A mild changed condition for fractional (g, f, n)-critical deleted graph

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Abstract

A graph G is fractional (g, f, n)-critical deleted if after deleting any edge e, the resulting graph is still a fractional (g, f, n)-critical graph. In Gao and Wang [8], it is presented that G is a fractional (g, f, n)-critical graph if $I(G) \geq \frac{b^2+bn-\Delta}{a}$, where $I(G)$ is the isolated toughness of graph G. The aim of this work is to reveal that after mild changing of this bound, we obtain an isolated toughness condition for a graph to be fractional (g, f, n)-critical deleted. Furthermore, we show that this conclusion can help to directly deduce an $I(G)$ bound for all fractional (g, f, n)-critical deleted graphs. Finally, we propose an open question for a more generalized case.

1 Introduction

We only discuss simple and finite graph throughout this article. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G) \subseteq V \times V$.

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We denote $d(v)$ as the degree of vertex $v \in V(G)$ and $N(v) = \{x : x \in V(G), xv \in E(G)\}$ as its neighborhood. For any $S \subseteq V(G)$, we use $G[S]$ to denote the induced subgraph by S, and set $G - S = G[V(G) \setminus S]$. Denote $e_G(S,T) = |\{e = uv | u \in S, v \in T\}|$ for any $S, T \subset V(G)$ with $S \cap T = \emptyset$. The minimum degree of G is expressed as $\delta(G) = \min\{d(v) | v \in V(G)\}$. Other notations and terminologies used but undefined in this paper can be found in the classic book by Bondy and Mutry [1].

Suppose that g and f are two non-negative integer-valued functions defined on $V(G)$ satisfying $g(v) \leq f(v)$ for any $v \in V(G)$. A *fractional (g,f)-factor* is considered as a function h which assigns to each edge e a number such that $0 \leq h(e) \leq 1$ and $g(v) \leq \sum_{e \in E(v)} h(e) \leq f(v)$ for each vertex v. If $g(v) = a$ and $f(v) = b$ for any $v \in V(G)$, then a fractional (g,f)-factor is a fractional $[a,b]$-factor. Furthermore, if $g(v) = f(v) = k$ ($k \geq 1$ is an integer constant) for all $v \in V(G)$, then a fractional (g,f)-factor becomes a fractional k-factor, which is the primitive definition of fractional factor.

A graph G is called a *fractional (g,f)-deleted graph* if after removing any edge $e \in E(G)$, the resulting graph admits a fractional (g,f)-factor. In the setting of $g(v) = f(v) = k$ for each $v \in V(G)$, the fractional (g,f)-deleted graph becomes a fractional k-deleted graph. A graph G is called a *fractional (g,f,n)-critical graph* if after delating any n vertices, the resulting graph admits a fractional (g,f)-factor. A graph is called a *fractional (g,f,n)-critical deleted graph* if after deleting any n vertices, the resulting graph is still a fractional (g,f)-deleted graph.

Stability is a significant factor of network designing in which the designers need to ensure the network to a certain degree of firmness at minimal cost. Yang et al. [13] introduced the notion of *isolated toughness* to measure the vulnerability of the network, which can be formulated as follows: if G is a complete graph, $I(G) = \infty$; otherwise,

$$I(G) = \min\{\frac{|S|}{i(G-S)}\Big| S \subset V(G), i(G-S) \geq 2\},$$

where $i(G-S)$ is the number of isolated vertices of $G - S$. Together with traditional toughness parameters, isolated toughness plays a vital role in network designing and cyber attacking.

Previous studies have found that there is a close relationship between isolated toughness and the existence of fractional factors. Several papers have contributed in this topic. Li et al. [11] presented the sharp isolated toughness bound for k-deleted graph. Zhou et al. [16] determined an isolated toughness condition for fractional (g,f)-factors. Zhou and Pan [17] studied an isolated toughness condition for a graph to be fractional (a, b, k)-critical. Very recently, Gao and Wang [8] derived an isolated toughness condition for a graph to be fractional (g, f, n)-critical. In addition, an isolated toughness condition for fractional (k, m)-deleted graphs is obtained by Gao et al. [6]. More results on the topic with fractional factor, fractional critical graphs, fractional deleted graphs and other applications can refer to Gao and Gao [4], Zhou et al. [14, 15, 18, 19, 20], Gao et al. [3, 5, 7], and Gao and Wang [9].

