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Decomposition of pseudo-radioactive chemical products with a mathematical approach

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Abstract The aim of this paper is to study the decomposition of pseudo–radioactive products that follow a dynamics determined by a trigonometric factor. In particular for maps of the form $e^{\cos(\pi t)}$ is proved that an asymptotic sampling recomposition property, generalizing the classical Shannon-Whittaker-Kotel'nikov Theorem, works.

Keywords Pseudo-radioactive · band-limited signal · Shannon's sampling theorem · Approximation theory.

1 Introduction and statement of the main result

In [4], we studied the decomposition of pseudo–radioactive products that follow a Gaussian dynamics in terms of a generalization of the well-known Shannon-Whittaker-Kotel'nikov Theorem (see, for instance, [7] and [8]) for a non-banded limited maps on $L^2(\mathbb{R})$, i.e. for Paley-Wiener signals.

One of the main characteristics of this kind of products is that their decomposition dynamics is unknown except for a little amount of laboratory

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temporal samples. Some experimental results have shown that, locally, their behaviors have a Gaussian adjustment, that is, their decomposition function is $f(t) = e^{-\lambda t^2}$, $\lambda > 0$. In [4] we saw that this type of functions satisfies an asymptotic sampling recomposition property called $\mathcal{P}$.

This paper follows the spirit of [4] and extends its results to pseudo–radioactive materials whose dynamics is not, strictly speaking, a Gaussian function. More precisely, we shall prove that the function $f(t) = e^{\cos(\pi t)}$ holds the property $\mathcal{P}$ for every t. Note that the fact that property $\mathcal{P}$ works for trigonometrical maps implies that is possible to use the recomposition property for chemical reactions models with oscillators, i.e., ordinary differential equations of order two.

2 On the property $\mathcal{P}$

We shall remember that a central result of the Signal Theory is the Shannon-Whittaker-Kotel′nikov′s Theorem (see [7] or [8]), based on the normalized cardinal sinus map defined by:

$$\operatorname{sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t} & \text{if } t \neq 0, \\ 1 & \text{if } t = 0. \end{cases}$$

Later, Middleton incorporated a new theorem dealing with band step functions (see [6]), and opened the door to important generalizations. Marvasti and Jain (see [5]) proved that the bandwidth of a signal can be compressed by a ratio of $\frac{1}{n}$ if and only if the signal has n^{th}-order zero crossings or zeros (if complex), and Agud and Catalán (see [1]) stated a new generalization where they prove that we can apply the SWK theorem to a particular kind of signals using less samples per unit of time . All of these generalizations and expansions tried to obtain approximations of non band-limited signals using band-limited ones by increasing their band size. In [4] we studied a different approach, because we kept constant the sampling frequency and generalized in the limit the results of Marvasti et al. and Agud et al. (see [4] and references inside).

Antuña et al. (see [2] and [3]) stated and proved, respectively, the following property $\mathcal{P}$ and theorem.

Property $\mathcal{P}$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a map and $\tau \in \mathbb{R}^+$. We say that f holds the property $\mathcal{P}$ for τ if

$$f(t) = \lim_{n\to\infty} \left(\sum_{k\in\mathbb{Z}} f^{\frac{1}{n}} \left(\frac{k}{\tau}\right) \operatorname{sinc}(\tau t - k) \right)^n \tag{1}$$

Theorem 1 *The Gaussian maps, i.e. maps of the form $e^{-\lambda t^2}$, hold property $\mathcal{P}$ for every given $\tau \in \mathbb{R}^+$.*

Now we shall prove an analogous result for the function $f(t) = e^{\cos(\pi t)}$.

3 Auxiliary results

Lemma 1 *The equality*

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \tag{2}$$

holds for all $z \in \mathbb{Z}$.

