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# Edge Irregular Reflexive Labeling for Disjoint Union of Generalized Petersen Graph

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**Abstract:** In the mathematical discipline of graph theory, a graph labeling is the assignment of labels, traditionally represented by integers, to the edges or vertices, or both, of a graph. Formally, given a graph $G = (V, E)$, a vertex labeling is a function from $V$ to a set of labels. A graph with such a function defined is called a vertex-labeled graph. Likewise, an edge labeling is a function of $E$ to a set of labels. In this case, the graph is called an edge-labeled graph. We study an edge irregular reflexive $k-$labeling for the disjoint union of the cycle related graphs and determine the exact value of the reflexive edge strength for the disjoint union of $s$ isomorphic copies of the cycle related graphs namely Generalized Peterson graphs.

**Keywords:** Edge irregular reflexive labeling; reflexive edge strength; Generalized Peterson graphs.

**MSC:** 05C12, 05C90

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## 1. Introduction.

All graphs considered in this paper are simple, finite and undirected. Chartrand et al. [1] proposed the following problem. Assign a positive integer label from the set $\{1, 2, ...k\}$ to the edges of a simple connected graph of order at least three in such a way that the graph becomes irregular, i.e the weight (label sum) at each vertex are distinct. What is the minimum value of the largest label $k$ over all such irregular assignments. This parameter of the graph $G$ is well known as the irregularity strength of the graph $G$. An excellent survey on the irregularity strength is given by Lahel in [2]. For recent results, see the papers by Amar and Togni in [3], Dimitz et al. in [4], Gyarfas in [5] and Nierhoff in [6].

Motivated by these papers, an edge irregular $k-$labeling as a vertex labeling $\phi : V(G) \rightarrow \{1, 2, ..., k\}$ was defined, such that for every two different edges $xy$ and $x^{'}y^{'}$ there is $w_{\phi}(xy) \neq w_{\phi}(x^{'}y^{'})$, where the weight of an edge $xy \in E(G)$ is $w_{\phi}(xy) = \phi(x) + \phi(y)$. The minimum $k$ for which the graph $G$ has an edge irregular $k-$labeling is called the edge irregularity strength of the graph $G$, denoted by $es(G)$. In [7] are estimated the bounds of the parameters $es(G)$, and the exact value of the edge irregularity strength for several families of graphs are determined, namely paths, stars, double stars and the Cartesian product of two paths.

Baca *et.al* [8], defined the total labeling $\phi : V(G) \cup E(G) \rightarrow \{1, 2, ..., k\}$ to be an edge irregular total $k-$labeling of the graph $G$ if for every two different edge $xy$ and $x^{'}y^{'}$ of $G$ one has $w_{\phi}(xy) = \phi(x) + \phi(xy) + \phi(y) \neq w_{\phi}(x^{'}y^{'}) = \phi(x^{'}) + \phi(x^{'}y^{'}) + \phi(y^{'})$. The total edge irregularity strength, $tes(G)$, is defined as the minimum $k$ for which $G$ has an edge irregular total $k-$labeling. Estimated of this parameters are obtained, which provides the precise values of the total edge irregularity strength for paths, cycles, stars, wheels and friendship graphs. Further results on the total irregularity strength can be found in [9–14].

The seminal problem for the irregular labeling arise from a consideration of graphs with distinct degree. In a simple graph, it is not possible to construct a graph in which every vertex has a unique degree; however, this is possible in multigraphs (graphs in which we allow multiple edges between the adjacent vertices). The question then became: "what is the smallest number of parallel edges between two vertices required to ensure that the graph display vertex irregularity?" This problem is equivalent to the labeling problem as described at the beginning of this section.

Ryan et.al, [15], decreed that the vertex labels should represent loops at the vertex. The consequence was two-fold; first, each vertex label was required to be an even integer, since each loop added two to the vertex degree; and second, unlike in total irregular labeling, the label 0 was permitted as representing a loopless vertex. Edges continued to be labelled by integers from one to $k$.

Thus, they defined labelings $f_e : E(G) \rightarrow \{1, 2, ..., k_e\}$ and $f_v : V(G) \rightarrow \{0, 2, ..., 2k_v\}$, and then, labeling $f$ is a total $k-$labeling of $G$ defined such that $f(x) = f_v(x)$ if $x \in V(G)$ and $f(x) = f_e(x)$ if $x \in E(G)$, where $k = max\{k_e, 2k_v\}$.