Let's review two known classic conclusions on tight isolated toughness bound of fractional k-factor and fractional k-deleted graph:

- G has a fractional k-factor for $k \geq 2$ if $\delta(G) \geq k$ and $I(G) \geq k$;
- G is a fractional k-deleted graph for $k \geq 2$ if $\delta(G) \geq k+1$ and $I(G) > k$.

Since the $I(G)$ bound in both two cases are sharp, it fully reveals that on the basis of the original fractional k-factor graph, deleting an edge has little effect on it, and it only requires a slight improvement in original isolated toughness condition to reach the requirement of the fractional k-deleted graph.

Recalling that in Gao and Wang [8], they determined that graph G is fractional (g, f, n)-critical if $\delta(G) \geq \frac{(b+2)^2}{4a} + \frac{bn}{a} + b - 1$ and $I(G) \geq \frac{b^2+bn-\Delta}{a}$, where $a \leq g(x) \leq f(x) \leq b$ with $1 \leq a \leq b$ and $b \geq 2$ for all $x \in V(G)$. It is natural to ask the question: whether we can get the isolated toughness bound for fractional (g, f, n)-critical deleted graph only by slightly changing the condition in Gao and Wang [8]? Intuitively, it should be possible. The main purpose of our paper is to give a positive answer to the above guesswork. Our main results to be proved in the third section can be stated as follows.

Theorem 1.1. *Let G be a graph and let g, f be two integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) \leq b$ with $1 \leq a \leq b$ and $b \geq 2$ for all $x \in V(G)$, where a, b are positive integers. Let n be a non-negative integer and $\Delta = b - a$. A graph G with $\delta(G) \geq \frac{(b+2)^2}{4a} + b + \frac{bn}{a} - 1$ is a fractional (g, f, n)-critical deleted graph if $I(G) > \frac{b^2+bn-\Delta}{a}$.*

In the field of computer science, the significance of engineering application of Theorem 1.1 lies in that in a data transmission network which satisfies the condition of isolated toughness. And the damage on a single channel will not have a big impact on the entire network.

The rest paper is organised as follows. First, we introduce some lemmas which will be used in the proof of Theorem 1.1; then we remark that we can get similarly conclusion for a graph to be all fractional (g, f, n)-critical deleted by means of same trick; at last, we propose an open question for the general case.

2 Useful lemmas

Let

$$\varepsilon(S,T) = \begin{cases} 2, & T \text{ is not an independent set} \\ 1, & T \text{ is an independent set and } e_G(T, V(G) \setminus (S \cup T)) \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

The proof of Theorem 1.1 depends heavily on the following lemma which characterises the necessary and sufficient condition of fractional (g, f, n)-critical deleted graphs.

Lemma 2.1. (Gao [2]) *Let G be a graph, g, f be two integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$. Let n be two non-negative integers. Then G is fractional (g, f, n)-critical deleted graph if and only if*

$$f(S) - g(T) + d_{G-S}(T) \geq \max_{U \subseteq S, |U|=n} \{f(U)\} + \varepsilon(S,T) \tag{2.1}$$

for all disjoint subsets S, T of $V(G)$ with $|S| \geq n$.

The next two lemmas were proposed by Liu and Zhang [12] which will be applied in our proof.

Lemma 2.2. (Liu and Zhang [12]) *Let G be a graph and let $H = G[T]$ such that $\delta(H) \geq 1$ and $1 \leq d_G(x) \leq k-1$ for every $x \in V(H)$ where $T \subseteq V(G)$ and $k \geq 2$. Let $T_1, \ldots, T_{k-1}$ be a partition of the vertices of H satisfying $d_G(x) = j$ for each $x \in T_j$ where we allow some T_j to be empty. If each component of H has a vertex of degree at most $k-2$ in G, then H has a maximal independent set I and a covering set $C = V(H) - I$ such that*

$$\sum_{j=1}^{k-1} (k-j)c_j \leq \sum_{j=1}^{k-1} (k-2)(k-j)i_j,$$

where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for $j = 1, \ldots, k-1$.

Obviously, Lemma 2.2 is also established for $\delta(H) \geq 0$. The second lemma is derived in light of the proving process of Lemma 2.2 in [12].

