Similarity dimension for IFS-attractors

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Abstract. Moran's Theorem is one of the milestones in Fractal Geometry. It allows the calculation of the similarity dimension of any (strict) self-similar set lying under the open set condition. Throughout a new fractal dimension we provide in the context of fractal structures, we generalize such a classical result for attractors which are required to satisfy no separation properties.

Keywords: Fractal, iterated function system, IFS-attractor, fractal structure, fractal dimension, box dimension, Hausdorff dimension.

1. Introduction

In this paper, we revisit a classical problem in Fractal Geometry from the viewpoint of fractal structures: how to calculate the similarity dimension of the attractor of an iterated function system. A solution for such a problem requires the open set condition to be satisfied by the corresponding iterated function system. Such a condition tries to control the overlapping among the self-similar copies of the whole attractor.

We have to trace back to the forties to find out the key result that allows the effective calculation of the similarity dimension of (strict) self-similar sets from their similarity ratios. It was proved by Australian mathematician P.A.P. Moran, a Besicovitch pupil at Cambridge (c.f. [12, Theorem II]).

A new viewpoint regarding fractals arises from Asymmetric Topology. In fact, a fractal structure is a kind of uniformity which provides better approaches of a space as deeper levels in its structure are explored.

In this article, we prove a generalized Moran's Theorem for attractors which, unlike Moran's Theorem, are not required to be under the open set condition.

The structure of this paper is as follows. In Section 2, we provide all the mathematical background to make this article self-contained. This includes the basics on IFS-attractors, the open set condition, fractal structures, and a brief description regarding both the Hausdorff and box dimensions. Section 3 explains how classical box dimension can be generalized by fractal dimension III in the context of fractal structures. Section 4 contains a generalized Moran's Theorem (c.f. Theorem 4.1), and finally, Section 5 summarizes the main conclusions (c.f. Theorem 5.1).

2. Preliminaries

2.1. IFS-attractors

Let $k \geq 2$. By an iterated function system (IFS), we shall understand a finite collection of similitudes on $\mathbb{R}^d$, $\mathcal{F} = \{f_1, \ldots, f_k\}$, where each self-map f_i satisfies the identity

$$d(f_i(x), f_i(y)) = c_i \cdot d(x, y), \text{ for all } x, y \in \mathbb{R}^d,$$

$c_i \in (0, 1)$ is the similarity ratio of f_i, and d refers to the Euclidean distance. Then there exists a unique compact (nonempty) subset $\mathcal{K} \subset \mathbb{R}^d$ satisfying the next Hutchinson's equation (c.f. [10]):

$$\mathcal{K} = \cup_{i=1}^k f_i(\mathcal{K}). \quad (1)$$

$\mathcal{K}$ is called the attractor (also the self-similar set generated by $\mathcal{F}$) and consists of (smaller) self-similar copies $\mathcal{K}_i$ of the whole attractor $\mathcal{K}$, also known as *pre-fractals* of $\mathcal{K}$ (c.f. [6]). In fact, $\mathcal{K}_i = f_i(\mathcal{K})$, for all $i = 1, \ldots, k$. We also denote $\mathcal{K}_{ij} = f_i(f_j(\mathcal{K}))$, and so on. In general, we will follow the notation used in Bandt's paper, c.f. [3]: let $n \in \mathbb{N}$ and $\Sigma = \{1, \ldots, k\}$ be a finite alphabet. Moreover, let $\Sigma^n = \{\mathbf{i} = i_1 \cdots i_n : i_j \in \Sigma, j = 1, \ldots, n\}$ be the collection of all $n-$length *words* from Σ. Further, we also write $f_{\mathbf{i}} = f_{i_1} \circ \cdots \circ f_{i_n}$, $c_{\mathbf{i}} = c_{i_1} \cdots c_{i_n}$, and $\mathcal{K}_{\mathbf{i}} = f_{\mathbf{i}}(\mathcal{K})$. Accordingly, Eq. (1) can be rewritten as $\mathcal{K} = \cup\{\mathcal{K}_{\mathbf{i}} : \mathbf{i} \in \Sigma^n\}$. Letting $n \to \infty$, the so-called *address map* $\pi : S^\infty \longrightarrow \mathcal{K}$ stands as a continuous map from Σ^∞, the set consisting of all infinite length words (sequences), onto the attractor $\mathcal{K}$.

