THE DYNAMICS OF THE RELATIVISTIC KEPLER PROBLEM

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Abstract. We deal with the Hamiltonian system (HS) provided by the correction given by the special relativity to the motion of the two–body problem, or by the first order correction to this problem coming from the general relativity. This Hamiltonian system is completely integrable with the Hamiltonian H and the angular momentum C. We have two objectives.

First, we describe the global dynamics of the Hamiltonian system (HS) in the following sense. Let I_h (respectively I_c) be the subset of the phase space where $H = h$ (respectively $C = c$). Since H and C are first integrals, the sets I_h, I_c and $I_{hc} = I_h \cap I_c$ are invariant under the flow of the Hamiltonian system (HS). We determine the global dynamics on those sets when h and c vary.

Second, recently Tudoran in [19] provided a criterion which detects when a non–degenerate equilibrium point of a completely integrable system is Lyapunov stable. Every equilibrium point q of the completely integrable Hamiltonian system (HS) is degenerate and has zero angular momentum, so the mentioned criterion cannot be applied to it. But we will show that this criterion is also satisfied when it is applied to the Hamiltonian system (HS) restricted to zero angular momentum.

1. Introduction

The Manev Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{a}{r} + \frac{b}{r^2},$$

where a and b are arbitrary constant. This Hamiltonian describes the motion of a two–body problem defined by the potential $a/r + b/r^2$, where r is the distance between the two particles.

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The analysis of the motion of the two-body problems has a long story. Just after the Newton's work on the two–body gravitational problem, some discrepancies appeared between the theoretical motions of the pericenters of the planets and the observed ones. Consequently some doubts on the accuracy of the Newtonian inverse square law of gravitation motivated alternative gravitational models and corrections trying to reconcile these discrepancies. In fact Newton was the first in considering what we call now the Manev systems, see the book I, section IX, proposition XLIV, theorem XIV and corollary 2 of the Principia.

Several authors tried to find a suitable model with the advantages of the Newtonian one but with the convenient corrections in order that the orbits coming close to the observe ones in the solar system. There were many pre– and post–relativistic attempts to obtain such models. This is the case of the Manev Hamiltonian introduced by Manev in [12, 13, 14, 15]. See for more information on the two–body problem for instance [1, 2, 3].

The correction given by the special relativity to the motion of the two–body problem, or by the first order correction to this problem coming from the general relativity is

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{1}{r} - \frac{\varepsilon}{2r^2},$$

where $\varepsilon > 0$ is small, for details see [6, 7, 9, 17, 18]. Note that when $\varepsilon = 0$ we have the rotating Kepler problem, for more information on this last problem see [16].

Note that the Hamiltonian H is a particular case of the Manev Hamiltonian $\mathcal{H}$ when $a = -1$ and $b = -\varepsilon/2$.

The Hamiltonian system defined by the Hamiltonian H is

$$\begin{aligned} \dot{r} &= p_r, \\ \dot{\theta} &= \frac{p_\theta}{r^2}, \\ \dot{p_r} &= \frac{p_\theta^2 - \varepsilon}{r^3} - \frac{1}{r^2}, \\ \dot{p_\theta} &= 0. \end{aligned} \tag{1}$$

For a fixed $\varepsilon < 0$ small this Hamiltonian system has the circle of equilibria

$$\{q(\theta) = (r, \theta, p_\theta, p_r) = (-\varepsilon, \theta, 0, 0) : \theta \in \mathbb{S}^1\}.$$

The Hamiltonian system (1) is completely integrable in the sense of Liouville–Arnold because it has two independent first integrals the Hamiltonian H and the angular momentum p_θ in involution, for more details on completely integrable Hamiltonian systems see [4, 5].

The objective of this paper is double. Our first objective is to describe the global dynamics of the Hamiltonian system (1) in the following sense. Let I_h (respectively I_c) be the subset of the phase space where H (respectively p_θ) takes the value h (respectively c). Due to the fact that H and p_θ are first integrals, the sets I_h, I_c and $I_{hc} = I_h \cap I_c$ are invariant under the flow of the Hamiltonian system (1). We determine the global dynamics on those sets when h and c vary. Moreover, we describe the foliation of the phase space by the invariant sets I_h, and the foliation of I_h by the invariant sets I_{hc}. See section 2.

In fact this first objective was studied inside the Manev Hamiltonian $\mathcal{H}$ with arbitrary values of a and b, see [10], but from there it is not easy to obtain the global dynamics for the particular case $a = -1$ and $b = -\varepsilon/2$. We do this work here.