Lemma 2.3. (Liu and Zhang [12]) *Let G be a graph and let $H = G[T]$ such that $d_G(x) = k-1$ for every $x \in V(H)$ and no component of H is isomorphic to K_k where $T \subseteq V(G)$ and $k \geq 2$. Then there exists an independent set I and the covering set $C = V(H) - I$ of H satisfying*

$$|V(H)| \leq \sum_{i=1}^{k}(k-i+1)|I^{(i)}| - \frac{|I^{(1)}|}{2}$$

and

$$|C| \leq \sum_{i=1}^{k}(k-i)|I^{(i)}| - \frac{|I^{(1)}|}{2}$$

where $I^{(i)} = \{x \in I, d_H(x) = k-i\}$ for $1 \leq i \leq k$ and $\sum\limits_{i=1}^{k}|I^{(i)}| = |I|$.

3 Proof of Theorem 1.1

If G is complete, then the conclusion is directly verified in terms of $\delta(G) \geq \frac{(b+2)^2}{4a} + b + \frac{bn}{a} - 1$ and the definition of fractional factor. We assume that G is not complete, and G satisfies the conditions of Theorem 1.1, but is not a fractional (g, f, n)-critical deleted graph. By Lemma 2.1 and $\varepsilon(S, T) \leq 2$ for any disjoint subsets $S, T \subseteq V(G)$, there exist disjoint subsets S and T of $V(G)$ satisfying

$$\begin{aligned} & a|S| - b|T| + \sum_{x \in T} d_{G-S}(x) - bn \\ (3.1)\quad \leq\ & f(S) - g(T) + d_{G-S}(T) - \max_{U \subseteq S, |U|=n}\{f(U)\} \leq \varepsilon(S,T) - 1 = 1. \end{aligned}$$

We select S and T such that $|T|$ is minimum. Thus, it's easy to check that $T \neq \emptyset$, and $d_{G-S}(x) \leq g(x) - 1 \leq b - 1$ for any $x \in T$.

Let l be the number of the components of $H' = G[T]$ which are isomorphic to K_b and let $T_0 = \{v \in V(H') | d_{G-S}(v) = 0\}$. Let H be the subgraph inferred from $H' - T_0$ by removing those l components isomorphic to K_b. Let S' be a set of vertices that contains exactly $b-1$ vertices in each component of K_b in H'.

If $|V(H)| = 0$, then using (3.1) we get $|S| \leq \frac{b(|T_0|+l)+bn+1}{a}$. Since $|T| \neq 0$, we have $|T_0| + l \geq 1$, and thus consider two cases according to the value of $|T_0|+l$. If $|T_0|+l = 1$, then $d_{G-S}(x)+|S| \geq d_G(x) \geq \delta(G) \geq \frac{(b+2)^2}{4a}+b+\frac{bn}{a}-1$ and

$$d_{G-S}(x) \geq \frac{(b+2)^2}{4a}+b+\frac{bn}{a}-1-|S| \geq \frac{(b+2)^2}{4a}+b+\frac{bn}{a}-1-\frac{b+bn+1}{a} > b-1,$$

which contradicts to $d_{G-S}(x) \leq b-1$ for any $x \in T$. It implies $i(G-S\cup S') \geq |T_0|+l \geq 2$ and $I(G) \leq \frac{|S\cup S'|}{i(G-S-S')} \leq \frac{b(|T_0|+l)+bn+1+la(b-1)}{a(|T_0|+l)} \leq \frac{b}{a}+b-1+\frac{bn+1}{2a}$. This contradicts $I(G) > \frac{b^2+bn-\Delta}{a}$.

In what follows, we consider $|V(H)| > 0$. Let $H = H_1 \cup H_2$ where H_1 is the union of components of H such that $d_{G-S}(v) = b-1$ for each vertex $v \in V(H_1)$ and $H_2 = H - H_1$. In view of Lemma 2.3, H_1 has a maximum independent set I_1 and the covering set $C_1 = V(H_1) - I_1$ satisfies

$$|V(H_1)| \leq \sum_{i=1}^{b}(b-i+1)|I^{(i)}| - \frac{|I^{(1)}|}{2}, \tag{3.2}$$

and

$$|C_1| \leq \sum_{i=1}^{b}(b-i)|I^{(i)}| - \frac{|I^{(1)}|}{2}, \tag{3.3}$$

where $I^{(i)} = \{v \in I_1, d_{H_1}(v) = b-i\}$ for $1 \leq i \leq b$ and $\sum_{i=1}^{b}|I^{(i)}| = |I_1|$. Let $T_j = \{v \in V(H_2)|d_{G-S}(v) = j\}$ for $1 \leq j \leq b-1$. In terms of the definition of H and H_2, each component of H_2 has a vertex of degree at most $b-2$ in $G-S$. By means of Lemma 2.2, H_2 has a maximal independent set I_2 and the covering set $C_2 = V(H_2) - I_2$ satisfies

