# MORE THAN SEVENTY YEARS FROM A MILESTONE IN FRACTAL GEOMETRY: MORAN'S THEOREM

M. FERNÁNDEZ-MARTÍNEZ[1], J. L. G. GUIRAO[2] AND G. RODRÍGUEZ-BERMÚDEZ[1]

Abstract. In this paper, we re-explore in detail the techniques employed in the original P.A.P. Moran's proof for a key result in Fractal Geometry allowing the calculation of the Hausdorff dimension of attractors of self-similar sets.

## 1. Introduction

In 1946, the Australian mathematician Patrick Alfred Pierce Moran (1917-1988) published the paper entitled "Additive functions of intervals and Hausdorff measure" (c.f. [1, 2]) containing one of the cornerstones in Fractal Geometry. That theorem allows calculating the Hausdorff dimension of any attractor, $\mathcal{K} \subseteq \mathbb{R}^q$, whose distinct pieces $f_i(\mathcal{K})$ do not overlap by $\dim_{\mathrm{H}}(\mathcal{K}) = s$, where $s$ is the (unique) solution of the equation $\sum_{i=1}^{k} c_i^s = 1$, which only involves the ratios $0 < c_i < 1$ associated with each similarity $f_i : \mathbb{R}^q \to \mathbb{R}^q$. It is worth mentioning that some Moran type theorems have been proved afterwards by relaxing the separation conditions required to the pieces of the self-similar set via discrete models of fractal dimension for a fractal structure. For additional details regarding them, we refer the reader to [3].

Several tributes regarding the figure of P.A.P. Moran have appeared in the literature. Among them, we may quote [4, 5, 6, 7, 8]. The main goal in the present article is to re-explore in detail the results and techniques contributed in that original Moran's paper which led to a remarkable advance in Fractal Geometry. Let this paper be a technical homage to his memory.

**Note.** We have followed the original notation of [1] as far as possible. However, the reader may find some exceptions. For instance, both Lemmas I and II in Section 3 have been interchanged with respect to the originals. It is worth mentioning that all the proofs have been redone in detail for a better understanding of the results.

## 2. The theorems

2.1. **Preliminaries.** All the results in the sequel stand in the Euclidean setting. Next, we recall the definitions of half-open intervals and figures in $\mathbb{R}^q$ and the Hausdorff measure (relative to a certain function), as well.

A half-open interval in $\mathbb{R}^q$ is a set of the form $[a_1, b_1) \times \cdots \times [a_q, b_q)$, where $a = (a_1, \ldots, a_q), b = (b_1, \ldots, b_q) \in \mathbb{R}^q$ with $a_i < b_i$ for all $i = 1, \ldots, q$. By a half-open figure, we shall understand a set that can be written as a finite union of half-open intervals in $\mathbb{R}^q$.

The diameter of a set $A$ in $\mathbb{R}^q$ is defined, as usual, by $\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}$, where $d$ denotes the Euclidean distance in $\mathbb{R}^q$.

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Let $\delta > 0$, $E$ be a bounded subset of $\mathbb{R}^q$, and $h : \mathbb{R} \to \mathbb{R}$ a continuous increasing function with $h(0) = 0$ and satisfying that $h(t) > 0$ for all $t > 0$. Consider the set of all $\delta-$covers of $E$, i.e., all the finite (resp., countable) sequences $\{U_i\}_{i\in I}$ of convex sets (open or closed) with diameters $d_i \leq \delta$ such that $E \subseteq \cup_{i\in I} U_i$. Define $\mathcal{H}^h_\delta(E) = \inf \sum_{\{U_i\}} h(d_i)$, where the sums are considered over all the $\delta-$covers of $E$. The set function $\mathcal{H}^h_\delta$ is non-decreasing as $\delta$ goes to 0. As such, $\mathcal{H}^h(E) := \lim_{\delta\to 0} \mathcal{H}^h_\delta(E)$ always exists and is named the Hausdorff measure of $E$ relative to the function $h(t)$. It holds that $\mathcal{H}^h(E) \in \{0, m, \infty\}$, where $0 < m < \infty$.