2.2. The open set condition

The open set condition (OSC) was first provided by P.A.P. Moran in [12] to prove that the Hausdorff measure of any attractor is positive. Interestingly, Schief proved in [14] its reciprocal: a positive Hausdorff measure implies the OSC. We say that an IFS $\mathcal{F} = \{f_1, \ldots, f_k\}$ is under the OSC if there exists a nonempty open subset $\mathcal{V} \subseteq \mathbb{R}^d$ (sometimes called a *feasible open set*, c.f. [3]) such that the $f_i(\mathcal{V})$ ($i = 1, \ldots, k$) are pairwise disjoint with all of them contained in $\mathcal{V}$, i.e., $\cup_{i=1}^k f_i(\mathcal{V}) \subseteq \mathcal{V} : f_i(\mathcal{V}) \cap f_j(\mathcal{V}) = \emptyset$, for all $i \neq j$.

On the other hand, observe that the feasible open set $\mathcal{V}$ and the attractor $\mathcal{K}$ may be disjoint. In these situations, the OSC may be too weak to prove theoretical results regarding the fractal dimension of $\mathcal{K}$. In this way, Lalley strengthened the definition of the OSC in the following sense [11]: the strong open set condition (SOSC) is fulfilled if it is also satisfied that $\mathcal{K} \cap \mathcal{V} \neq \emptyset$. It is worth pointing out that Schief also proved that the OSC and the SOSC are equivalent on Euclidean subsets (c.f. [14, Theorem 2.2]).

2.3. Fractal structures

The concept of a fractal structure was first introduced by Bandt and Retta in [4], and formalized afterwards by Arenas and Sánchez-Granero in [1] to characterize non-Archimedeanly quasi-metrizable spaces. They play a relevant role in asymmetric topology and constitute an ideal context where new models of fractal dimension can be provided (c.f. [7]). A family Γ of subsets of X is said to be a covering (of X) if $X = \cup\{A : A \in \Gamma\}$. Let Γ_1 and Γ_2 be two coverings of X. By $\Gamma_1 \prec \Gamma_2$, we understand that Γ_1 is a refinement of Γ_2, i.e., for all $A \in \Gamma_1$, there exists $B \in \Gamma_2$ such that $A \subseteq B$. In addition, $\Gamma_1 \prec\prec \Gamma_2$ means that $\Gamma_1 \prec \Gamma_2$, and also that for all $B \in \Gamma_2$, $B = \cup\{A \in \Gamma_1 : A \subseteq B\}$. Thus, a fractal structure on X is a countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ such that $\Gamma_{n+1} \prec\prec \Gamma_n$, for all $n \in \mathbb{N}$. Covering Γ_n is called level n of $\mathbf{\Gamma}$.

To simplify, we shall allow that a set can appear twice or more in a level of a fractal structure. A fractal structure is finite if all its levels are finite coverings.

It is worth pointing out that there exists a natural fractal structure for each attractor. Its description can be stated as follows.

Definition 2.1 (c.f. [2], Definition 4.4)**.** *Let $\mathcal{F}$ be an IFS whose attractor is $\mathcal{K}$. The natural fractal structure on $\mathcal{K}$ as a self-similar set is the countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$, where $\Gamma_n = \{f_{\boldsymbol{i}}(\mathcal{K}) : \boldsymbol{i} \in \Sigma^n\}$.*

Equivalently, the levels of the natural fractal structure on $\mathcal{K}$ as a self-similar set can be described as $\Gamma_1 = \{f_i(\mathcal{K}) : i \in \Sigma\}$, and $\Gamma_{n+1} = \{f_i(A) : A \in \Gamma_n, i \in \Sigma\}$.

On the other hand, it holds that any Euclidean space $\mathbb{R}^d$ can be always endowed with a natural fractal structure whose levels are given by (c.f. [9, Definition 3.1]):

$$\Gamma_n = \left\{ \left[\tfrac{k_1}{2^n}, \tfrac{k_1+1}{2^n}\right] \times \cdots \times \left[\tfrac{k_d}{2^n}, \tfrac{k_d+1}{2^n}\right] : k_i \in \mathbb{Z} \right\},$$

where $i = 1, \ldots, d$. Note that such a fractal structure is a tiling consisting of $2^{-n}-$cubes on $\mathbb{R}^d$.

