C_3^i) &= \alpha_2(a_i) + \alpha_2(a_{i+1}) + \alpha_2(b_i) \\ &= 2i + 1. && \text{for } 1 \le i \le n-1. \end{aligned}$$

It shows that the weights of the first type of cycles C_3^i are odd and increasing. The weights of the second type of cycles C_3^i, $1 \le i \le n-1$, are of the form

$$\begin{aligned} wt_{\alpha_2}(C_3^i) &= \alpha_2(b_i) + \alpha_2(b_{i+1}) + \alpha_2(a_{i+1}) \\ &= 2i + 2. && \text{for } 1 \le i \le n-1. \end{aligned}$$

It shows that the weights of the second type of cycles C_3^i are even and increasing. Combining these two cases of weights we obtained that the weights are different for any two distinct cycles C_3. Thus $\text{vhs}(TL_n, C_3) \le \lceil \frac{2n}{3} \rceil$. This completes the proof. □

The diagonal ladder graph DL_n is obtained from a ladder L_n by adding the edges $\{a_ib_{i+1}, a_{i+1}b_i : 1 \le i \le n-1\}$. So the diagonal ladder DL_n contains $2n$ vertices and $5n-4$ edges.

Theorem 5. *Let DL_n, $n \ge 2$, be a diagonal ladder. Then*

$$\text{vhs}(DL_n, K_4) = \left\lceil \frac{n+3}{4} \right\rceil.$$

Proof. The diagonal ladder DL_n, $n \geq 2$, admits a K_4- covering with exactly $(n-1)$ complete graphs K_4. The K_4^i has the vertex set $V(K_4^i) = \{a_i, b_i, a_{i+1}, b_{i+1} : 1 \leq i \leq n-1\}$ and the edge set $E(C_3^i) = \{a_i a_{i+1}, a_i b_i, a_i b_{i+1}, b_i b_{i+1}, b_i a_{i+1}, a_{i+1} b_{i+1} : 1 \leq i \leq n-1\}$. Then with respect to Theorem 1 it follows that $\text{vhs}(DL_n, K_4) \geq \lceil \frac{n+3}{4} \rceil$. To show that $\text{vhs}(DL_n, K_4) \leq \lceil \frac{n+3}{4} \rceil$ we define a K_4-irregular vertex labeling $\alpha_3 : V(DL_n) \to \{1, 2, \ldots, \lceil \frac{n+3}{4} \rceil\}$, in the following way:

$$\alpha_3(a_i) = \left\lceil \frac{i}{4} \right\rceil, \qquad \text{for} \quad 1 \leq i \leq n,$$

$$\alpha_3(b_i) = \begin{cases} 1 & \text{for} \;\; i = 1, 2, \\ \lceil \frac{i+2}{4} \rceil & \text{for} \;\; 3 \leq i \leq n. \end{cases}$$

One can verify that under the labeling α_3 all vertices labels are at least 1 and at most $\lceil \frac{n+3}{4} \rceil$. To show that α_3 is vertex K_4-irregular labeling it will be enough to show that $wt_{\alpha_3}(K_4^i) < wt_{\alpha_3}(K_4^{i+1})$. The weights of the K_4^i, for $i = 1, 2, \ldots, n-1$, under the labeling α_3, are:

$$\begin{aligned} wt_{\alpha_3}(K_4^i) &= \sum_{v \in V(K_4^i)} \alpha_3(v), \\ &= \alpha_3(a_i) + \alpha_3(a_{i+1}) + \alpha_3(b_i) + \alpha_3(b_{i+1}) \\ &= i + 3, \quad \text{for } 1 \leq i \leq n-1. \end{aligned}$$

This proves that $wt_{\alpha_3}(K_4^{i+1}) = wt_{\alpha_3}(K_4^i) + 1$. Therefore, α_3 is a vertex K_4-irregular labeling of DL_n. Thus $\text{vhs}(DL_n, K_4) \leq \lceil \frac{n+3}{4} \rceil$. This concludes the proof. □

4 Results for Antiprism A_n

The antiprism A_n [6], $n \geq 3$, is a 4-regular graph (Archimedean convex polytope), consisting $2n$ vertices and $4n$ edges. The vertex and edge set of A_n are defined as: $V(A_n) = \{x_i, y_i : 1 \leq i \leq n\}$, $E(A_n) = \{x_i y_i : 1 \leq i \leq n\} \cup \{x_i x_{i+1} : 1 \leq i \leq n\} \cup \{y_i x_{i+1} : 1 \leq i \leq n\} \cup \{y_i y_{i+1} : 1 \leq i \leq n\}$, with indices taken modulo n.