$$\sum_{j=1}^{b-1}(b-j)c_j \leq \sum_{j=1}^{b-1}(b-2)(b-j)i_j, \tag{3.4}$$

where $c_j = |C_2 \cap T_j|$ and $i_j = |I_2 \cap T_j|$ for each $j = 1, \ldots, b-1$. We set $W = V(G)-S-T$ and $U = S\cup S'\cup C_1\cup(N_G(I_1)\cap W))\cup C_2\cup(N_G(I_2)\cap W)$. Let $t_0 = |T_0|$. We deduce

$$|U| \leq |S| + l(b-1) + |C_1| + \sum_{j=1}^{b-1} j i_j + \sum_{i=1}^{b}(i-1)|I^{(i)}| \tag{3.5}$$

and

$$i(G-U) \geq t_0 + l + |I_1| + \sum_{j=1}^{b-1} i_j. \tag{3.6}$$

If $i(G-U) > 1$, we yield

$$|U| \geq I(G)i(G-U). \tag{3.7}$$

If $i(G-U)=1$, then $G[T]$ is a clique with $|V(G[T])|<b$. According to (3.1), we derive

$$\begin{aligned}|S| &\leq \frac{b|T|-d_{G-S}(T)+bn+1}{a} \leq \frac{b|T|-|T|(|T|-1)+bn+1}{a}\\ &\leq \frac{b\frac{b+1}{2}-\frac{b+1}{2}(\frac{b+1}{2}-1)+bn+1}{a} = \frac{bn+1}{a}+\frac{(b+1)^2}{4a}\end{aligned}$$

and

$$\begin{aligned}b-1 &\geq d_{G-S}(v) \geq \frac{(b+2)^2}{4a}+b+\frac{bn}{a}-1-|S|\\ &\geq \frac{(b+2)^2}{4a}+b+\frac{bn}{a}-1-(\frac{bn+1}{a}+\frac{(b+1)^2}{4a}) > b-1\end{aligned}$$

for any $v \in T$. This leads to a contradiction.

Following from (3.5)-(3.7), we yield

$$\begin{aligned}|S|+|C_1| &\geq \sum_{j=1}^{b-1}(I(G)-j)i_j+I(G)(t_0+l)+I(G)|I_1|\\ &\quad -\sum_{i=1}^{b}(i-1)|I^{(i)}|-l(b-1). \end{aligned} \tag{3.8}$$

In light of $b|T|-d_{G-S}(T) \geq a|S|-bn-1$, we get

$$bt_0+bl+|V(H_1)|+\sum_{j=1}^{b-1}(b-j)i_j+\sum_{j=1}^{b-1}(b-j)c_j \geq a|S|-bn-1.$$

Integrate (3.8) together, we derive

$$\begin{aligned}&|V(H_1)|+\sum_{j=1}^{b-1}(b-j)c_j+a|C_1| \qquad (3.9)\\ \geq &\sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j+(aI(G)-b)(t_0+l)+aI(G)|I_1|\\ &-a\sum_{i=1}^{b}(i-1)|I^{(i)}|-bn-1-la(b-1).\end{aligned}$$

In light of (3.2) and (3.3), we yield

$$|V(H_1)|+a|C_1| \leq \sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}|-\frac{(a+1)|I^{(1)}|}{2}. \tag{3.10}$$

From (3.4), (3.9) and (3.10), we obtain

$$
\begin{aligned}
(3.11)\qquad & \sum_{j=1}^{b-1}(b-2)(b-j)i_j+\sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}| \\
\geq\ & \sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j+(aI(G)-b)(t_0+l)+aI(G)|I_1| \\
& +\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|-bn-1-la(b-1).
\end{aligned}
$$

The following discussion is divided into two cases in view of the value of t_0+l.

Case 1. $t_0+l\geq 1$**.** In this case, by (3.11) and $(aI(G)-b)(t_0+l)-bn-1-la(b-1)>(b^2+bn-\Delta-b)(t_0+l)-bn-1-la(b-1)\geq -1$, we have

$$
\begin{aligned}
(3.12)\qquad & \sum_{j=1}^{b-1}(b-2)(b-j)i_j+\sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}| \\
>\ & \sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j+aI(G)|I_1|+\frac{(a+1)|I^{(1)}|}{2} \\
& -a\sum_{i=1}^{b}(i-1)|I^{(i)}|-1.
\end{aligned}
$$

If $\sum_{j=1}^{b-1}(b-2)(b-j)i_j>\sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j-1$, then at least one j satisfy $(b-2)(b-j)>aI(G)-aj-b+j-1$, i.e., $I(G)<\frac{b^2-b+(a-b+1)j+1}{a}$. If $a=b=k\geq 2$, then $I(G)<k$, which contradicts to $I(G)>k+n$. If $b\geq a+1$, then $I(G)<\frac{b^2-b+1}{a}$ also contradicts to $I(G)>\frac{b^2-b+a+bn}{a}$. Thus, we deduce $\sum_{j=1}^{b-1}(b-2)(b-j)i_j\leq\sum_{j=1}^{b-1}(aI(G)-aj-b+j)i_j-1$.