Second we shall show that every equilibrium point $q(\theta)$ of the Hamiltonian system (1) restricted to zero angular momentum is Lyapunov stable. But our interest in this Lyapunov stability becomes for showing that the criterion provided recently by Tudoran in [19] which detects when a non–degenerate equilibrium point of a completely integrable system is Lyapunov stable, also can be extended to the degenerate equilibrium points of the completely integrable system (1) restricted to zero angular momentum where the equilibrium points of system (1) live, see section 3.

2. On the global dynamics

2.1. **Critical points and critical values.** Let us adopt the notation and terminology of [10], as consequence, we suppose that I_h (respectively I_c) be the subset of the phase space where H (respectively p_θ) takes the value h (respectively c). Due to the fact that H and p_θ are first integrals, the sets I_h, I_c and $I_{hc} = I_h \cap I_c$ are invariant under the flow of the Hamiltonian system (1), i.e. if an orbit of the Hamiltonian system has a point on the set I_{hc} then the whole orbit is contained in this set.

Let $r \in \mathbb{R}^+ = (0, \infty)$, $\theta \in \mathbb{S}^1$, $(p_r, p_\theta) \in \mathbb{R}^2$ and $E = \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R}^2$, where

$$
\begin{aligned}
I_h = & \ \{(r, \theta, p_r, p_\theta) \in E : H(r, \theta, p_r, p_\theta) = h\}, \\
I_c = & \ \{(r, \theta, p_r, p_\theta) \in E : p_\theta = c\}.
\end{aligned}
$$

In this case the set of critical points of H is

$$\mathcal{C} = \{(r, \theta, p_r, p_\theta) \in E : r + \varepsilon = 0, \theta \in \mathbb{S}^1\}$$

since $r > 0$, then

$$\mathcal{C} = \left\{ \begin{array}{cl} \emptyset & \text{if } \varepsilon \geq 0, \\ -\varepsilon & \text{if } \varepsilon < 0. \end{array} \right\}$$

hence the critical value of H is $1/2\varepsilon$ if $\varepsilon < 0$

2.2. **Hill Regions.** Let E and $\mathcal{R} = \mathbb{R}^+ \times \mathbb{S}^1$ be the phase space and the configuration space of the Hamiltonian system (1), and let $\Gamma : E \to \mathcal{R}$ be the natural projection from E to $\mathcal{R}$. Then for each h belongs to real set $\mathbb{R}$, the regions of motion R_h (*Hill regions*) of I_h are defined by $\Gamma(I_h) = R_h$, for more details see [8, 10], hence

$$
\begin{aligned}
R_h = & \ \{(r, \theta) \in \mathcal{R} : -\frac{1}{r} - \frac{\varepsilon}{2r^2} \leq h\} \\
= & \ \{r \in R^+ : 2hr^2 + 2r + \varepsilon \geq 0\} \times \mathbb{S}^1.
\end{aligned}
$$

Therefore, if $h < 0$ then R_h is homeomorphic to

$$
\begin{array}{ll}
\emptyset & \text{if } \varepsilon < \dfrac{1}{2h}, \\
\left\{-\dfrac{1}{h}\right\} \times \mathbb{S}^1 & \text{if } \varepsilon = \dfrac{1}{2h}, \\
\left[-\dfrac{1 - \sqrt{1 - 2h\varepsilon}}{h}, -\dfrac{1 + \sqrt{1 - 2h\varepsilon}}{h}\right] \times \mathbb{S}^1 & \text{if } \dfrac{1}{2h} < \varepsilon < 0, \\
\left(0, -\dfrac{1 + \sqrt{1 - 2h\varepsilon}}{h}\right] \times \mathbb{S}^1 & \text{if } \varepsilon \geq 0.
\end{array}
\tag{2}
$$

If $h = 0$ then R_h is homeomorphic to

$$
\begin{array}{ll}
\left[-\dfrac{\varepsilon}{2}, \infty\right) \times \mathbb{S}^1 & \text{if } \varepsilon < 0, \\
\mathbb{R}^+ \times \mathbb{S}^1 & \text{if } \varepsilon \geq 0.
\end{array}
\tag{3}
$$