Theorem 6. *Let A_n, $n \geq 3$, be an antiprism. Then*

$$vhs(A_n, C_3) = \left\lceil \frac{2n+2}{3} \right\rceil.$$

Proof. The antiprism A_n, $n \geq 3$, admits a C_3- covering with exactly $2n$ cycles C_3. The first type of the cycle C_3^i has the vertex set $V(C_3^i) = \{a_i, a_{i+1}, b_i : 1 \leq i \leq n\}$ and the edge set $E(C_3^i) = \{a_i b_{i+1}, a_i b_i, b_i a_{i+1} : 1 \leq i \leq n\}$. The second type of the cycle C_3^i has the vertex set $V(C_3^i) = \{b_i, b_{i+1}, a_{i+1} : 1 \leq i \leq n\}$ and the edge set $E(C_3^i) = \{b_i b_{i+1}, b_i a_{i+1}, b_{i+1} a_{i+1} : 1 \leq i \leq n\}$. From Theorem 1 it follows that $\text{vhs}(A_n, C_3) \geq \lceil \frac{2n+2}{3} \rceil$. To show that $\lceil \frac{2n+2}{3} \rceil$ is an upper bound for the vertex C_3-irregularity strength of A_n we define a C_3-irregular vertex labeling $\alpha_4 : V(A_n) \to \{1, 2, \ldots, \lceil \frac{2n+2}{3} \rceil\}$, by letting $k = \lceil \frac{2n+2}{3} \rceil$, in the following way:

Case I. When $n \equiv 0 \text{ mod}(3)$;

$$\alpha_4(a_i)=\begin{cases}\dfrac{4i-3}{3} & \text{for } 1\le i\le \lceil \frac{n}{2}\rceil \text{ and } i\equiv 0 \pmod 3,\\ \dfrac{4i-1}{3} & \text{for } 1\le i\le \lceil \frac{n}{2}\rceil \text{ and } i\equiv 1 \pmod 3,\\ \dfrac{4i-5}{3} & \text{for } 1\le i\le \lceil \frac{n}{2}\rceil \text{ and } i\equiv 2 \pmod 3,\\ k & \text{for } i=\lceil \frac{n}{2}\rceil+1,\\ \dfrac{4(n-i)+9}{3} & \text{for } \lceil \frac{n}{2}\rceil+2\le i\le n \text{ and } i\equiv 0 \pmod 3,\\ \dfrac{4(n-i)+7}{3} & \text{for } \lceil \frac{n}{2}\rceil+2\le i\le n \text{ and } i\equiv 1 \pmod 3,\\ \dfrac{4(n-i)+8}{3} & \text{for } \lceil \frac{n}{2}\rceil+2\le i\le n \text{ and } i\equiv 2 \pmod 3.\end{cases}$$

$$\alpha_4(b_i)=\begin{cases}\dfrac{4i-3}{3} & \text{for } 1\le i\le t-1 \text{ and } i\equiv 0 \pmod 3,\\ \dfrac{4i-1}{3} & \text{for } 1\le i\le t-1 \text{ and } i\equiv 1 \pmod 3,\\ \dfrac{4i+1}{3} & \text{for } 1\le i\le t-1 \text{ and } i\equiv 2 \pmod 3,\\ k & \text{for } i=t,\\ \dfrac{4(n-i)+6}{3} & \text{for } t+1\le i\le n \text{ and } i\equiv 0 \pmod 3,\\ \dfrac{4(n-i)+7}{3} & \text{for } t+1\le i\le n \text{ and } i\equiv 1 \pmod 3,\\ \dfrac{4(n-i)+5}{3} & \text{for } t+1\le i\le n \text{ and } i\equiv 2 \pmod 3,\end{cases}$$

$$\text{where } t=\begin{cases}\lceil \frac{n}{2}\rceil & \text{if } n \text{ is odd},\\ \lceil \frac{n}{2}\rceil+1 & \text{if } n \text{ is even}.\end{cases}$$

One can easily verify that under the labeling α_4 and for $n \equiv 0 \pmod 3$ all edge labeled at least 1 and at most $\lceil 2n+2/3\rceil$. It is not difficult to see that under the edge labeling α_4 the weights of the first type of cycles C_3^i, $1 \le i \le n$, are of the form:

$$\begin{aligned} wt_{\alpha_4}(C_3^i) &= \alpha_4(a_i)+\alpha_4(a_{i+1})+\alpha_4(b_i)\\ &= \begin{cases}4i-1 & \text{for } 1\le i\le \lceil \frac{n}{2}\rceil,\\ 4(n-i)+6 & \text{for } \lceil \frac{n}{2}\rceil+1\le i\le n.\end{cases}\end{aligned}$$