2.4. The Hausdorff and box dimensions

Let (X, ρ) be a metric space. Along the sequel, $\mathrm{diam}\,(A)$ will denote the diameter of any subset A of X, i.e., $\mathrm{diam}\,(A) = \sup\{\rho(x, y) : x, y \in A\}$. Moreover, let $F \subseteq X$ and $\delta > 0$. By a $\delta-$cover of F, we shall understand a countable family of subsets $\{U_i : i \in I\}$ such that $F \subseteq \cup_{i \in I} U_i$ with $\mathrm{diam}\,(U_i) \leq \delta$. In addition, let $\mathcal{C}_\delta(F)$ be the collection of all $\delta-$covers of F and define

$$\mathcal{H}_\delta^s(F) = \inf \left\{ \sum_{i \in I} \mathrm{diam}\,(U_i)^s : \{U_i\}_{i \in I} \in \mathcal{C}_\delta(F) \right\}.$$

Hence, $\mathcal{H}_{\mathrm{H}}^\alpha(F) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(F)$ always exists and is called the ($s-$dimensional) Hausdorff measure of F. It allows to define the Hausdorff dimension of F as the (unique) critical point $s \geq 0$ where $\mathcal{H}_{\mathrm{H}}^s(F)$ "jumps" from ∞ to zero, i.e., $\dim_{\mathrm{H}}(F) = \sup\{s : \mathcal{H}_{\mathrm{H}}^s(F) = \infty\} = \inf\{s : \mathcal{H}_{\mathrm{H}}^s(F) = 0\}$. In particular,

$$\mathcal{H}_{\mathrm{H}}^{\dim_{\mathrm{H}}(F)}(F) \in \{0, d, \infty : d \in (0, \infty)\}.$$

Though the Hausdorff dimension is the most accurate model of fractal dimension since its definition is based on a measure, the box dimension is more appropriate to be applied in empirical contexts. The (lower/upper) box dimension of $F \subseteq \mathbb{R}^d$ is defined by the following (lower/upper) limit:

$$\dim_{\mathrm{B}}(F) = \lim_{\delta \to 0} \frac{\log \mathcal{N}_\delta(F)}{-\log \delta},$$

where $\mathcal{N}_\delta(F)$ is the number of $\delta-$cubes that intersect F. A $\delta-$cube in $\mathbb{R}^d$ is a set of the form $\{[k_1\delta, (k_1+1)\delta] \times \cdots \times [k_d\delta, (k_d+1)\delta] : k_1, \ldots, k_d \in \mathbb{Z}\}$. In particular, we can choose $\delta = 2^{-n}$. It is worth mentioning that $\mathcal{N}_\delta(F)$ can be calculated by one of the expressions displayed in [6, Equivalent definitions 3.1]. In particular, $\mathcal{N}_\delta(F)$ can be chosen as the smallest number of sets of diameter $\leq \delta$ that cover F.

3. Generalizing box dimension by fractal dimension III

In this section, we explain how to generalize the classical box dimension on Euclidean subsets from the viewpoint of fractal structures. To deal with, we shall define a fractal dimension model for fractal structures and prove that if the natural fractal structure for Euclidean subsets is fixed, then both dimensions coincide.

Let $\mathbf{\Gamma}$ be a fractal structure. We define $\mathcal{A}_n(F) = \{A \in \Gamma_n : A \cap F \neq \emptyset\}$ as the collection of all the elements in level n of $\mathbf{\Gamma}$ that intersect a given subset F of X, $\operatorname{diam}(\Gamma_n) = \sup\{\operatorname{diam}(A) : A \in \Gamma_n\}$, and $\operatorname{diam}(F, \Gamma_n) = \sup\{\operatorname{diam}(A) : A \in \mathcal{A}_n(F)\}$.

Definition 3.1 (c.f. [8], Definition 4.2)**.** *Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X, ρ), F be a subset of X, assume that $\operatorname{diam}(F, \Gamma_n) \to 0$, and define $\mathcal{H}^s_{n,3}(F)$ by*

$$\inf \left\{ \sum_{i \in I} \operatorname{diam}(A_i)^s : \{A_i\}_{i \in I} \in \mathcal{A}_{n,3}(F) \right\},$$

where $\mathcal{A}_{n,3}(F) = \{\{A : A \in \mathcal{A}_l(F)\} : l \geq n\}$. Moreover, let $\mathcal{H}^s_3(F) = \lim_{n \to \infty} \mathcal{H}^s_{n,3}(F)$. The fractal dimension III of F is given as the (unique) non-negative real number such that

$$\begin{aligned} \dim^3_{\mathbf{\Gamma}}(F) &= \sup\{s \geq 0 : \mathcal{H}^s_3(F) = \infty\} \\ &= \inf\{s \geq 0 : \mathcal{H}^s_3(F) = 0\}. \end{aligned}$$

It is worth pointing out that fractal dimension III always exists, since the sequence $\{\mathcal{H}^s_{n,3}(F) : n \in \mathbb{N}\}$ is monotonic in $n \in \mathbb{N}$. We assume that $\inf \emptyset = \infty$ in Definition 3.1. For instance, if there exists $F \subset X$ such that $\mathcal{A}_{n,3}(F) = \emptyset$, then we have $\dim^3_{\mathbf{\Gamma}}(F) = \infty$.