In order to prove this lemma, we need, previously, the following one:

Lemma 2 (The additive Herglotz Lemma) *Let* f *be an entire function such that*

$$f(z) = \frac{1}{2} f\left(\frac{z}{2}\right) + \frac{1}{2} f\left(\frac{z+1}{2}\right), \quad \forall z \in \mathbb{C}. \tag{3}$$

Then f *is constant.*

Proof Assume that f is an entire function and satisfies (3), and let D_r be the disk

$$D_r = \{z \in \mathbb{C} : |z| \leq r\},$$

with $r > 1$. It is clear that if $z \in D_r$ then $\frac{z}{2}, \frac{z+1}{2} \in D_r$.

Let $M = \max_{z \in D_r} \{|f'(z)|\}$. If we differentiate the expression (3), we obtain:

$$f'(z) = \frac{1}{4} f'\left(\frac{z}{2}\right) + \frac{1}{4} f'\left(\frac{z+1}{2}\right) \qquad \forall z \in D_r$$

so,

$$4|f'(z)| = \left| f'\left(\frac{z}{2}\right) + f'\left(\frac{z+1}{2}\right) \right| \leq 2M$$

Hence, $|f'(z)| \leq \frac{M}{2}$, for all z, in contradiction with the hypothesis, unless $M = 0$. In this case, $f'(z) = 0$ in D_r, and so f is constant. □

We can now prove Lemma 1.

Proof (Lemma 1) Let us consider the function

$$g(z) = \lim_{n \to \infty} \sum_{k=-n}^{n} \frac{1}{z+k} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

It is clear that $\pi \cot(\pi z)$ y $g(z)$ are meromorphic functions, $\mathbb{Z}-$periodic, with simple poles at $z = n$, $n \in \mathbb{Z}$.

It is immediate that $\cot(\pi z)$ satisfies (3), since

$$\cot(\pi z) = \frac{1}{2} \cot \frac{\pi z}{2} + \frac{1}{2} \cot \frac{\pi(z+1)}{2}$$

Similarly, as $\sum_{k=-n}^{n} \frac{1}{z+k}$ satisfies as well (3), up to a remainder term that for $n \to \infty$ tends to 0, we can state that the function $f(z) = g(z) - \pi \cot(\pi z)$ is an entire function that satisfies Lemma 2. Hence, $f(z)$ is constant. But $f\left(\frac{1}{2}\right)=0$, since $\pi\cot(\pi z)$ vanishes at $z=\frac{1}{2}$ and the sum $g\left(\frac{1}{2}\right)$ is a real telescopic series

$$g\left(\frac{1}{2}\right)=2+\sum_{n=1}^{\infty}\frac{4}{1-4n^2}=0,$$

we have that $f(z)=0$. □

From the equation (2), a couple of related identities can be obtained:

Lemma 3 *The equalities*

$$\begin{aligned}\pi\tan\tfrac{\pi z}{2} &= \sum_{n=1}^{\infty}\tfrac{4z}{(2n-1)^2-z^2}\\ \sum_{n\in\mathbb{N}}\tfrac{(-1)^{n+1}}{n^2-z^2} &= \tfrac{-1}{z}+\tfrac{\pi}{2z\sin(\pi z)}\end{aligned} \tag{4}$$

hold for all $z\in\mathbb{C}$.

Proof Having in mind that $\pi\tan\frac{\pi z}{2}=\pi\cot\frac{\pi z}{2}-2\pi\cot(\pi z)$, we have

$$\pi\cot\frac{\pi z}{2}-2\pi\cot(\pi z)=\sum_{n=1}^{\infty}\frac{z}{\left(\frac{z}{2}\right)^2-n^2}-\sum_{n=1}^{\infty}\frac{4z}{z^2-n^2}$$

Splitting the last series into even and odd terms, we have:

$$\sum_{n=1}^{\infty}\frac{4z}{z^2-4n^2}-\sum_{n=0}^{\infty}\frac{4z}{z^2-(2n+1)^2}-\sum_{n=1}^{\infty}\frac{4z}{z^2-4n^2}=\sum_{n=0}^{\infty}\frac{4z}{(2^n-1)^2-z^2}$$