On the other hand, the weights of the second type of cycles C_3^i, $1 \le i \le n$, are of the form

$$\begin{aligned} wt_{\alpha_4}(C_3^i) &= \alpha_4(b_i)+\alpha_4(b_{i+1})+\alpha_4(a_{i+1})\\ &= \begin{cases}4i+1 & \text{for } 1\le i\le s,\\ 4(n-i+1) & \text{for } s+1\le i\le n,\end{cases}\end{aligned}$$

$$\text{where } s=\begin{cases}\lceil \frac{n}{2}\rceil-1 & \text{if } n \text{ is odd},\\ \lceil \frac{n}{2}\rceil & \text{if } n \text{ is even}.\end{cases}$$

Case II. When $n \equiv 1$ mod(3);

$$\alpha_4(a_i)=\begin{cases}1 & \text{for } i=1,\\ \dfrac{4i-3}{3} & \text{for } 1\le i\le \lceil \frac{n}{2}\rceil \text{ and } i\equiv 0 \pmod 3,\\ \dfrac{4i-4}{3} & \text{for } 1\le i\le \lceil \frac{n}{2}\rceil \text{ and } i\equiv 1 \pmod 3,\\ \dfrac{4i-5}{3} & \text{for } 1\le i\le \lceil \frac{n}{2}\rceil \text{ and } i\equiv 2 \pmod 3.\end{cases}$$

- when n is even $$\alpha_4(a_i)=\begin{cases}k & \text{for } i=\lceil \frac{n}{2}\rceil+1 \text{ and } i=\lceil \frac{n}{2}\rceil+2,\\ \dfrac{4(n-i)+11}{3} & \text{for } \lceil \frac{n}{2}\rceil+3\le i\le n \text{ and } i\equiv 0 \pmod 3,\\ \dfrac{4(n-i)+9}{3} & \text{for } \lceil \frac{n}{2}\rceil+3\le i\le n \text{ and } i\equiv 1 \pmod 3,\\ \dfrac{4(n-i)+7}{3} & \text{for } \lceil \frac{n}{2}\rceil+3\le i\le n \text{ and } i\equiv 2 \pmod 3.\end{cases}$$

- when n is odd $$\alpha_4(a_i)=\begin{cases}k & \text{for } i=\lceil \frac{n}{2}\rceil+1,\\ \dfrac{4(n-i)+11}{3} & \text{for } \lceil \frac{n}{2}\rceil+2\le i\le n \text{ and } i\equiv 0 \pmod 3,\\ \dfrac{4(n-i)+9}{3} & \text{for } \lceil \frac{n}{2}\rceil+2\le i\le n \text{ and } i\equiv 1 \pmod 3,\\ \dfrac{4(n-i)+7}{3} & \text{for } \lceil \frac{n}{2}\rceil+2\le i\le n \text{ and } i\equiv 2 \pmod 3.\end{cases}$$

- when n is even $$\alpha_4(b_i)=\begin{cases}\dfrac{4i-3}{3} & \text{for } 1\le i\le \lceil \frac{n}{2}\rceil \text{ and } i\equiv 0 \pmod 3,\\ \dfrac{4i-1}{3} & \text{for } 1\le i\le \lceil \frac{n}{2}\rceil \text{ and } i\equiv 1 \pmod 3,\\ \dfrac{4i-2}{3} & \text{for } 1\le i\le \lceil \frac{n}{2}\rceil \text{ and } i\equiv 2 \pmod 3,\\ k & \text{for } i=\lceil \frac{n}{2}\rceil+1.\end{cases}$$

- when n is odd $$\alpha_4(b_i)=\begin{cases}\dfrac{4i-3}{3} & \text{for } \lceil \frac{n}{2}\rceil+3\le i\le n \text{ and } i\equiv 0 \pmod 3,\\ \dfrac{4i-1}{3} & \text{for } \lceil \frac{n}{2}\rceil+3\le i\le n \text{ and } i\equiv 1 \pmod 3,\\ \dfrac{4i-2}{3} & \text{for } \lceil \frac{n}{2}\rceil+3\le i\le n \text{ and } i\equiv 2 \pmod 3,\\ k & \text{for } i=\lceil \frac{n}{2}\rceil,\ \lceil \frac{n}{2}\rceil+1.\end{cases}$$

$$\alpha_4(b_i)=\begin{cases}\dfrac{4(n-i)+5}{3} & \text{for } \lceil \frac{n}{2}\rceil+2\le i\le n \text{ and } i\equiv 0 \pmod 3,\\ \dfrac{4(n-i)+9}{3} & \text{for } \lceil \frac{n}{2}\rceil+2\le i\le n \text{ and } i\equiv 1 \pmod 3,\\ \dfrac{4(n-i)+7}{3} & \text{for } \lceil \frac{n}{2}\rceil+2\le i\le n \text{ and } i\equiv 2 \pmod 3.\end{cases}$$