The total $k-$labeling $f$ is called an edge irregular reflexive $k-$labeling of the graph $G$ if for every two different edges $xy$ and $x^{'}y^{'}$ of $G$, one has $wt(xy) = f_v(x) + f_e(xy) + f_v(y) \neq wt(x^{'}y^{'}) = f_v(x^{'}) + f_e(x^{'}y^{'}) + f_v(y^{'})$. The smallest value of $k$ for which such labeling exists is called the reflexive edge strength of the graph $G$ and is denoted by $res(G)$. For recent results see [16,17].

The result of this variation was not widely manifest in the labeling strength, but did produce some important outcomes:

$tes(K_5) = 5$ whereas $res(K_5) = 4$

The effect of this change was immediate in the following conjecture where we were able to remove the pesky exception see [18].

**Conjecture 1.** *Any graph $G$ with maximum degree $\Delta(G)$ other than $K_5$ satisfies*

$$\text{tes}(G) = \max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta+1}{2}\right\rceil\right\}.$$

In term of reflexive edge irregularity strength, Baca et al. [19] purpose the following conjecture and prove the Theorem 1.

**Conjecture 2.** *Any graph $G$ with maximum degree $\Delta(G)$ satisfies*

$$\text{res}(G) = \max\left\{\left\lceil\frac{|E(G)|}{3}+r\right\rceil, \left\lfloor\frac{\Delta+2}{2}\right\rfloor\right\},$$

where $r = 1$ for $|E(G)| \equiv 2,3(mod\ 6)$, and zero otherwise.

**Theorem 1.** *For every graph G,* $res(C_n) = \begin{cases} \lceil \frac{|E(G)|}{3} \rceil, & \text{if } n \not\equiv 2,3(mod\ 6) \\ \lceil \frac{|E(G)|}{3} \rceil + 1, & \text{if } n \equiv 2,3(mod\ 6). \end{cases}$

## 2. Constructing an Edge Irregular Reflexive Labeling

Let us recall the following lemma.

**Lemma 1.** *For every graph G,* $res(G) \geq \begin{cases} \lceil \frac{|E(G)|}{3} \rceil, & \text{if } n \not\equiv 2,3(mod\ 6) \\ \lceil \frac{|E(G)|}{3} \rceil + 1, & \text{if } n \equiv 2,3(mod\ 6). \end{cases}$

The lower bound for $res(G)$ follows from the fact that the minimal edge weight under an edge irregular reflexive labeling in one, and the minimum of the maximal edge weight, that is $|E(G)|$ can be achieved only as the sum of three numbers, at least two of which are even.

In this paper, we investigate the reflexive edge irregularity strength for disjoint union of $s$ isomorphic copies of Generalized Peterson graphs

## 3. Generalized Petersen Graph

The generalized Peterson graph $P(n,m)$ has been studied extensively in recent years. Generalized Peterson graphs were first defined by Watkins [20]. Mominul Haque [21] determined the irregular total labelings of generalized Peterson graphs, Jendrol and Žoldák [22] determined the irregularity strength of generalized Peterson graphs and Chunling *et. al* [23] determined the total edge irregularity strength of generalized Peterson graphs. In this paper, we investigate the reflexive edge irregularity strength of complete disjoint union of $s$ isomorphic copies of the generalized Peterson graphs. First we define the vertex set and edge set of disjoint union of $s$ isomorphic copies of generalized Peterson graph $P(n,m)$ in the following way.

$$V(sP(n,m)) = \{x_i^j, y_i^j : 1 \leq i \leq n; 1 \leq j \leq s\}$$

$$E(sP(n,m)) = \{x_i^j x_{i+m}^j, x_i^j y_i^j, y_i^j y_{i+1}^j : 1 \leq i \leq n; 1 \leq j \leq s\}$$

where the subscripts $i$ and $i+m$ are taken under modulo $n$.