If $h > 0$ then R_h is homeomorphic to

$$\begin{aligned} &\left[\frac{\sqrt{1-2h\varepsilon}-1}{h}, \infty\right) \times \mathbb{S}^1 \quad \text{if } \varepsilon < 0, \\ &\mathbb{R}^+ \times \mathbb{S}^1 \qquad\qquad\qquad\quad \text{if } \varepsilon \geq 0. \end{aligned} \tag{4}$$

2.3. **The sets I_h.** Now we determine the topology of the invariant energy levels I_h by using the fact:

$$\begin{aligned} I_h = &\ \{(r, \theta, p_r, p_\theta) \in E : g(r, p_r, p_\theta) = h\} \\ \approx &\ \{g^{-1}(h)\} \times \mathbb{S}^1, \end{aligned}$$

where

$$g(r, p_r, p_\theta) = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{1}{r} - \frac{\varepsilon}{2r^2},$$

If h is a regular value of the map $g : \mathbb{R}^+ \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ and the set of points $\{g^{-1}(h)\} \neq \emptyset$, then the set $\{g^{-1}(h)\}$ is a surface in $\mathbb{R}^+ \times \mathbb{R}^2$. Hence the intersection of $\{g^{-1}(h)\}$ with $\{r = r_0 = \text{constant}\}$, is
either an ellipse if $\dfrac{1}{r_0} + \dfrac{\varepsilon}{2r_0^2} + h > 0$,
or a point if $\dfrac{1}{r_0} + \dfrac{\varepsilon}{2r_0^2} + h = 0$,
or the empty set if $\dfrac{1}{r_0} + \dfrac{\varepsilon}{2r_0^2} + h < 0$.

¿From the definition of the energy levels I_h we obtain

$$I_h = \bigcup_{(r,\theta)\in R_h} E_{(r,\theta)} \tag{5}$$

where

$$E_{(r,\theta)} = \{(r, \theta, p_r, p_\theta) \in E : p_r^2 + \frac{p_\theta^2}{r^2} = \frac{1}{4r^2}\big(2hr^2 + 2r + \varepsilon\big)\}.$$

For each point $(r, \theta) \in R_h$ the set $E_{(r,\theta)}$ is an ellipse, a point or the empty set if the point (r, θ) belongs to the interior of the Hill region R_h, to the boundary of R_h, or does not belong to R_h, respectively. Therefore every energy level I_h of the planar relativistic Kepler problem is homeomorphic, from (2) and (6), to

$$\begin{array}{ll} \emptyset & \text{if } \varepsilon < \dfrac{1}{2h}, \\ \mathbb{S}^1 & \text{if } \varepsilon = \dfrac{1}{2h}, \\ \mathbb{S}^2 \times \mathbb{S}^1 & \text{if } \dfrac{1}{2h} < \varepsilon < 0, \\ (\mathbb{S}^2 \times \mathbb{S}^1) \setminus \mathbb{S}^1 & \text{if } 0 \leq \varepsilon, \end{array}$$

if $h < 0$; and from (3), (4) and (6) to

$$\begin{array}{ll} (\mathbb{S}^2 \times \mathbb{S}^1) \setminus \mathbb{S}^1 & \text{if } \varepsilon < 0, \\ \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{S}^1 & \text{if } \varepsilon \geq 0, \end{array}$$

if $h \geq 0$.

2.4. **The sets I_c.** Since $I_c = \{(r, \theta, p_r, p_\theta) \in E : p_\theta = c\}$, we get that I_c is homeomorphic to $\mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R}$ for all $c \in \mathbb{R}$.

2.5. **The foliation of I_h by I_{hc}.** We can compute the invariant set I_{hc} from knowing the set $\{g^{-1}(h)\}$ and

$$\begin{aligned} I_{hc} = & \ I_h \cap I_c \\ = & \ I_h \cap \{p_\theta = c\} \\ = & \ (\{g^{-1}(h)\} \cap \{p_\theta = c\}) \times \mathbb{S}^1 \end{aligned} \tag{6}$$

Hence the foliation of I_h by I_{hc} can be described when h varies through the following cases:

Case 1: $h < 0$. Then the surface $g^{-1}(h)$ is the topological plane $\mathbb{R}^2$ of Fig. 1(a). The curves $\gamma_{hc} = \{g^{-1}(h)\} \cap \{p_\theta = c\}$ for each $|c| \leq c_1 = \sqrt{(2\varepsilon h - 1)/(2h)}$ are homeomorphic to:

- one component homeomorphic to $\mathbb{R}$ if $0 \leq c \leq c_2$,
- one component homeomorphic to $\mathbb{S}^1$ if $c_2 = \sqrt{\varepsilon} < |c| < c_1$, and
- one component homeomorphic to a point if $|c| = c_1$.