A sufficient condition leading to the function $\frac{h(t)}{t}$ being non-increasing is that $h(t)$ is concave from below. Such a condition is usually required in the case $q = 1$. On the other hand, for $q > 1$, it may be assumed that $\frac{h(t)}{t^q}$ is non-increasing as $t$ increases. In particular, one may take $h(t) = t^p$ for $p \leq q$, leading to the standard definition of Hausdorff measure.

2.2. **Statements of the results.** The main result in this paper was proved by Moran as forthcoming Theorem I. Given a subset $E$ of $\mathbb{R}^q$ with a finite Hausdorff measure, it characterizes that measure being positive in terms of the existence of an additive function of half-open figures satisfying certain metric properties.

**Theorem I.** *Let $E$ be a closed and bounded subset of $\mathbb{R}^q$ with $\mathcal{H}^h(E) < \infty$. The two following statements are equivalent:*
*(1) $\mathcal{H}^h(E) > 0$.*
*(2) There exists an additive function of half-open figures, $\varphi$, satisfying the three following conditions:*
*(i) $\varphi(R) \geq 0$ for all half-open figure $R$.*
*(ii) If $E \subseteq R$, then $\varphi(R) \geq b > 0$ with $b$ being a fixed constant.*
*(iii) There exists a constant $k \neq 0$ such that $\varphi(R) \leq k \cdot h(\mathrm{diam}\,(R))$.*
*In that case, $\mathcal{H}^h(E) \geq \frac{b}{k}$.*

The classical Moran's theorem, which allows the calculation of the Hausdorff dimension of attractors of iterated function systems of similarities, holds as a consequence of Theorem I. However, following the original ideas of Moran, we shall prove it in Section 3 from a more general construction described in Theorem III.

**Theorem II.** *Let $E = \cup_{i=1}^n E_i$ be a closed bounded set, where the $E_i$ are closed and non-overlapping sets geometrically similar to $E$ but reduced in the ratios $t_i$. Then $\dim_{\mathrm{H}}(E) = p_0$ and $0 < \mathcal{H}^{p_0}(E) < \infty$, where $p_0$ is the root of the equation $\sum_{i=1}^n t_i^p = 1$.*

Notice that the set $E$ in Theorem II (understood by Moran as a generalization of the Cantor's ternary set) is self-similar. Further, $p_0$ is sometimes called as similarity dimension of $E$. In particular, if all the similarity ratios are the same, say $t_i = t$ for all $i = 1, \ldots, n$, then we have $p_0 = \log \frac{1}{n} / \log t = \log n / \log \frac{1}{t}$.

**Theorem III.** *Let $O_1$ be an open bounded set and $O_2^i \subseteq O_1 : i = 1, \ldots, n$ be $n$ non-overlapping open sets geometrically similar to $O_1$ but reduced in the ratios $t_i$. Similarly, let $O_3^{ij} : i, j = 1, \ldots, n$ be $n^2$ open sets satisfying the same relations to $O_2^i$ as each $O_2^i$ does to $O_1$, and so on. Let $P_1 = \overline{O_1}, P_2 = \overline{\cup_{i=1}^n O_2^i}, P_3 = \overline{\cup_{i,j=1}^n O_3^{ij}}, \ldots$. Then $E = \cap_{n\geq 1} P_n$ is closed and bounded with $0 < \mathcal{H}^{p_0}(E) < \infty$, where $p_0$ is the root of the equation $\sum_{i=1}^n t_i^p = 1$.*

Notice that in previous theorem, the open sets $O_3^{ij}$ are similar to $O_1$, i.e., $O_3^{ij} = f_{ij}(O_1)$, though the similarity $f_{ij}$ may not be a composition of the similarities that map $O_1$ to each $O_2^i$. As such, Theorem III remains also valid for *statistically self-similar sets* whenever the open sets do not overlap.