Regarding the second identity, note that it is equivalent to prove that

$$\frac{\pi}{\sin(\pi z)}=\frac{1}{z}+\sum_{n\in\mathbb{N}}\frac{(-1)^n 2z}{z^2-n^2}$$

But as $\frac{\pi}{\sin(\pi z)}=\pi\cot(\pi z)+\pi\tan\frac{\pi z}{2}$, using the formulae above, we obtain:

$$\begin{aligned}\frac{\pi}{\sin(\pi z)} &= \pi\cot(\pi z)+\pi\tan\frac{\pi z}{2}=\\ &= \frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}+\sum_{n=0}^{\infty}\frac{4z}{(2n+1)^2-z^2}=\\ &= \frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-(2n)^2}+\sum_{n=0}^{\infty}\frac{2z}{z^2-(2n+1)^2}-\sum_{n=0}^{\infty}\frac{4z}{z^2-(2n+1)^2}=\\ &= \frac{1}{z}+\sum_{n=1}^{\infty}\frac{(-1)^n 2z}{z^2-n^2}\end{aligned}$$

□

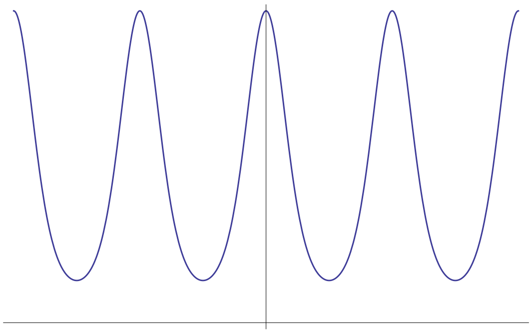

Fig. 1 $f(t) = e^{\cos(\pi t)}$

4 Main result

Theorem 2 *The function $f(z) = e^{\cos(\pi t)}$ satisfies the property $\mathcal{P}$.*

Proof If we define $\lambda_k = e^{(-1)^k}$, $k \in \mathbb{Z}$, it follows from the expansion (2) of the cotangent that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \log(\lambda_k)\,\mathrm{sinc}(t-k) = \log(\lambda_0)\,\mathrm{sinc}(t) + \frac{2t\sin(\pi t)}{\pi}\sum_{k\in\mathbb{N}}\frac{(-1)^k\log(\lambda_k)}{t^2-k^2} = \\ = \mathrm{sinc}(t) + \frac{2t\sin(\pi t)}{\pi}\left(\frac{\pi\cot(\pi t)}{2t} - \frac{1}{2t^2}\right) = \\ = \mathrm{sinc}(t)(1+\pi t\cot(\pi t)-1) = \\ = \cos(\pi t)\end{aligned}$$

hence,

$$f(t) = \prod_{k\in\mathbb{Z}} \lambda_k^{\mathrm{sinc}(t-k)} = e^{\cos(\pi t)},$$

whose graphical representation is shown in Fig.1.

It is clear that f is analytic. Now we show that f satisfies $\mathcal{P}$. Let us now see that

$$\lim_{n\to\infty}\left(\sum_{k\in\mathbb{Z}} \lambda_k^{\frac{1}{n}}\,\mathrm{sinc}(t-k)\right)^n = \prod_{k\in\mathbb{Z}} \lambda_k^{\mathrm{sinc}(t-k)} \tag{5}$$

It is clear that if $t \in \mathbb{Z}$, (5) holds. So, we may assume that $t \notin \mathbb{Z}$. Using the formulae of Lemma 3, we can define the functions:

$$\begin{aligned} A(t) &= \sum_{k\in\mathbb{N}}\frac{1}{(2k-1)^2-t^2} = \frac{\pi}{4t}\tan\left(\frac{\pi t}{2}\right) + \frac{1}{2t^2} - \frac{\pi}{2t\sin(\pi t)} \\ B(t) &= \sum_{k\in\mathbb{N}}\frac{1}{(2k-1)^2-t^2} = \frac{\pi}{4t}\tan\left(\frac{\pi t}{2}\right)\end{aligned} \tag{6}$$