The manifold I_h is homeomorphic to a solid torus without its boundary. The dynamics on I_h can be obtained by rotating Fig. 1(b) around the $e-$axis. From this figure we have

- one periodic orbit (topologically a circle) I_{hc} for $|c| = c_1$,
- one two-dimensional torus I_{hc} for $c_2 < |c| < c_1$, and
- one cylinder I_{hc} for $0 < |c| < c_2$,

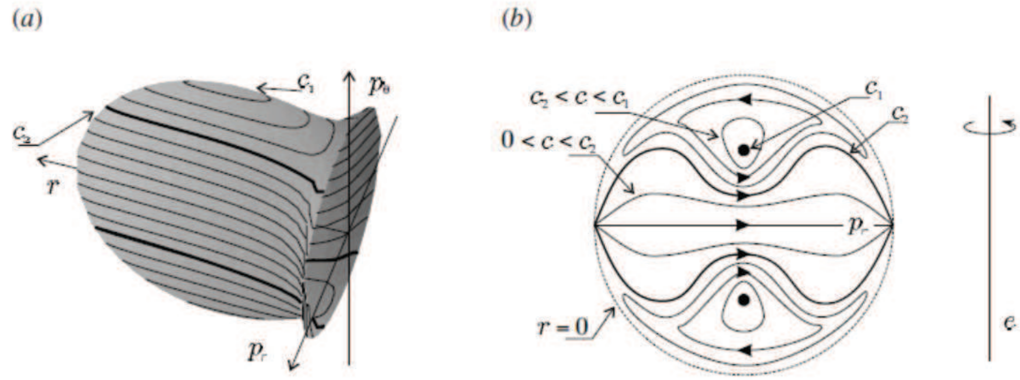

FIGURE 1. When $h < 0$: (a) The surface $g^{-1}(h)$. (b) Manifold $I_h/\mathbb{S}^1$.

which foliate I_h.

Case 2: $h \geq 0$ and $\varepsilon \geq 0$. Then $g^{-1}(h)$ is homeomorphic to a cylinder $\mathbb{R} \times \mathbb{S}^1$ see Fig. 2(a), and the curves γ_{hc} are formed by

- two components each of them homeomorphic to $\mathbb{R}$ if $c = 0$, and
- one component homeomorphic to $\mathbb{R}$ if $c \geq 0$.

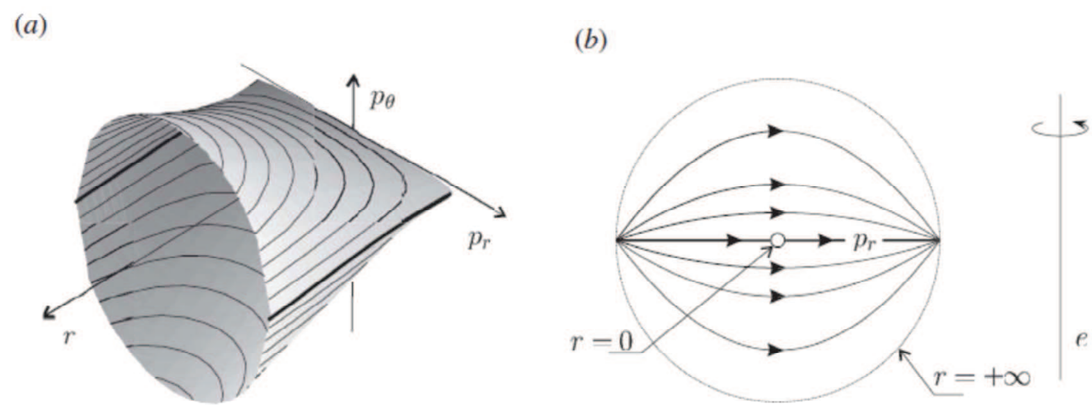

FIGURE 2. When $h \geq 0$ and $\varepsilon \geq 0$: (a) The surface $g^{-1}(h)$. (b) Manifold $I_h/\mathbb{S}^1$.

The manifold I_h is homeomorphic to a solid torus without its boundary and without its central circular axis. The dynamics on I_h can be obtained by rotating Fig. 2(b) around the e axis. From this figure we obtain one cylinder I_{hc} for every $c \in \mathbb{R} \setminus \{0\}$, and two cylinders I_{hc} for every $c = 0$, which foliate I_h.