**Theorem 2.** *For $s \geq 1$, $n \geq 3$ and $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$. we have*

$$res(sP(n,m)) = \begin{cases} ns+1, & \text{if } s \text{ is odd} \\ ns, & \text{if } s \text{ is even} \end{cases}$$

**Proof.** Since $(sP(n,m))$ has $3ns$ edges. Therefore From Lemma 1, we get

$$(sP(n,m)) \geq \begin{cases} ns+1, & \text{if } s \text{ is odd} \\ ns, & \text{if } s \text{ is even} \end{cases}$$

Next, we will show that $res(sP(n,m)) \leq \begin{cases} ns+1, & \text{if } s \text{ is odd} \\ ns, & \text{if } s \text{ is even} \end{cases}$

For this we define a $f-$labeling on $(sP(n,m))$ as follow:

$$k = \begin{cases} ns+1, & \text{if } s \text{ is odd} \\ ns, & \text{if } s \text{ is even} \end{cases}$$

For $s = 1$, $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$. and $i = 1, 2, ..., n$ we have the following labeling of vertices and edges

along with their weights.
$f(x_i^1) = 0, f(x_i^1 x_{i+1}^1) = i,$

$$f(y_i^1) = \begin{cases} n-1, & \text{for } i = 1 \\ k, & \text{for } i = 2,3,...,n \end{cases}$$

$$f(x_i^1 y_i^1) = \begin{cases} 1, & \text{for } i = 1 \\ i-1, & \text{for } i = 2,3,...,n \end{cases}$$

$$f(y_i^1 y_{i+1}^1) = \begin{cases} 1, & \text{for } i = 1 \\ i-1, & \text{for } i = 2,3,...,n-1 \\ 2, & \text{for } i = n \end{cases}$$

$wt(x_i^1 x_{i+1}^1) = i, wt(x_i^1 y_i^1) = n + i$

$$wt(y_i^1 y_{i+1}^1) = \begin{cases} 2n+1, & \text{for } i = 1 \\ 2n+1+i, & \text{for } i = 2,3,...,n-1 \\ 2n+2, & \text{for } i = n \end{cases}$$

For $s \geq 2$ and $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$. we have the following labeling of vertices and edges along with their weights:

$$f(x_i^j) = \begin{cases} ns-n+1, & \text{if } 1 \leq i < n, (s \ \ is \ \ even) \\ ns-n, & \text{if } 1 \leq i < n, (s \ \ is \ \ odd) \end{cases}$$

$$f(y_i^j) = \begin{cases} ns, & \text{if } 1 \leq i < n, (s \ \ is \ \ even) \\ ns+1, & \text{if } 1 \leq i < n, (s \ \ is \ \ odd) \end{cases}$$

$$f(x_i^j x_{i+m}^j) = \begin{cases} ns-n-2+i, & \text{if } 1 \leq i < n, (s \ \ is \ \ even) \\ ns-n+i, & \text{if } 1 \leq i < n, (s \ \ is \ \ odd) \end{cases}$$

$$f(x_i^j y_i^j) = \begin{cases} ns-n-1+i, & \text{if } 1 \leq i < n, (s \ \ is \ \ even) \\ ns-n+1+i, & \text{if } 1 \leq i < n, (s \ \ is \ \ odd) \end{cases}$$

$$f(y_i^j y_{i+1}^j) = \begin{cases} ns-n+i, & \text{if } 1 \leq i < n, (s \ \ is \ \ even) \\ ns-n-2+i, & \text{if } 1 \leq i < n, (s \ \ is \ \ odd) \end{cases}$$

For $1 \leq i < n, \quad s \geq 2$ we have

$wt(x_i^j x_{i+m}^j) = 3ns - 3n + i, wt(x_i^j y_i^j) = 3ns - 2n + i, wt(y_i^j y_{i+1}^j) = 3ns - n + i,$

It is easy to check that there are no two edges of the same weight. So, $f$ is an edge irregular reflexive labeling of $(sP(n_j, m))$ for $1 \leq m \leq \lfloor \frac{n-1}{2}. \rfloor$ and for $n \geq 3$. Which completes the proof. □

**Theorem 3.** *Let $(sP(n, \frac{n}{2}))$, $s \geq 1$ be isomorphic copies of the generalized Petersen graphs with n even, and $n = 4, 6, 8$. Then*

$$res(s(P(n, \frac{n}{2}))) = \begin{cases} \lceil \frac{5ns}{6} \rceil, & \text{for } \lceil \frac{5ns}{2} \rceil \not\equiv 2, 3(mod \ \ 6) \\ \lceil \frac{5ns}{6} \rceil + 1, & \text{for } \lceil \frac{5ns}{2} \rceil \equiv 2, 3(mod \ \ 6) \end{cases}$$