Observe that the condition $\operatorname{diam}(F, \Gamma_n) \to 0$, though necessary in previous Definition 3.1, is not too restrictive.

Remark 3.2. *Let $\mathcal{K}$ be any attractor endowed with its natural fractal structure as a self-similar set. Then $\operatorname{diam}(\mathcal{K}, \Gamma_n) \to 0$ since $\{\operatorname{diam}(\Gamma_n) : n \in \mathbb{N}\}$ decreases geometrically.*

The following result gives a handier expression to deal with the effective calculation of fractal dimension III.

Theorem 3.3 (c.f. [8], Theorem 4.7)**.** *Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X, ρ), F be a subset of X, and assume that $\mathcal{H}^s(F) = \lim_{n \to \infty} \mathcal{H}^s_n(F)$ exists, where $\mathcal{H}^s_n(F) = \sum \{\operatorname{diam}(A)^s : A \in \mathcal{A}_n(F)\}$. Then*

$$\begin{aligned} \dim^3_{\mathbf{\Gamma}}(F) &= \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\} \\ &= \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\}. \end{aligned}$$

Next, we show that fractal dimension III generalizes box dimension. To deal with, we will prove even a more general result, aimed by the next Euclidean property: for each $\delta > 0$ and all Euclidean subset F with $\operatorname{diam}(F) \leq \delta$, there are at most 3^d $\delta-$cubes in $\mathbb{R}^d$ that are intersected by F. This fact motivates the following definition.

Definition 3.4. *Let $\mathbf{\Gamma}$ be a fractal structure on a metrizable space X and F be a subset of X. We say that $\mathbf{\Gamma}$ is under the $\kappa-$condition if there exists a natural number κ such that for all $n \in \mathbb{N}$, every subset A of X with $\operatorname{diam}(A) \leq \operatorname{diam}(F, \Gamma_n)$ intersects at most to κ elements in level n of $\mathbf{\Gamma}$.*

Hence, the following result can be stated.

Theorem 3.5 (c.f. [8], Theorem 4.17)**.** *Let $\mathbf{\Gamma}$ be a fractal structure under the $\kappa-$condition on a metric space (X, ρ) and F be a subset of X. Moreover, assume that there exists $\dim_{\mathrm{B}}(F)$. If for all $A \in \mathcal{A}_n(F)$ it holds that $\operatorname{diam}(A) = \operatorname{diam}(F, \Gamma_n)$ with $\operatorname{diam}(F, \Gamma_n) \to 0$, then the fractal dimension III of F equals the box dimension of F, i.e., $\dim_{\mathrm{B}}(F) = \dim^3_{\mathbf{\Gamma}}(F)$.*

The next corollary follows immediately from Theorem 3.5.

Corollary 3.6 (c.f. [8], Theorem 4.15)**.** *Let $F \subset \mathbb{R}^d$ endowed with its natural fractal structure as a Euclidean subset. Assume that there exists $\dim_{\mathrm{B}}(F)$. Then $\dim_{\mathrm{B}}(F) = \dim^3_{\mathbf{\Gamma}}(F)$.*

To justify that, just notice that the natural fractal structure on any Euclidean subset consists of $2^{-n}-$diameter elements in level n of $\mathbf{\Gamma}$.

4. A Moran's Theorem for fractal dimension III

Moran's Theorem is a milestone in Fractal Geometry. It was first contributed by P.A.P. Moran (1946), who required the pre-fractals $\mathcal{K}_{\mathbf{i}}$ of an attractor $\mathcal{K}$ not to overlap among them (which is equivalent to require $\mathcal{F}$ to satisfy the OSC), to prove that the Hausdorff dimension of $\mathcal{K}$ follows as the (unique) solution of an equation only involving the similarity ratios c_i associated with each $f_i \in \mathcal{F}$. However, it is worth pointing out that such a result still remains quite powerful, since without a wide amount of effort, the Hausdorff dimension of a wide class of self-similar sets can be easily calculated. Next, we recall that classical result.