On the other hand, we consider

$$
\begin{aligned}
& \sum_{i=1}^{b}(ab-ai+b-i+1)|I^{(i)}| \\
>\ & aI(G)|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|-1 \\
>\ & (b^2+bn-b+a)|I_1|+\frac{(a+1)|I^{(1)}|}{2}-a\sum_{i=1}^{b}(i-1)|I^{(i)}|-1,
\end{aligned}
$$

which implies

$$
\sum_{i=1}^{b}(ab+2b-2a-i+1-b^2)|I^{(i)}|-\frac{(a+1)|I^{(1)}|}{2}+1>0.
$$

Notice that $ab + 2b - 2a - i + 1 - b^2 = (b - a)(2 - b) - i + 1 \le 0$ and $(b-a)(2-b) \le 0$. If $\sum_{i=2}^{b} |I^{(i)}| > 0$, then $\sum\limits_{i=1}^{b}(ab+2b-2a-i+1-b^2)|I^{(i)}|$ is a negative integer, a contradiction. If $|I^{(1)}| > 0$, then $\frac{(a+1)|I^{(1)}|}{2}$ is a negative number, a contradiction.

Therefore, whatever $|I_1| = 0$, or $|I_2| = 0$, or both $|I_1| \ge 1$ and $|I_2| \ge 1$, we get a contradiction.

Case 2. $t_0 + l = 0$**.** In this case, (3.11) becomes

$$\begin{aligned} &\sum_{j=1}^{b-1}(b-2)(b-j)i_j + \sum_{i=1}^{b}(ab - ai + b - i + 1)|I^{(i)}| \\ \ge\ &\sum_{j=1}^{b-1}(aI(G) - aj - b + j)i_j + aI(G)|I_1| + \frac{(a+1)|I^{(1)}|}{2} \\ &-a\sum_{i=1}^{b}(i-1)|I^{(i)}| - bn - 1. \end{aligned} \tag{3.13}$$

We discuss the following three situations.

Subcase 2.1. $|I_1| = 0$.

We have $|I_2| > 0$ by $|V(H)| > 0$, and (3.13) becomes

$$\sum_{j=1}^{b-1}(b-2)(b-j)i_j > \sum_{j=1}^{b-1}(aI(G) - aj - b + j)i_j + aI(G)|I_1| - bn - 1.$$

Note that $(b-2)(b-j)-(aI(G)-aj-b+j) < (a-b+1)j-a-bn \le -bn-1$, a contradiction.

Subcase 2.2. $|I_2| = 0$.

We get $|I_1| > 0$ by $|V(H)| > 0$, and (3.13) becomes

$$\begin{aligned} &\sum_{i=1}^{b}(ab - ai + b - i + 1)|I^{(i)}| - aI(G)|I_1| - \frac{(a+1)|I^{(1)}|}{2} \\ &+a\sum_{i=1}^{b}(i-1)|I^{(i)}| + bn + 1 \ge 0. \end{aligned}$$

By means of $I(G) > \frac{b^2+bn-\Delta}{a}$, we yield

$$\begin{aligned} 0 &< \sum_{i=1}^{b}(ab + 2b - 2a - i + 1 - b^2 - bn)|I^{(i)}| - \frac{(a+1)|I^{(1)}|}{2} + bn + 1 \\ &= \sum_{i=1}^{b}((b-2)(a-b) - i + 1)|I^{(i)}| - \frac{(a+1)|I^{(1)}|}{2} + bn(1 - |I_1|) + 1 \\ &\le \sum_{i=1}^{b}((b-2)(a-b) - i + 1)|I^{(i)}| - \frac{(a+1)|I^{(1)}|}{2} + 1 \le 0, \end{aligned}$$

a contradiction.

Subcase 2.3. $|I_1| \geq 1$ and $|I_2| \geq 1$.