## 3. Proofs

*Proof of Theorem I (Sufficiency).* Let $\{U_i\}_{i\in I}$ be a covering of $E$ by open sets with $\operatorname{diam}(U_i)=\delta_i$. We affirm that $\sum h(\delta_i)\geq \frac{b}{k}$, where the sum is considered over the class of such coverings. In fact, let $\varepsilon>0$ and $\{U_i\}_{i\in I}$ be any of such coverings. Notice that $I$ can be assumed to be finite since $E$ is compact. Also, the $U_i$ can be considered to be convex, so let us assume that $U_i\subseteq U_i'$ for each $i\in I$ with $U_i'$ being a half-open figure. Let $\operatorname{diam}(U_i')=\delta_i'$ be so close to $\delta_i$ that

$$h(\delta_i')<(1+\varepsilon)\cdot h(\delta_i) \text{ for all } i\in I. \tag{3.1}$$

By condition (iii), there exists $k\neq 0$ such that $h(\delta_i')\geq \frac{1}{k}\cdot\varphi(U_i')$ for each $U_i'$. Hence, we have

$$\begin{aligned}\sum h(\delta_i)&\geq \frac{1}{1+\varepsilon}\cdot\sum h(\delta_i')\geq \frac{1}{k\cdot(1+\varepsilon)}\cdot\sum\varphi(U_i')\\ &\geq \frac{1}{k\cdot(1+\varepsilon)}\cdot\varphi(\cup_{i\in I}U_i'),\end{aligned}$$

where the first inequality is due to Eq. (3.1), the second stands by applying the condition (iii), and the last one holds since $\varphi(\cup_{i\in I}U_i')\leq\sum_{i\in I}\varphi(U_i')$. In fact, the $U_i'$ may overlap. Notice also that the condition (ii) leads to $\varphi(\cup_{i\in I}U_i')\geq b$ since $E_1\subseteq\cup_{i\in I}U_i'$. As such, $\sum h(\delta_i)\geq\frac{b}{k\cdot(1+\varepsilon)}$ for each $\varepsilon>0$ and all covering $\{U_i\}_{i\in I}$ of $E$. Hence, $\sum h(\delta_i)\geq\frac{b}{k}$, so $\mathcal{H}^h(E)\geq\frac{b}{k}$. □

It is worth noting that it suffices with the set function $\varphi$ being finitely additive in previous Theorem I. Moreover, it has been defined on half-open figures, which is enough since $E$ is compact.

The proof of the necessity for Theorem I is more awkward. First of all, let us introduce the concept of upper $h-$density of a subset of $\mathbb{R}^q$.

**Definition 3.1.** Let $E$ be a subset of $\mathbb{R}^q$. We define the upper $h-$density of $E$ at a point $a\in E$, $\overline{D}_h(a)$, in the following terms:

$$\overline{D}_h(a)=\overline{\lim_{r\to 0}}\,\frac{\mathcal{H}^h(C(a,r)\cap E)}{h(2r)},$$

where $C(a,r)$ denotes an open ball in $\mathbb{R}^q$ centred at $a$ whose radius equals $r$.

Along the sequel, we shall denote $V(d_0)$ the class of all the sequences by open sets, $\{U_i\}_{i\in I}$, where $\operatorname{diam}(U_i)<d_0$ for all $i\in I$. Also, given a set $A$, let us define $U(A,\rho)=\{\{U_i\}_{i\in I}\in V(\rho): A\subseteq\cup_{i\in I}U_i\}$. In other words, $U(A,\rho)$ is the collection consisting of all the $\rho-$covers of $A$. Next, we shall prove the two following technical lemmas.