**Proof.** For $m = \frac{n}{2}$, $(sP(n, \frac{n}{2}))$ has $\frac{5ns}{2}$ edges. From Lemma 1, we get

$$res(s(P(n, \tfrac{n}{2}))) \geq \begin{cases} \lceil \frac{5ns}{6} \rceil, & \text{for } \lceil \frac{5ns}{2} \rceil \not\equiv 2,3(mod\ \ 6) \\ \lceil \frac{5ns}{6} \rceil + 1, & \text{for } \lceil \frac{5ns}{2} \rceil \equiv 2,3(mod\ \ 6) \end{cases}$$

Next, we will show that

$$res(s(P(n, \tfrac{n}{2}))) \leq \begin{cases} \lceil \frac{5ns}{6} \rceil, & \text{for } \lceil \frac{5ns}{2} \rceil \not\equiv 2,3(mod\ \ 6) \\ \lceil \frac{5ns}{6} \rceil + 1, & \text{for } \lceil \frac{5ns}{2} \rceil \equiv 2,3(mod\ \ 6) \end{cases}$$

For $n = 4, 6, 8$ and $j = 1$ we have the following labelings and weights of vertices and edges as followed:

$f(x_i^1) = 0$, $f(x_i^1 x_{i+\frac{n}{2}}^1) = i$,

$wt(x_i^1 x_{i+\frac{n}{2}}^1) = i$, $wt(x_i^1 y_i^1) = \frac{n}{2} + i$

For $n = 4$ and $j = 1$

$$f(y_i^1) = \begin{cases} 2, & \text{for } i = 1,2 \\ 4, & \text{for } i = 3,4 \end{cases}$$

$$f(x_i^1 y_i^1) = \begin{cases} i, & \text{for } i = 1,2 \\ 4 - i, & \text{for } i = 3,4 \end{cases}$$

$$f(y_i^1 y_{i+1}^1) = \begin{cases} n - 1, & \text{for } i = 1,2 \\ n - 2, & \text{for } i = 3,4 \end{cases}$$

$wt(x_i^1 x_{i+\frac{n}{2}}^1) = i$, $wt(x_i^1 y_i^1) = 2 + i$

$$wt(y_i^1 y_{i+1}^1) = \begin{cases} 2n - 1, & \text{for } i = 1 \\ 2n - 3 + i, & \text{for } i = 2,3 \\ 2n, & \text{for } i = 4 \end{cases}$$

For $n = 6$ and $j = 1$

$$f(y_i^1) = \begin{cases} 2, & \text{for } i = 1 \\ 4, & \text{for } i = 2,3 \\ 6, & \text{for } i = 4,5,6 \end{cases}$$

$$f(x_i^1 y_i^1) = \begin{cases} 2, & \text{for } i = 1 \\ 1, & \text{for } i = 2 \\ i - \frac{n}{2}, & \text{for } i = 3,4,5,6 \end{cases}$$

$$f(y_i^1 y_{i+1}^1) = \begin{cases} \frac{n}{2} + 2 - i, & \text{for } i = 1,3 \\ i - \frac{n}{2} + 1, & \text{for } i = 2,4,5 \\ n - 1, & \text{for } i = 6 \end{cases}$$

$$wt(y_i^1 y_{i+1}^1) = \begin{cases} n + 3 + i, & \text{for } i = 1,2 \\ n + 4 + i, & \text{for } i = 3,4,5 \\ \frac{3n}{2} + 4, & \text{for } i = 6 \end{cases}$$