The last situation can be dealt using the trick similar to the discussion above.

Therefore, we complete the proof of the desired result. □

4 Isolated condition for all fractional (g, f, n)-critical deleted graphs

The purpose of this section is to show that we can directly infer the isolated condition for all fractional (g, f, n)-critical deleted graphs.

We say G has all fractional (g, f)-factors if G has a fractional p-factor for each $p : V(G) \to \mathbb{N}$ with $g(v) \leq p(v) \leq f(v)$ for any $v \in V(G)$. Similar to fractional (g, f)-deleted graph and fractional (g, f, n)-critical deleted graph, the concepts of all fractional (g, f)-deleted graph and all fractional (g, f, n)-critical deleted graph can be well defined. That is to say, a graph G is to be all fractional (g, f)-deleted if removing any edge e from G, the resulting graph still has all fractional (g, f)-factors, and a graph G is an all fractional (g, f, n)-critical deleted graph if after deleting any n vertices, the remaining graph is an all fractional (g, f)-deleted graph. Very recently, Gao et al. [10] presented the sufficient and necessary condition for a graph to be all fractional (g, f, n)-critical deleted which presented as follows.

Lemma 4.1. (Gao et al. [10]) *Let n be nonnegative integer, and $g, f : V(G) \to \mathbb{Z}^+$ be two valued functions with $g(x) \leq f(x)$ for each $x \in V(G)$. Then G is all fractional (g, f, n)-critical deleted if and only if*

$$g(S) - f(T) + \sum_{x \in T} d_{G-S}(x) \geq \max\{g(U) : U \subseteq S, |U| = n\} + \varepsilon(S, T),$$

for any non-disjoint subsets $S, T \subseteq V(G)$ with $|S| \geq n$.

Through careful observation, it can be found that the necessary and sufficient conditions of the fractional (g, f, n)-critical deleted graphs and all fractional (g, f, n)-critical deleted graphs are very similar in its expression form, which makes us confirm that the proving process of the Theorem 1.1 manifested in last section can also help to obtain the isolated toughness condition of all fractional (g, f, n)-critical deleted graph. In fact, such conclusion can be obtained directly.

Theorem 4.2. *Let G be a graph and let g, f be two integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) \leq b$ with $1 \leq a \leq b$ and $b \geq 2$ for all $x \in V(G)$, where a, b are positive integers. Let n be a non-negative integer and $\Delta = b - a$. A graph G with $\delta(G) \geq \frac{(b+2)^2}{4a} + b + n - 1$ is an all fractional (g, f, n)-critical deleted graph if $I(G) > \frac{b^2-\Delta}{a} + n$.*

We discover that in the proofing of Theorem 4.2, inequality (3.1) is changed to the following fashion.

$$\begin{aligned} & a|S| - b|T| + \sum_{x \in T} d_{G-S}(x) - an \\ \leq\ & a(|S| - n) + \sum_{x \in T} (d_{G-S}(x) - b) \\ \leq\ & g(S - U) + \sum_{x \in T} (d_{G-S}(x) - f(x)) \leq -1. \end{aligned}$$

Thus, the latter proof process is the same as Section 3 except that bn is replaced by an. This is why Theorem 4.2 can be yielded immediately.

5 Conclusion and discussion

In this work, we focus on the relationship between isolated toughness and fractional (g, f, n)-critical deleted graphs. We argue that only slight change the original $I(G)$ condition on fractional (g, f, n)-critical graphs, then the isolated toughness bound for fractional (g, f, n)-critical deleted graphs is confirmed. Our conclusion has potential engineering applications which implies that only one channel damaged will not have fatal influence on the entire data transmission network.

Let's think about the more general case. If two channels, three channels, $\cdots$, or m channels are damaged or blocked in data transmission network, then what will happened? Still by simple transformation of the current network can this problem be solved? This question corresponds to fractional (g, f, m)-deleted graphs (or fractional (g, f, n, m)-critical deleted graphs), where m is the number of deleted edges. Intuitively, we give a negative answer. As m increases, the isolated toughness condition will make a fundamental change in order to satisfy the condition of fractional (g, f, m)-deleted graphs (or fractional (g, f, n, m)-critical deleted graphs). Through preliminary attempts, we confirmed that the optimal bound contain a function of m, but we can't determine the specific expression so far. Thus, we propose an open question to finish this paper.

Problem 5.1. *What is the tight $I(G)$ bound for fractional (g, f, n, m)-critical deleted graphs (res. all fractional (g, f, n, m)-critical deleted graphs).*

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