Case 3: $h \geq 0$ and $\varepsilon < 0$. Now $g^{-1}(h)$ is homeomorphic to a plane $\mathbb{R}^2$ see Fig. 3(a), and the curves γ_{hc} are homeomorphic to $\mathbb{R}$.

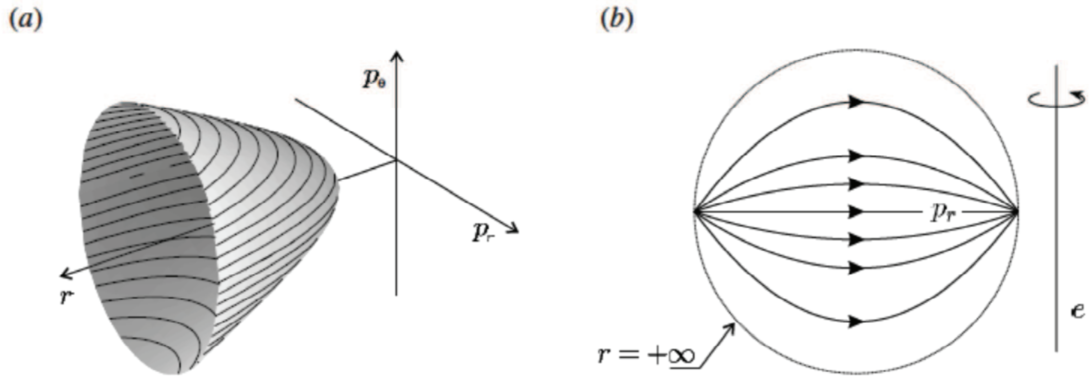

FIGURE 3. When $h \geq 0$ and $\varepsilon < 0$: (a) The surface $g^{-1}(h)$. (b) Manifold $I_h/\mathbb{S}^1$.

The manifold I_h is homeomorphic to a solid torus without its boundary. The dynamics of I_h can be be obtained by rotating Fig. 3(b) around the e axis. From this figure the foliation of I_h is done by the cylinders I_{hc} for every $c \in \mathbb{R} \setminus \{0\}$.

3. On the Lyapunov stability of the equilibria

Let $\phi_t(p)$ be the solution of a differential system (S) such that $\phi_t(p) = p$, i.e. $\phi_t(p)$ is the flow defined by the system (S). An equilibrium point q of the differential system (S) is called *Lyapunov stable* if for every open neighborhood U of q, there exists an open neighborhood $V \subseteq U$ of q such that $\phi_t(p) \in U$ for any $p \in V$ and any $t \geq 0$. An equilibrium state which is not Lyapunov stable is called *unstable.*

For $\varepsilon < 0$ sufficiently small we restrict the dynamics of the Hamiltonian system (1) to the space $p_\theta = 0$, where we have the circle of equilibria $q(\theta)$ for $\theta \in \mathbb{S}^1$. Then we have the differential system

$$\begin{aligned} \dot{r} &= p_r, \\ \dot{\theta} &= 0, \\ \dot{p_r} &= -\frac{1}{r^2} - \frac{\varepsilon}{r^3}. \end{aligned} \tag{7}$$

This differential system is completely integrable because it has the two functionally independent first integrals

$$C_1 = \theta \quad \text{and} \quad C_2 = \frac{1}{2}p_r^2 - \frac{1}{r} - \frac{\varepsilon}{2r^2}.$$

According with [19] for our system (7) a non-degenerate equilibrium point is a point which satisfies that the determinant of the Hessian of the function C_2 is non–zero at that equilibrium, see Definition 3.3 of [19]. Then every equilibrium point $(-\varepsilon, \theta, 0)$ of system (7) is degenerate, because the determinant of the Hessian of the function C_2 at this equilibrium is zero. Indeed, the mentioned Hessian is

$$\left.\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \dfrac{1}{\varepsilon^3} & 0 & 0 \end{pmatrix}\right|_{(r,\theta,p_r)=(-\varepsilon,\theta,0)}, \tag{8}$$

and clearly its determinant is zero.

In [19] to a non-degenerate equilibria q is associated a real number $I(q)$ as follows. If $P(x)$ is the characteristic polynomial defined by the linear part of the complete integrable differential system at q, then $P(x) = (-x)^{n-2}(x^2 + I(q))$ where n is the dimension of the differential system, see Theorem 5.4 of [19].