One can easily verify that under the labeling α_4 and for $n\equiv 1 \pmod 3$ all edge labeled at least 1 and at most $\lceil 2n+2/3\rceil$. It is not difficult to see that under the edge labeling α_4 the weights of the first type of cycles C_3^i, $1\le i\le n$, are of the form:

$$wt_{\alpha_4}(C_3^i) = \alpha_4(a_i) + \alpha_4(a_{i+1}) + \alpha_4(b_i)$$

$$= \begin{cases} 3 & \text{for } i = 1, \\ 4i-2 & \text{for } 2 \le i \le \lceil \frac{n}{2} \rceil - 1, \\ 3k-5 & \text{for } i = \lceil \frac{n}{2} \rceil \text{ and } n \text{ is even}, \\ 3k-2 & \text{for } i = \lceil \frac{n}{2} \rceil \text{ and } n \text{ is odd}, \\ 3k & \text{for } i = \lceil \frac{n}{2} \rceil + 1 \text{ and } n \text{ is even}, \\ 3k-1 & \text{for } i = \lceil \frac{n}{2} \rceil + 1 \text{ and } n \text{ is odd}, \\ 3k-4 & \text{for } i = \lceil \frac{n}{2} \rceil + 2 \text{ and } n \text{ is even}, \\ 3k-7 & \text{for } i = \lceil \frac{n}{2} \rceil + 2 \text{ and } n \text{ is odd}, \\ 4(n-i)+7 & \text{for } \lceil \frac{n}{2} \rceil + 3 \le i \le n. \end{cases}$$

On the other hand, the weights of the second type of cycles C_3^i, $1 \le i \le n$, are of the form:

$$wt_{\alpha_4}(C_3^i) = \alpha_4(b_i) + \alpha_4(b_{i+1}) + \alpha_4(a_{i+1})$$

$$= \begin{cases} 4i & \text{for } 1 \le i \le \lceil \frac{n}{2} \rceil - 2, \\ 3k-8 & \text{for } i = \lceil \frac{n}{2} \rceil - 1 \text{ and } n \text{ is even}, \\ 3k-5 & \text{for } i = \lceil \frac{n}{2} \rceil - 1 \text{ and } n \text{ is odd}, \\ 3k-2 & \text{for } i = \lceil \frac{n}{2} \rceil \text{ and } n \text{ is even}, \\ 3k & \text{for } i = \lceil \frac{n}{2} \rceil \text{ and } n \text{ is odd}, \\ 3k-1 & \text{for } i = \lceil \frac{n}{2} \rceil + 1 \text{ and } n \text{ is even}, \\ 3k-4 & \text{for } i = \lceil \frac{n}{2} \rceil + 1 \text{ and } n \text{ is odd}, \\ 3k-7 & \text{for } i = \lceil \frac{n}{2} \rceil + 2 \text{ and } n \text{ is even}, \\ 3k-9 & \text{for } i = \lceil \frac{n}{2} \rceil + 2 \text{ and } n \text{ is odd}, \\ 4(n-i)+5 & \text{for } \lceil \frac{n}{2} \rceil + 3 \le i \le n. \end{cases}$$

Case III. When $n \equiv 2 \text{ mod}(3)$;

There are three more subcases in this case and we shall discuss these one by one.

Case III.A. When $n \equiv 2 \text{ mod}(3)$ and $k \equiv 2 \pmod 3$.

$$\alpha_4(a_i) = \left\lceil \frac{2i-1}{3} \right\rceil, \quad \text{for } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil$$

$$\alpha_4(a_i) = \begin{cases} k & \text{for } i = n-2,\ n-1, \\ k-1 & \text{for } i = n. \end{cases}$$