**Lemma I.** *Let $\varepsilon>0$ and $A\subseteq\mathbb{R}^q$ be a $h-$measurable set with $\mathcal{H}^h(A)<\infty$. There exists $d_0>0$ such that*

$$\mathcal{H}^h(A\cap V)\leq\sum_{V(d_0)}h(d_i)+\varepsilon \textit{ for all } V\in V(d_0).$$

*Proof.* Let $\varepsilon > 0$. Then there exists a number depending on both $A$ and $\varepsilon$, say $d_0$, such that $\mathcal{H}^h(A) - \mathcal{H}^h_{d_0}(A) < \frac{\varepsilon}{2}$. Hence, it holds that

$$\mathcal{H}^h(A) - \frac{\varepsilon}{2} < \mathcal{H}^h_{d_0}(A) < \sum_{U(A,d_0)} h(d_i). \tag{3.2}$$

Observe that $A \cap V$ is $h-$measurable for all $V \in V(d_0)$ since all the sets in $V(d_0)$ are open. In addition, we can write $A = (A \setminus (A \cap V)) \cup (A \cap V)$ for all $V \in V(d_0)$, where that union remains disjoint. As such, we have

$$\mathcal{H}^h(A) = \mathcal{H}^h(A \cap V) + \mathcal{H}^h(A \setminus (A \cap V)) \text{ for all } V \in V(d_0). \tag{3.3}$$

Replacing Eq. (3.3) into Eq. (3.2), it holds that

$$\sum_{U(A,d_0)} h(d_i) > \mathcal{H}^h(A \cap V) + \mathcal{H}^h(A \setminus (A \cap V)) - \frac{\varepsilon}{2}. \tag{3.4}$$

Notice that each $d_0-$cover of $A$ is also a $d_0-$cover of $A \setminus (A \cap V)$ for all $V \in V(d_0)$. If we denote $U_1 := U(A \setminus (A \cap V), d_0)$ for short, then

$$\sum_{U_1} h(d_i) \leq \mathcal{H}^h(A \setminus (A \cap V)) + \frac{\varepsilon}{2}. \tag{3.5}$$

Replacing Eq. (3.5) into Eq. (3.4), the following inequality stands:

$$\sum_{U(A,d_0)} h(d_i) \geq \mathcal{H}^h(A \cap V) + \sum_{U_1} h(d_i) - \varepsilon,$$

or equivalently,

$$\sum_{U_1} h(d_i) \leq \mathcal{H}^h(A \cap V) + \sum_{U_1} h(d_i) \leq \sum_{U(A,d_0)} h(d_i) + \varepsilon$$

since $\mathcal{H}^h(A \cap V) \geq 0$. Hence, $U_1 \subseteq U(A, d_0)$, so we can write $U(A, d_0) = U_1 \subseteq U_1 \cup V(d_0)$. Accordingly,

$$\sum_{U(A,d_0)} h(d_i) \leq \sum_{V(d_0)} h(d_i) + \sum_{U_1} h(d_i) \leq \sum_{V(d_0)} h(d_i) + \mathcal{H}^h(A \setminus (A \cap V)) + \frac{\varepsilon}{2} \tag{3.6}$$

with the second inequality above being due to Eq. (3.5). As such,

$$\sum_{V(d_0)} h(d_i) \geq \sum_{U(A,d_0)} h(d_i) - \mathcal{H}^h(A \setminus (A \cap V)) - \frac{\varepsilon}{2} \geq \mathcal{H}^h(A \cap V) - \varepsilon$$

by applying both Eqs. (3.4) and (3.6). □

**Lemma II.** *Let $E \subseteq \mathbb{R}^q$ be a $h-$measurable set with $\mathcal{H}^h(E) < \infty$. Then $\overline{D}_h(a) \leq 1$ for all $a \in E$ except, possibly, for a set of zero $h-$measure.*