For $n = 8$ and $j = 1$

$$f(y_i^1) = \begin{cases} 4, & \text{for } i = 1,2 \\ 6, & \text{for } i = 3,4 \\ 8, & \text{for } i = 5,6,7,8 \end{cases}$$

$$f(x_i^1 y_i^1) = \begin{cases} 1, & \text{for } i = 1,3 \\ 2, & \text{for } i = 2,4 \\ i - \frac{n}{2}, & \text{for } i = 5,6,7,8 \end{cases}$$

$$f(y_i^1 y_{i+1}^1) = \begin{cases} \frac{n}{2} + 2 - i, & \text{for } 1 \le i \le \frac{n}{2} \\ i - \frac{n}{2} + 1, & \text{for } \frac{n}{2} + 1 \le i \le n \end{cases}$$

$$wt(y_i^1 y_{i+1}^1) = \begin{cases} n + 3 + i, & \text{for } i = 1,2 \\ n + 4 + i, & \text{for } i = 3,4,5 \\ \frac{3n}{2} + 5, & \text{for } i = 6 \end{cases}$$

For $j \ge 2$ and $n = 4$ we define a $f-$labeling on vertices and edges of $(s(P(4,2)))$ as follow:

$$f(x_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil, & \text{if } 1 \le i < 4, \ s \equiv 1,2(mod\ 3) \\ \lceil \frac{5n(s-1)}{6} \rceil + 1, & \text{if } 1 \le i < 4, \ s \equiv 0(mod\ 3) \end{cases}$$

$$f(y_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil + 1, & \text{if } 1 \le i < 4, \ s \equiv 2(mod\ 3) \\ \lceil \frac{5n(s-1)}{6} \rceil, & \text{if } 1 \le i < 4, \ s \equiv 0,1(mod\ 3) \end{cases}$$

$$f(x_i^j x_{i+\frac{n}{2}}^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 2 + i, & \text{if } 1 \le i < 2, \ s \equiv 2(mod\ 3) \\ \lceil \frac{5n(s-1)}{6} \rceil - 3 + i, & \text{if } 1 \le i < 2, \ s \equiv 0(mod\ 3) \\ \lceil \frac{5n(s-1)}{6} \rceil + i, & \text{if } 1 \le i < 2, \ j \equiv 1(mod\ 3) \end{cases}$$

$$f(x_i^j y_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 4 + i, & \text{if } 1 \le i < 4, \ s \equiv 2(mod\ 3) \\ \lceil \frac{5n(s-1)}{6} \rceil - 3 + i, & \text{if } 1 \le i < 4, \ s \equiv 0(mod\ 3) \\ \lceil \frac{5n(s-1)}{6} \rceil - 2 + i, & \text{if } 1 \le i < 4, \ s \equiv 1(mod\ 3) \end{cases}$$

$$f(y_i^j y_{i+1}^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 4 + i, & \text{if } 1 \le i < 4, \ s \equiv 2(mod\ 3) \\ \lceil \frac{5n(s-1)}{6} \rceil - 1 + i, & \text{if } 1 \le i < 4, \ s \equiv 0(mod\ 3) \\ \lceil \frac{5n(s-1)}{6} \rceil - 2 + i, & \text{if } 1 \le i < 4, \ s \equiv 1(mod\ 3) \end{cases}$$

$$wt(x_i^j x_{i+\frac{n}{2}}^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil - 2 + i, & \text{if } 1 \le i < 2, \ s \equiv 2(mod\ 3) \\ \lceil \frac{5n(s-1)}{2} \rceil + i, & \text{if } 1 \le i < 2, \ s \equiv 1(mod\ 3) \\ \lceil \frac{5n(s-1)}{2} \rceil - 1 + i, & \text{if } 1 \le i < 2, \ s \equiv 0(mod\ 3) \end{cases}$$

$$wt(x_i^j y_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil - 2 + i + ns, & \text{if } 1 \le i < 4, \ s \equiv 0(mod\ 3) \\ \lceil \frac{5n(s-1)}{2} \rceil + i + ns, & \text{if } 1 \le i < 4, \ s \equiv 1(mod\ 3) \\ \lceil \frac{5n(s-1)}{2} \rceil - 1 + i + ns, & \text{if } 1 \le i < 4, \ s \equiv 2(mod\ 3) \end{cases}$$

$$wt(y_i^j y_{i+1}^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil - 4 + i + 2ns, & \text{if } 1 \le i < 4, \ s \equiv 0(mod\ 3) \\ \lceil \frac{5n(s-1)}{2} \rceil - 2 + i + 2ns, & \text{if } 1 \le i < 4, \ s \equiv 1,2(mod\ 3) \end{cases}$$