• when n is odd and $\lceil \frac{n}{2} \rceil + 1 \le i \le k-2$,

$$\alpha_4(a_i) = \begin{cases} \lceil \frac{n}{3} \rceil - 1 + 2 \left\lceil \frac{i-1-\lceil \frac{n}{2} \rceil}{3} \right\rceil & \text{foi} \equiv 1 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 2 \left\lceil \frac{i-2-\lceil \frac{n}{2} \rceil}{3} \right\rceil + 3 & \text{for } i \equiv 2 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 3 + 2 \left\lceil \frac{i-3-\lceil \frac{n}{2} \rceil}{3} \right\rceil & \text{for } i \equiv 0 \pmod 3. \end{cases}$$

• when n is odd and $k-1 \leq i \leq n-3$,

$$\alpha_4(a_i) = \begin{cases} \lceil \frac{n}{3} \rceil + 2 \left\lceil \frac{k-5-\lceil \frac{n}{2} \rceil}{3} \right\rceil + 2 \lceil \frac{i-k+1}{3} \rceil + 1 & \text{for } i \equiv 1 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 5 + 2 \left\lceil \frac{k-5-\lceil \frac{n}{2} \rceil}{3} \right\rceil + 2 \lceil \frac{i-k}{3} \rceil & \text{for } i \equiv 2 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 2 \left\lceil \frac{k-5-\lceil \frac{n}{2} \rceil}{3} \right\rceil + 2 \lceil \frac{i-k-1}{3} \rceil + 6 & \text{for } i \equiv 0 \pmod 3, \end{cases}$$

• when n is even and $\lceil \frac{n}{2} \rceil + 1 \leq i \leq k-2$,

$$\alpha_4(a_i) = \begin{cases} \lceil \frac{n}{3} \rceil + 2 \left\lceil \frac{i-1-\lceil \frac{n}{2} \rceil}{3} \right\rceil + 2 & \text{for } i \equiv 2 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 2 + 2 \left\lceil \frac{i-2-\lceil \frac{n}{2} \rceil}{3} \right\rceil & \text{for } i \equiv 0 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 2 \left\lceil \frac{i-3-\lceil \frac{n}{2} \rceil}{3} \right\rceil & \text{for } i \equiv 1 \pmod 3. \end{cases}$$

• when n is even and $k-1 \leq i \leq n-3$,

$$\alpha_4(a_i) = \begin{cases} \lceil \frac{n}{3} \rceil + 2 \left\lceil \frac{k-4-\lceil \frac{n}{2} \rceil}{3} \right\rceil + 2 \lceil \frac{i-k+1}{3} \rceil & \text{for } i \equiv 1 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 4 + 2 \left\lceil \frac{k-4-\lceil \frac{n}{2} \rceil}{3} \right\rceil + 2 \lceil \frac{i-k}{3} \rceil & \text{for } i \equiv 2 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 2 \left\lceil \frac{k-4-\lceil \frac{n}{2} \rceil}{3} \right\rceil + 2 \lceil \frac{i-k-1}{3} \rceil + 5 & \text{for } i \equiv 0 \pmod 3. \end{cases}$$

The labels of the b_i vertices are given as under:

• when n is odd:

$$\alpha_4(b_i) = \begin{cases} \lfloor \frac{2i+2}{3} \rfloor & \text{for } 1 \leq i \leq \lceil \frac{k}{2} \rceil \\ \lceil \frac{2i+1}{3} \rceil & \text{for } \lceil \frac{k}{2} \rceil + 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1 \\ k & \text{for } i = n-2, \\ k-1 & \text{for } i = n-1, \ n. \end{cases}$$

• when n is odd and $\lceil \frac{n}{2} \rceil + 1 \leq i \leq k-3$;

$$\alpha_4(b_i) = \begin{cases} \lceil \frac{n}{3} \rceil + 2 \left\lceil \frac{i-\lceil \frac{n}{2} \rceil}{3} \right\rceil + 2 & \text{for } i \equiv 0 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 2 + 2 \left\lceil \frac{i-1-\lceil \frac{n}{2} \rceil}{3} \right\rceil & \text{for } i \equiv 1 \pmod 3, \\ \lceil \frac{n}{3} \rceil + 2 \left\lceil \frac{i-2-\lceil \frac{n}{2} \rceil}{3} \right\rceil & \text{for } i \equiv 2 \pmod 3. \end{cases}$$

• when n is odd and $k-2 \leq i \leq n-3$;

$$\alpha_4(b_i)=\begin{cases}\left\lceil\frac{n}{3}\right\rceil+2\left\lceil\frac{k-5-\lceil\frac{n}{2}\rceil}{3}\right\rceil+2\left\lceil\frac{i-k+2}{3}\right\rceil+4 & \text{for } i\equiv 0 \pmod 3,\\ \left\lceil\frac{n}{3}\right\rceil+5+2\left\lceil\frac{k-5-\lceil\frac{n}{2}\rceil}{3}\right\rceil+2\left\lceil\frac{i-k+1}{3}\right\rceil & \text{for } i\equiv 1 \pmod 3,\\ \left\lceil\frac{n}{3}\right\rceil+2\left\lceil\frac{k-5-\lceil\frac{n}{2}\rceil}{3}\right\rceil+2\left\lceil\frac{i-k}{3}\right\rceil+2 & \text{for } i\equiv 2 \pmod 3.\end{cases}$$