*Proof.* Notice that for each $u > 0$, the set $\{a \in E : \overline{D}_h(a) > u\}$ is $h-$measurable. In this way, let $A' = \{a \in E : \overline{D}_h(a) > 1\}$ and assume (looking for a contradiction) that $\mathcal{H}^h(A') > 0$. Accordingly, there exists $b > 0$ such that $\mathcal{H}^h(A'') > 0$, where $A'' = \{a \in E : \overline{D}_h(a) > 1 + b\}$. By Lemma I, there exists $\rho_1 > 0$ such that the inequality provided by Lemma I stands for $\varepsilon = \min\left\{\frac{1}{2}\,\mathcal{H}^h(A''), \frac{b}{2\cdot 3^q}\,\mathcal{H}^h(A'')\right\}$. Hence, we can write $A'' = \cup_{i\geq 1} A_i$, where each $A_i$ is the set of points $a \in A''$ for which an open circle centred at $a$ with a radius being equal to $r$, $C(a, r)$, can be drawn, where

$$\frac{\rho_1}{3+i} \leq 2r < \frac{\rho_1}{2+i} \text{ and } \frac{\mathcal{H}^h(A \cap C(a, r))}{h(2r)} > 1 + b.$$

Observe that $A_i$ is $h-$measurable for each $i \geq 1$. The circle $C(a, r)$ defined above is named density circle of class $i$. For each $a \in A_1$, let us draw both a circle of both radius and class equal to 1 as well as a concentric circle with radius equal to $3r$. Also, for all $a \in A_1$ outside the two previous circles, we shall draw two concentric circles similarly. Such a process can be repeated iteratively so at each stage, a point outside the already drawn circles is selected. This way, the density circle of the lowest possible class can be drawn together with a concentric circle of three times the radius. It holds that only a finite number of non-overlapping density circles (of any order) can be drawn since $\mathcal{H}^h(A) < \infty$. As such, both a finite (resp., countably) set containing non-overlapping circles, $C$, and a set of concentric circles, $C_1$ heve been constructed. It holds that $C_1$ covers $A''$. If the function $\frac{h(r)}{r^q}$ is non-increasing of $r$, then it is clear that $h(6r) \leq 3^q \cdot h(2r)$ for all $r > 0$. Hence,

$$\sum_{C} h(2r) \geq \frac{1}{3^q} \cdot \sum_{C_1} h(6r). \tag{3.7}$$

The radius of each circle in $C_1$ satisfies that $3r < \rho_1$. The following expression follows:

$$\sum_{C_1} h(6r) \geq \mathcal{H}^h(A'') - \varepsilon \geq \frac{1}{2} \cdot \mathcal{H}^h(A''), \tag{3.8}$$

where the first inequality above is due to the definition of $\mathcal{H}^h(A'')$ and the second one uses that $\varepsilon \leq \frac{1}{2} \cdot \mathcal{H}^h(A'')$. From both Eqs. (3.7) and (3.8), it holds that

$$\sum_{C} h(2r) \geq \frac{1}{2 \cdot 3^q} \cdot \mathcal{H}^h(A''). \tag{3.9}$$

The circles in $C$ do not overlap. As such,

$$\mathcal{H}^h(A'' \cap C) > (1+b) \cdot \sum_{C} h(2r) \geq \sum_{C} h(2r) + \frac{b}{2 \cdot 3^q} \cdot \mathcal{H}^h(A'') > \sum_{C} h(2r) + \varepsilon, \tag{3.10}$$

where the second inequality is due to Eq. (3.9). However, on the other hand, one may take $C = V(\rho_1)$. Thus, Lemma I would lead to

$$\mathcal{H}^h(A'' \cap C) \leq \sum_{C} h(2r) + \varepsilon,$$

a contradiction with Eq. (3.10). Consequently, $\mathcal{H}^h(A'') = 0$. □

The two lemmas above being proved, next we can tackle with the proof of the necessity of the conditions in Theorem I.