In Theorem 5.4 of [19] it is also shown that a non-degenerate equilibrium point q is unstable if $I(q) < 0$, and in Theorem 5.6 it is proved that a non-degenerate equilibrium point q is Lyapunov stable if $I(q) > 0$.

From (8) the characteristic polynomial of the linear part of system (7) at the equilibrium is $(-\varepsilon, \theta, 0)$ is

$$\lambda\left(\lambda^2 - \frac{1}{\varepsilon^3}\right).$$

So applying the Turoban criterium (which in fact we cannot apply because the equilibrium $(-\varepsilon, \theta, 0)$ is degenerate) we obtain that

$$I(-\varepsilon, \theta, 0) = -\frac{1}{\varepsilon^3} > 0. \tag{9}$$

because $\varepsilon < 0$, and by the mentioned criterium the equilibrium $(-\varepsilon, \theta, 0)$ would be Lyapunov stable.

Now we shall prove that really the equilibrium point $(-\varepsilon, \theta, 0)$ is Lyapunov stable for all $\theta \in \mathbb{S}^1$. Consequently we have proved that the criterion of Tudoran also works for the degenerate equilibrium points of the completely integrable system (7).

We note that we cannot apply the so-called "Arnol'd stability test" to the equilibrium $(-\varepsilon, \theta, 0)$ because the second condition of that test does not hold, see the statement of this test in Theorem 5.5 of [19]. From

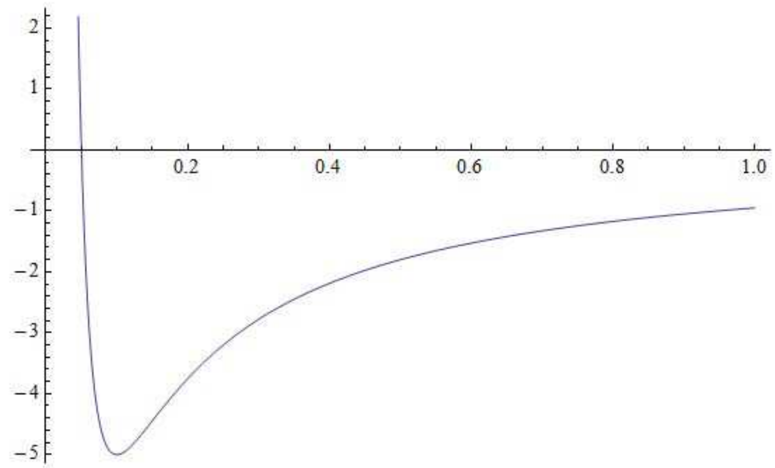

FIGURE 4. The graphic of the potential $V(r) = -1/r - \varepsilon/r^2$ for $-\varepsilon = 1/10$.

the differential system (7) it is clear that the variable θ is constant for any solution. So for studying the Lyapunov stability at an equilibrium $(-\varepsilon, \theta, 0)$ of system (7) we can restrict to study it at the equilibrium $(-\varepsilon, 0)$ of the Hamiltonian system with one degree of freedom

$$
\begin{aligned}
\dot{r} &= p_r, \\
\dot{p_r} &= -\frac{1}{r^2} - \frac{\varepsilon}{r^3}.
\end{aligned} \tag{10}
$$

with Hamiltonian C_2. Since the potential $V(r) = -1/r - \varepsilon/r^2$ of system (10) has the graphic of the Figure 4, and the minimum of this graphic takes place at $r = -\varepsilon$, recall that $\varepsilon < 0$ is fixed and small, it follows that the equilibrium point $(-\varepsilon, 0)$ of system (10) is a center, i.e. all the orbits in a convenient neighborhood of it are periodic with the exception of the equilibrium point (see [11] for more information). Hence given any neighborhood U of $(-\varepsilon, 0)$ in the plane (r, p_r) there is another neighborhood $V \subseteq U$ of $(-\varepsilon, 0)$ formed by sufficiently small periodic orbits surrounding the point $(-\varepsilon, 0)$, and consequently contained in U. Hence the equilibrium point $(-\varepsilon, 0)$ is Lyapunov stable for system (10), and consequently for the system (7).

In summary, we have proved that the degenerate equilibrium points of the completely integrable system (7) satisfies the Tudoran criterion (see (9)) and are Lyapunov stable.

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