- when n is even:

$$\alpha_4(b_i)=\begin{cases}\left\lfloor\frac{2i+2}{3}\right\rfloor & \text{for } 1\le i\le\left\lceil\frac{k}{2}\right\rceil\\ \left\lceil\frac{2i+1}{3}\right\rceil & \text{for } \left\lceil\frac{k}{2}\right\rceil+1\le i\le\left\lceil\frac{n}{2}\right\rceil\\ k & \text{for } i=n-2,\\ k-1 & \text{for } i=n-1,\ n.\end{cases}$$

- when n is even and $\left\lceil\frac{n}{2}+1\right\rceil+1\le i\le k-3$;

$$\alpha_4(b_i)=\begin{cases}\left\lceil\frac{n}{3}\right\rceil+2\left\lceil\frac{i-1-\lceil\frac{n}{2}\rceil}{3}\right\rceil-1 & \text{for } i\equiv 2 \pmod 3,\\ \left\lceil\frac{n}{3}\right\rceil+3+2\left\lceil\frac{i-2-\lceil\frac{n}{2}\rceil}{3}\right\rceil & \text{for } i\equiv 0 \pmod 3,\\ \left\lceil\frac{n}{3}\right\rceil+2\left\lceil\frac{i-3-\lceil\frac{n}{2}\rceil}{3}\right\rceil+3 & \text{for } i\equiv 1 \pmod 3.\end{cases}$$

- when n is even and $k-2\le i\le n-3$;

$$\alpha_4(b_i)=\begin{cases}\left\lceil\frac{n}{3}\right\rceil+2\left\lceil\frac{k-7-\lceil\frac{n}{2}\rceil}{3}\right\rceil+2\left\lceil\frac{i-k+2}{3}\right\rceil+5 & \text{for } i\equiv 0 \pmod 3,\\ \left\lceil\frac{n}{3}\right\rceil+6+2\left\lceil\frac{k-7-\lceil\frac{n}{2}\rceil}{3}\right\rceil+2\left\lceil\frac{i-k+1}{3}\right\rceil & \text{for } i\equiv 1 \pmod 3,\\ \left\lceil\frac{n}{3}\right\rceil+2\left\lceil\frac{k-7-\lceil\frac{n}{2}\rceil}{3}\right\rceil+2\left\lceil\frac{i-k}{3}\right\rceil+3 & \text{for } i\equiv 2 \pmod 3.\end{cases}$$

One can easily verify that under the labeling α_4 in the Case III.A all edge labeled at least 1 and at most $\lceil 2n+2/3\rceil$. It is not difficult to see that under the edge labeling α_4 the weights of the first type of cycles C_3^i, $1\le i\le n$, when n is odd, are of the form:

$$\begin{aligned} wt_{\alpha_4}(C_3^i) &= \alpha_4(a_i)+\alpha_4(a_{i+1})+\alpha_4(b_i)\\ &=\begin{cases}2i+1 & \text{for } 1\le i\le\left\lceil\frac{k}{2}\right\rceil-1,\\ 2\left\lceil\frac{k}{2}\right\rceil+2+2(i-\left\lceil\frac{k}{2}\right\rceil) & \text{for } \left\lceil\frac{k}{2}\right\rceil\le i\le\left\lceil\frac{n}{2}\right\rceil-1,\\ 2\left\lceil\frac{n}{2}\right\rceil+1 & \text{for } i=\left\lceil\frac{n}{2}\right\rceil,\\ 2\left\lceil\frac{n}{2}\right\rceil+4+2(i-\left\lceil\frac{n}{2}\right\rceil-1) & \text{for } \left\lceil\frac{n}{2}\right\rceil+1\le i\le k-2,\\ 2k+1+2(i-k+1) & \text{for } k-1\le i\le n-3,\\ 3k & \text{for } i=n-2,\\ 3k-2 & \text{for } i=n-1,\\ 2k-1 & \text{for } i=n.\end{cases}\end{aligned}$$