Computing, and using again the notation

$$h(t,n) = \sum_{k\in\mathbb{Z}} \lambda_k^{\frac{1}{n}} \operatorname{sinc}(t-k) \tag{7}$$

we have

$$\begin{aligned} h(t,n) &= \lambda_0 \operatorname{sinc}(t) + \frac{2t\sin(\pi t)}{\pi} \sum_{k\in\mathbb{N}} \frac{(-1)^k \lambda_k^{\frac{1}{n}}}{t^2-k^2} = \\ &= e^{\frac{1}{n}} \operatorname{sinc}(t) + \frac{2t\sin(\pi t)}{\pi} \left(-e^{\frac{1}{n}} A(t) + e^{-\frac{1}{n}} B(t)\right) \end{aligned}$$

So, taking limit when n tends to infinity in expression above, it is

$$\begin{aligned} \lim_{n\to\infty} h(t,n) &= \lim_{n\to\infty} \sum_{k\in\mathbb{Z}} \lambda_k^{\frac{1}{n}} \operatorname{sinc}(t-k) = \\ &= \operatorname{sinc}(t) + \frac{2t\sin(\pi t)}{\pi} \left(\frac{\pi}{2t\sin(\pi t)} - \frac{1}{2t^2} \right) = 1 \end{aligned}$$

On the other hand, developing the exponential in a power series and using the identity above

$$\operatorname{sinc}(t) - \frac{2t\sin(\pi t)}{\pi} A(t) + \frac{2t\sin(\pi t)}{\pi} B(t) - 1 = 0,$$

we have

$$\begin{aligned} n(h(t,n)-1) &= ne^{\frac{1}{n}} \left(\operatorname{sinc}(t) - \frac{2t\sin(\pi t)}{\pi} A(t) \right) + ne^{-\frac{1}{n}} \frac{2t\sin(\pi t)}{\pi} B(t) - n = \\ &= ne^{-\frac{1}{n}} \left(\operatorname{sinc}(t) - \frac{2t\sin(\pi t)}{\pi} A(t) + \frac{2t\sin(\pi t)}{\pi} B(t) \right) + \\ &\quad + n\frac{2t\sin(\pi t)}{\pi} B(t) \left(e^{-\frac{1}{n}} - e^{\frac{1}{n}} \right) - n = \\ &= n\left(e^{\frac{1}{n}} - 1\right) + n\frac{2t\sin(\pi t)}{\pi} B(t) \left(e^{-\frac{1}{n}} - e^{\frac{1}{n}} \right) = \\ &= n\left(\frac{1}{n} + \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) \right) + n\frac{2t\sin(\pi t)}{\pi} B(t) \left(\frac{-2}{n^2} + o\left(\frac{1}{n^2}\right) \right) = \\ &= 1 - \frac{4t\sin(\pi t)}{\pi} B(t) + \frac{1}{2n} + o\left(\frac{1}{n}\right) \end{aligned}$$

so, by (6), we have:

$$\begin{aligned} \lim_{n\to\infty} n(h(t,n)-1) &= 1 - \frac{4t\sin(\pi t)}{\pi} B(t) = \\ &= 1 - \tan\left(\frac{\pi t}{2}\right) \sin(\pi t) = 1 - 2\sin^2\left(\frac{\pi t}{2}\right) = \\ &= \cos(\pi t) \end{aligned}$$

concluding that

$$\lim_{n\to\infty} e^{\lim_{n\to\infty} n(h(t,n)-1} = e^{\cos(\pi t)} = \prod_{k\in\mathbb{Z}} \lambda_k^{\operatorname{sinc}(t-k)}$$

□

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