For $j \ge 2$ and $n = 6$ we define a $f-$labeling on vertices and edges of $(s(P(6,3)))$ as follow:

$$f(x_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil + 1, & \text{if } 1 \le i < 6, \ s \equiv 0,2,4(mod\ 6) \\ \lceil \frac{5n(s-1)}{6} \rceil, & \text{if } 1 \le i < 6, \ s \equiv 1,3,5(mod\ 6) \end{cases}$$

$$f(y_i^j) = \begin{cases} \lceil \frac{5ns}{6} \rceil, & \text{if } 1 \le i < 6, \ s \equiv 0,2,4(mod\ 6) \\ \lceil \frac{5ns}{6} \rceil + 1, & \text{if } 1 \le i < 6, \ s \equiv 1,3,5(mod\ 6) \end{cases}$$

$$f(x_i^j x_{i+\frac{n}{2}}^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 2 + i, & \text{if } 1 \le i < 3, \quad (s \ \ is \ \ even) \\ \lceil \frac{5n(s-1)}{6} \rceil + i, & \text{if } 1 \le i < 3, \quad (s \ \ is \ \ odd) \end{cases}$$

$$f(x_i^j y_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 3 + i, & \text{if } 1 \le i < 6, \end{cases}$$

$$f(y_i^j y_{i+1}^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 1 + i, & \text{if } 1 \le i < 6, \quad s \equiv 0 (mod \ \ 2) \\ \lceil \frac{5n(s-1)}{6} \rceil - 3 + i, & \text{if } 1 \le i < 6, \quad s \equiv 1 (mod \ \ 2) \end{cases}$$

$$wt(x_i^j x_{i+\frac{n}{2}}^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil + i, & \text{if } 1 \le i < 3 \end{cases}$$

$$wt(x_i^j y_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil - 3 + i + \frac{5ns}{6}, & \text{if } 1 \le i < 6, \quad s \equiv 2 (mod \ \ 6) \\ \lceil \frac{5n(s-1)}{2} \rceil - 2 + i + \frac{5ns}{6}, & \text{if } 1 \le i < 6, \quad s \equiv 0,1,3,4,5 (mod \ \ 6) \end{cases}$$

$$wt(y_i^j y_{i+1}^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil - 1 + i + \frac{5ns}{3}, & \text{if } 1 \le i < 6 \end{cases}$$

For $j \ge 2$ and $n = 8$ we define a $f-$labeling on vertices and edges of $(s(P(8,4)))$ as follow:

$$f(x_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil + 1, & \text{if } 1 \le i < 8, \quad s \equiv 0,2 (mod \ \ 6) \\ \lceil \frac{5n(s-1)}{6} \rceil, & \text{if } 1 \le i < 8, \quad s \equiv 1,3,4,5 (mod \ \ 6) \end{cases}$$

$$f(y_i^j) = \begin{cases} \lceil \frac{5ns}{6} \rceil, & \text{if } 1 \le i < 8, \quad s \equiv 2,3,4,5 (mod \ \ 6) \\ \lceil \frac{5ns}{6} \rceil + 1, & \text{if } 1 \le i < 8, \quad s \equiv 0,1 (mod \ \ 6) \end{cases}$$

$$f(x_i^j x_{i+\frac{n}{2}}^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 2 + i, & \text{if } 1 \le i < 4, \quad (s \ \ is \ \ even) \\ \lceil \frac{5n(s-1)}{6} \rceil + i, & \text{if } 1 \le i < 4, \quad (s \ \ is \ \ odd) \end{cases}$$

$$f(x_i^j y_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 5 + i, & \text{if } 1 \le i < 8, \quad s \equiv 2 (mod \ \ 6) \\ \lceil \frac{5n(s-1)}{6} \rceil - 4 + i, & \text{if } 1 \le i < 8, \quad s \equiv 0,1,3,4,5 (mod \ \ 6) \end{cases}$$

$$f(y_i^j y_{i+1}^j) = \begin{cases} \lceil \frac{5n(s-1)}{6} \rceil - 3 + i, & \text{if } 1 \le i < 8, \quad s \equiv 2,5 (mod \ \ 6) \\ \lceil \frac{5n(s-1)}{6} \rceil - 2 + i, & \text{if } 1 \le i < 8, \quad s \equiv 0,3 (mod \ \ 6) \\ \lceil \frac{5n(s-1)}{6} \rceil - 2 + i, & \text{if } 1 \le i < 8, \quad s \equiv 1,4 (mod \ \ 6) \end{cases}$$