*Proof of Theorem I (Necessity).* Let $E_1 = \{a \in E : \overline{D}_h(a) > 1\}$. Thus, $E_1$ can be contained in an open set, $F_1$, with $\mathcal{H}^h(E \cap F_1) \leq \frac{1}{2} \cdot \mathcal{H}^h(E)$. Hence, the set $E_2 := E \setminus (E \cap F_1)$ is closed (in $E$) and we also have that $\mathcal{H}^h(E_2) > 0$. By Lemma II, it holds that $\overline{D}_h(a) \leq 1$ for all $a \in E_2$, where that density is relative to $E_2$. Notice also that $\overline{D}_h(p) = 0$ for all $p \in E \setminus E_2$ since $E_2$ is closed. As such, for all $a \in E_2$, there exists $r_0$ such that $\frac{\mathcal{H}^h(E_2 \cap C(a,r))}{h(2r)} \leq 2$ whenever $r \leq r_0$. Let $r_0 > 0$ and define

$$G(r_0) := \left\{ a \in E_2 : \frac{\mathcal{H}^h(E_2 \cap C(a,r))}{h(2r)} \leq 2 \text{ for all } r \leq r_0 \right\}.$$

It holds that $G(r_0)$ is $h-$measurable and $\lim_{r_0 \to 0} G(r_0) = E_2$. In this way, take $r_0$ such that $\mathcal{H}^h(G(r_0)) > 0$. We shall define $\Phi(R) = \mathcal{H}^h(G(r_0) \cap R)$ for each half-open figure $R$. Clearly, $\Phi$ is an additive function of half-open figures. Next, assume that $G(r_0) \cap R \neq \varnothing$ with $\operatorname{diam}(R) < \frac{1}{4} \cdot r_0$. Thus, $R$ can be contained in a circle centred at some point of $G(r_0)$ with radius equal to $\operatorname{diam}(R)$. Hence,

$$\Phi(R) = \mathcal{H}^h(G(r_0) \cap R) \leq 2 \cdot h(2 \operatorname{diam}(R)) \leq 2 \cdot 2^q \cdot h(\operatorname{diam}(R)).$$

If $G(r_0) \cap R = \varnothing$, we have $\Phi(R) = 0$. Since $\mathcal{H}^h(G(r_0))$ is bounded above, then $\Phi(R)$ also is. □

*Proof of Theorem III.* Firstly, let us verify that $\mathcal{H}^{p_0}(E) < \infty$. Fix $\delta = \operatorname{diam}(O_1)$ and let $d$ be such that $\delta < d < \infty$. Then $P_m$ can be covered by $n^m$ open sets with diameters of the form $t_i \cdots t_k\, d$. It is clear that that $\sum (t_i \cdots t_k\, d)^{p_0} = d^{p_0}$ since $p_0$ is the root of the equation $\sum_{i=1}^n t_i^p = 1$. Notice also that the largest diameter of all the covering sets can be made as small as desired since $t_i < 1$ for all $i = 1, \ldots, n$. As such, $\mathcal{H}^{p_0}(E) < d^{p_0}$ for all $d > \delta$. Hence, $\mathcal{H}^{p_0}(E) \leq \delta^{p_0} < \infty$.

Next, we shall prove that $\mathcal{H}^{p_0}(E) > 0$. Assume that $\operatorname{diam}(O_1) \leq 1$. For each $m-$order set, $O_m^{ij\ldots k}$, let us define $\varphi(O_m^{ij\ldots k}) = (t_i t_j \cdots t_k)^{p_0}$ as well as