The weights, under the labeling α_4, of the second type of cycles C_3^i, $1 \le i \le n$, when n is odd, are of the form:

$$wt_{\alpha_4}(C_3^i) = \alpha_4(b_i) + \alpha_4(b_{i+1}) + \alpha_4(a_{i+1})$$
$$= \begin{cases} 2i+2 & \text{for } 1 \le i \le \lceil \frac{k}{2} \rceil - 1, \\ 2\lceil \frac{k}{2} \rceil + 3 + 2(i - \lceil \frac{k}{2} \rceil) & \text{for } \lceil \frac{k}{2} \rceil \le i \le \lceil \frac{n}{2} \rceil - 1, \\ 2\lceil \frac{n}{2} \rceil + 3 & \text{for } i = \lceil \frac{n}{2} \rceil, \\ 2\lceil \frac{n}{2} \rceil + 5 + 2(i - \lceil \frac{n}{2} \rceil - 1) & \text{for } \lceil \frac{n}{2} \rceil + 1 \le i \le k-3, \\ 2k + 2(i-k+2) & \text{for } k-2 \le i \le n-3, \\ 3k-1 & \text{for } i = n-2, \\ 3k-3 & \text{for } i = n-1, \\ k+1 & \text{for } i = n. \end{cases}$$

The weights, under the labeling α_4, of the first type of cycles C_3^i, $1 \le i \le n$, when n is even, are of the form:

$$wt_{\alpha_4}(C_3^i) = \alpha_4(a_i) + \alpha_4(a_{i+1}) + \alpha_4(b_i)$$
$$= \begin{cases} 2i+1 & \text{for } 1 \le i \le \lceil \frac{k}{2} \rceil - 1, \\ 2\lceil \frac{k}{2} \rceil + 2 + 2(i - \lceil \frac{k}{2} \rceil) & \text{for } \lceil \frac{k}{2} \rceil \le i \le \lceil \frac{n}{2} \rceil - 1, \\ n+3 & \text{for } i = \lceil \frac{n}{2} \rceil, \\ n+4+2(i - \lceil \frac{n}{2} \rceil - 1) & \text{for } \lceil \frac{n}{2} \rceil + 1 \le i \le k-2, \\ 2k+2+2(i-k+1) & \text{for } k-1 \le i \le n-3, \\ 3k & \text{for } i = n-2, \\ 3k-2 & \text{for } i = n-1, \\ 2k-1 & \text{for } i = n. \end{cases}$$

The weights, under the labeling α_4, of the second type of cycles C_3^i, $1 \le i \le n$, when n is even, are of the form:

$$wt_{\alpha_4}(C_3^i) = \alpha_4(b_i) + \alpha_4(b_{i+1}) + \alpha_4(a_{i+1})$$
$$= \begin{cases} 2i+2 & \text{for } 1 \le i \le \lceil \frac{k}{2} \rceil - 1, \\ 2\lceil \frac{k}{2} \rceil + 3 + 2(i - \lceil \frac{k}{2} \rceil) & \text{for } \lceil \frac{k}{2} \rceil \le i \le \lceil \frac{n}{2} \rceil - 1, \\ 2\lceil \frac{n}{2} \rceil + 2 & \text{for } i = \lceil \frac{n}{2} \rceil, \\ 2\lceil \frac{n}{2} \rceil + 5 + 2(i - \lceil \frac{n}{2} \rceil - 1) & \text{for } \lceil \frac{n}{2} \rceil + 1 \le i \le k-3, \\ 2k + 2(i-k+2) & \text{for } k-2 \le i \le n-3, \\ 3k-1 & \text{for } i = n-2, \\ 3k-3 & \text{for } i = n-1, \\ k+1 & \text{for } i = n. \end{cases}$$

One can check that the weights in this case are distinct under the labeling α_4.

Case III.B. When $n \equiv 2 \bmod(3)$ and $k \equiv 1 \pmod 3$;

$$\alpha_4(a_i)=\begin{cases}\left\lceil\frac{2i-1}{3}\right\rceil & \text{for } 1\le i\le\left\lceil\frac{k}{2}\right\rceil+1,\\ \frac{k+5}{3}+\left\lfloor\frac{2(i-\lceil\frac{k}{2}\rceil-2)+1}{3}\right\rfloor & \text{for } \left\lceil\frac{k}{2}\right\rceil+2\le i\le k-1,\\ \frac{k+5}{3}+\left\lfloor\frac{k-5}{3}\right\rfloor+1 & \text{for } i=k,\\ \left\lfloor\frac{2k+6}{3}\right\rfloor+\left\lfloor\frac{2(i-k-1)+1}{3}\right\rfloor+1 & \text{for } k+1\le i\le n-1,\\ k & \text{for } i=n.\end{cases}$$