$$wt(x_i^j x_{i+\frac{n}{2}}^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil - 1 + i, & \text{if } 1 \le i < 4, \quad s \equiv 2 (mod \ \ 6) \\ \lceil \frac{5n(s-1)}{2} \rceil - 2 + i, & \text{if } 1 \le i < 4, \quad s \equiv 3 (mod \ \ 6) \\ \lceil \frac{5n(s-1)}{2} \rceil + i, & \text{if } 1 \le i < 4, \quad s \equiv 0,1,4 (mod \ \ 6) \\ \lceil \frac{5n(s-1)}{2} \rceil - 3 + i, & \text{if } 1 \le i < 4, \quad s \equiv 5 (mod \ \ 6) \end{cases}$$

$$wt(x_i^j y_i^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil - 4 + i + \lceil \frac{5ns}{6} \rceil, & \text{if } 1 \le i < 8, \quad s \equiv 2,3,4,5 (mod \ \ 6) \\ \lceil \frac{5n(s-1)}{2} \rceil - 2 + i + \lceil \frac{5ns}{6} \rceil, & \text{if } 1 \le i < 8, \quad s \equiv 0 (mod \ \ 6) \\ \lceil \frac{5n(s-1)}{2} \rceil - 3 + i + \lceil \frac{5ns}{6} \rceil, & \text{if } 1 \le i < 8, \quad s \equiv 1 (mod \ \ 6) \end{cases}$$

$$wt(y_i^j y_{i+1}^j) = \begin{cases} \lceil \frac{5n(s-1)}{2} \rceil - 3 + i + \lceil \frac{5ns}{3} \rceil, & \text{if } 1 \le i < 8 \quad s \equiv 2,5 (mod \ \ 6) \\ \lceil \frac{5n(s-1)}{2} \rceil - 2 + i + \lceil \frac{5ns}{3} \rceil, & \text{if } 1 \le i < 8 \quad s \equiv 1,3 (mod \ \ 6) \\ \lceil \frac{5n(s-1)}{2} \rceil - 4 + i + \lceil \frac{5ns}{3} \rceil, & \text{if } 1 \le i < 8 \quad s \equiv 4 (mod \ \ 6) \\ \lceil \frac{5n(s-1)}{2} \rceil + i + \lceil \frac{5ns}{3} \rceil, & \text{if } 1 \le i < 8 \quad s \equiv 0 (mod \ \ 6) \end{cases}$$

It is easy to check that there are no two edges of the same weight. So, $f$ is an edge irregular reflexive labeling of $(sP(n, \frac{n}{2}))$ for $m = \frac{n}{2}$. and for $n = 4.6, 8$ Which completes the proof. □

## 4. Conclusion

In this paper we have determined the edge irregular reflexive labeling for disjoint union of $s$ isomorphic copies of generalizes Petersen graphs $P(n, m)$ for $s \geq 1$, $n \geq 3$ and $1 \leq m < \frac{n}{2}$. and $(P(n, m))$ for $n = 4, 6, 8$ with $m = \frac{n}{2}$ We tried to find the edge irregular reflexive labeling for disjoint union of $s$ isomorphic copies of generalized Petersen graphs $(P(n, m))$ for $n \geq 10$ with $m = \frac{n}{2}$ and $n$ even but so far without success. So we conclude the paper with the following open problem.

## Open Problem:

Determined the edge irregular reflexive labeling for disjoint union of $s$ isomorphic copies of generalized Petersen graphs $(P(n, m))$ for $n \geq 10$ with $m = \frac{n}{2}$ and $n$ even. i.e

$$res(s(P(n, \tfrac{n}{2}))) = \begin{cases} \lceil \frac{5ns}{6} \rceil, & \text{for } \lceil \frac{5ns}{2} \rceil \not\equiv 2, 3 (mod\ \ 6) \\ \lceil \frac{5ns}{6} \rceil + 1, & \text{for } \lceil \frac{5ns}{2} \rceil \equiv 2, 3 (mod\ \ 6) \end{cases}$$

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