$$\Phi(A) = \lim_{m \to \infty} \sum \varphi(O_m^{ij\ldots k}),$$

where these sums are considered over all the $m-$order sets with closures entirely contained in $A$. In addition, let $A_\eta = \{(x, y) : (x + h, y + k) \in A, 0 \leq h, k \leq \eta\}$. It holds that $A_\eta$ is also half-open and $\psi(A) = \lim_{\eta \to 0} \Phi(A_\eta)$ is an additive function of half-open intervals. Next, we shall prove that $\psi$ satisfies the conditions of Theorem I. Suppose that $t_1 \geq t_2 \geq \ldots \geq t_n$. Let the similarity ratios of the sets $O_2^i, O_3^{ij}, \ldots$ be arranged decreasingly and denoted as $T_1 \geq T_2 \geq \ldots$. Let $A$ be a set and $r$ such that $T_{r+1} \leq \operatorname{diam}(A) \leq T_r$. Let $C$ be a ball centred at $x \in A$ whose radius equals $2 \cdot \operatorname{diam}(A)$ and consider all the sets $O$ contained in $C$ with reduction ratios $c$, where $T_{r+1}\, t_n \leq c \leq T_{r+1}$. We shall consider only those sets with the largest reduction ratios and denote such a collection as $\{P_i\}_{i=1}^N$. We affirm that

$$N \leq \frac{4\pi\, (\operatorname{diam}(A))^2}{(T_{r+1} t_n)^2\, \mu(O_1)},$$

where $\mu$ denotes the Lebesgue measure. For each $i = 1, \ldots, N$, it holds that

$$\mu(P_i) = \mu(O_m^{ij\ldots k}) = c^2\, \mu(O_1) \geq (T_{r+1} t_n)^2\, \mu(O_1).$$

Hence,

$$4\pi d^2 = \mu(C) \geq \sum_{i=1}^N \mu(P_i) \geq N\, (T_{r+1} t_n)^2\, \mu(O_1).$$

Since $\varphi(P_i) \leq T_r^{p_0}$ for all $i = 1, \ldots, N$, then we have

$$\begin{aligned}
\psi(A) &\leq \sum_{i=1}^N \varphi(P_i) \leq N \cdot T_r^{p_0} \leq \frac{4\pi\, (\operatorname{diam}(A))^2}{(T_{r+1}\, t_n)^2\, \mu(O_1)} \cdot T_r^{p_0} \\
&= \frac{4\pi\, (\operatorname{diam}(A))^{2-p_0}\; T_r^2}{T_{r+1}^2\, T_r^{2-p_0}\, t_n^2\, \mu(O_1)} \cdot (\operatorname{diam}(A))^{p_0} \\
&\leq \frac{4\pi}{t_n^4\, \mu(O_1)}\, (\operatorname{diam}(A))^{p_0} = \kappa \cdot (\operatorname{diam}(A))^{p_0}.
\end{aligned}$$

Accordingly, there exists $\kappa \neq 0$ such that $\psi(A) \leq \kappa \cdot (\operatorname{diam}(A))^{p_0}$ for each half-open figure $A$. Hence, Theorem I gives $\mathcal{H}^{p_0}(E) > 0$. □

Finally, we can prove Theorem II from stronger Theorem III.

*Proof of Theorem II.* Define the open set $O_1 = \{x : d(x, E) < \frac{\eta}{3}\}$, where $\eta = \inf\{d_{\mathrm{H}}(E_i, E_j) : 1 \leq i, j \leq n\}$. Next, define $O_2^i : i = 1, \ldots, n$ to satisfy the same relations to $E_i$ as $O_1$ does to $E$, and so on. It is worth noting that such sets allow defining $E$ by means of the construction provided in Theorem III. Hence, $0 < \mathcal{H}^{p_0}(E) < \infty$ and $p_0 = \dim_{\mathrm{H}}(E)$. □

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[1] University Centre of Defence at the Spanish Air Force Academy, MDE-UPCT, 30720 Santiago de la Ribera, Murcia (SPAIN)
*E-mail address*: manuel.fernandez-martinez@cud.upct.es, german.rodriguez@cud.upct.es

[2] Departamento de Matemática Aplicada y Estadística. Universidad Politécnica de Cartagena, Hospital de Marina, 30203-Cartagena, Región de Murcia, Murcia (SPAIN)–Corresponding Author–
*E-mail address*: juan.garcia@upct.es