$$\alpha_4(b_i)=\begin{cases}1 & \text{for } i=1\\ \left\lceil\frac{2i}{3}\right\rceil & \text{for } 2\le i\le\left\lceil\frac{k}{2}\right\rceil-1,\\ \frac{k+2}{3}+\left\lfloor\frac{2(i-\lceil\frac{k}{2}\rceil)+1}{3}\right\rfloor & \text{for } \left\lceil\frac{k}{2}\right\rceil\le i\le k-3,\\ \frac{k+2}{3}+\left\lfloor\frac{k-5}{3}\right\rfloor+1 & \text{for } i=k-2,\\ \frac{k+2}{3}+\left\lfloor\frac{k-5}{3}\right\rfloor+2 & \text{for } i=k-21,\\ \left\lfloor\frac{2k+3}{3}\right\rfloor+\left\lceil\frac{2(i-k)+1}{3}\right\rceil & \text{for } k\le i\le n-2,\\ k & \text{for } i=n-1,\ n.\end{cases}$$

Case III.C. When $n\equiv 2 \bmod(3)$ and $k\equiv 0 \pmod 3$;

$$\alpha_4(a_i)=\begin{cases}\left\lceil\frac{2i-1}{3}\right\rceil & \text{for } 1\le i\le\left\lceil\frac{k}{2}\right\rceil+2,\\ \frac{k+9}{3}+2\left\lfloor\frac{i-\lceil\frac{k}{2}\rceil-3}{3}\right\rfloor & \text{for } \left\lceil\frac{k}{2}\right\rceil+3\le i\le k+1,\\ \frac{2k+6}{3}+\left\lfloor\frac{2(i-k-2)}{3}\right\rfloor & \text{for } k+2\le i\le n,\end{cases}$$

$$\alpha_4(b_i)=\begin{cases}1 & \text{for } i=1,\\ \left\lceil\frac{2i}{3}\right\rceil & \text{for } 2\le i\le\left\lceil\frac{k}{2}\right\rceil,\\ \frac{k+6}{3}+2\left\lfloor\frac{2(i-\lceil\frac{k}{2}\rceil)-1}{3}\right\rfloor & \text{for } \left\lceil\frac{k}{2}\right\rceil+1\le i\le k-1,\\ \frac{2k+3}{3}+\left\lfloor\frac{2(i-k)+1}{3}\right\rfloor & \text{for } k\le i\le n.\end{cases}$$

It is easily to verify that under the labeling α_4 in the Case III.B and Case III.C all edge labeled at least 1 and at most $\lceil 2n+2/3\rceil$. It is not difficult to see that under the edge labeling α_4 the weights of the first type of cycles C_3^i, $1\le i\le n$ are of the form:

$$\begin{aligned} wt_{\alpha_4}(C_3^i) &= \alpha_4(a_i)+\alpha_4(a_{i+1})+\alpha_4(b_i)\\ &= \begin{cases}2i+1 & \text{for } 1\le i\le\left\lceil\frac{k}{2}\right\rceil,\\ k+4+2(i-\left\lceil\frac{k}{2}\right\rceil-1) & \text{for } \left\lceil\frac{k}{2}\right\rceil+1\le i\le k-1,\\ 2k+3+2(i-k) & \text{for } k\le i\le n-1,\\ 2k+1 & \text{for } i=n.\end{cases}\end{aligned}$$

Under the edge labeling α_4 the weights of the second type of cycles C_3^i, $1\le i\le n$ are of the form:

$$
\begin{aligned}
wt_{\alpha_4}(C_3^i) &= \alpha_4(b_i) + \alpha_4(b_{i+1}) + \alpha_4(a_{i+1}) \\
&= \begin{cases} 2i+2 & \text{for } 1 \le i \le \lceil \frac{k}{2} \rceil - 1, \\ k+3+2(i-\lceil \frac{k}{2} \rceil) & \text{for } \lceil \frac{k}{2} \rceil \le i \le k-2, \\ 2k+2+2(i-k+1) & \text{for } k-1 \le i \le n-1, \\ k+2 & \text{for } i = n. \end{cases}
\end{aligned}
$$

It is a matter of routine to check that the weights in this case are distinct under the labeling α_4. Thus the α_4 is a vertex C_3-irregular labeling. This concludes the theorem. □

5 Conclusion

In this paper we have investigate the vertex H-irregularity strength of graphs. We have found the exact values of this parameter for several families of graphs namely, prisms, antiprisms, triangular ladders and diagonal ladders.

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