Moran's Theorem (1946). *Let $\mathcal{F}$ be an IFS whose attractor is $\mathcal{K}$. Let c_i be the similarity ratio associated with each similarity $f_i \in \mathcal{F}$, and assume that $\mathcal{F}$ is under the OSC. If α is the solution of the equation $\sum_{i\in I} c_i^s = 1$, then $\dim_{\mathrm{B}}(\mathcal{K}) = \dim_{\mathrm{H}}(\mathcal{K}) = \alpha$, and $0 < \mathcal{H}^{\alpha}_{\mathrm{H}}(\mathcal{K}) < \infty$.*

The unique (positive) solution of $\sum_{i\in I} c_i^s = 1$ is called the similarity dimension of $\mathcal{K}$. Along the sequel, the similarity dimension of an IFS-attractor will be denoted by α. Thus, we always have $\mathcal{H}^{\alpha}_{\mathrm{H}}(\mathcal{K}) \in (0,\infty)$. A proof for Moran's Theorem can be found in Falconer's book (c.f. [6, Subsection 9.2]), though the reader may check that the proof regarding a lower bound of the Hausdorff dimension becomes quite awkward. Moreover, whether the OSC is not fulfilled by $\mathcal{F}$, then the calculation of the Hausdorff dimension of $\mathcal{K}$ becomes harder and only some partial results are known (c.f., e.g., [5,13]). However, even in that case, it holds that both the box and the Hausdorff dimensions of $\mathcal{K}$ can be approximated by fractal dimension III, which still equals the similarity dimension. Next, we provide the main theoretical result in this section, which provides a generalized version of the classical Moran's Theorem in terms of fractal dimension III.

Theorem 4.1 (c.f. [8], Theorem 4.20 and Corollary 4.22)**.** *Let $\mathcal{F}$ be an IFS whose associated attractor is $\mathcal{K}$. Assume that c_i is the similarity ratio associated with each similarity $f_i \in \mathcal{F}$ and let $\mathbf{\Gamma}$ be the natural fractal structure on $\mathcal{K}$ as a self-similar set. If α is the similarity dimension of $\mathcal{K}$, then*

(i) $\dim^3_{\mathbf{\Gamma}}(\mathcal{K}) = \alpha$ *and* $0 < \mathcal{H}^{\alpha}_{3}(\mathcal{K}) < \infty$.
(ii) In addition, if $\mathcal{F}$ is under the OSC, then $\dim_{\mathrm{B}}(\mathcal{K}) = \dim^3_{\mathbf{\Gamma}}(\mathcal{K}) = \dim_{\mathrm{H}}(\mathcal{K}) = \alpha$. *Also,* $\mathcal{H}^{\alpha}_{\mathrm{H}}(\mathcal{K}), \mathcal{H}^{\alpha}_{3}(\mathcal{K}) \in (0,\infty)$.

Proof. Firstly, it is clear that the attractor $\mathcal{K}$ is the unique non-empty compact subset of $\mathbb{R}^d$ satisfying the following Hutchinson's equation:

$$\mathcal{K} = \cup\{\mathcal{K}_{\mathbf{i}} : \mathbf{i} \in \Sigma^n\}.$$

Hence,

(i) Notice that $\mathcal{A}_{n,3}(\mathcal{K}) = \{\Gamma_m : m \geq n\}$. Further, let $\alpha \geq 0$ be such that $\sum_{i\in I} c_i^{\alpha} = 1$. In addition, observe that $c_{\mathbf{i}}$ is the similarity ratio associated with $f_{\mathbf{i}}$. Thus, $\mathrm{diam}\,(\mathcal{K}_{\mathbf{i}}) = c_{\mathbf{i}} \cdot \mathrm{diam}\,(\mathcal{K})$ for all $\mathbf{i} \in \Sigma^l$. It is also worth pointing out that

$$\begin{aligned}\sum\{c_{\mathbf{i}}^{\alpha} : \mathbf{i} \in \Sigma^l\} &= \sum_{i_1\in\Sigma} c_{i_1}^{\alpha} \cdots \sum_{i_l\in\Sigma} c_{i_l}^{\alpha} \\ &= \sum_{\mathbf{i}\in\Sigma^l} c_{\mathbf{i}}^{\alpha} = 1,\end{aligned}$$

for all $\mathbf{i} = i_1 \cdots i_l \in \Sigma^l$. Accordingly, for all $n \in \mathbb{N}$, the following calculations hold: $\mathcal{H}^{\alpha}_{n,3}(\mathcal{K}) = \inf\{\sum \mathrm{diam}\,(A)^{\alpha} : A \in \Gamma_m, m \geq n\} = \inf\{\sum \mathrm{diam}\,(\mathcal{K}_{\mathbf{i}})^{\alpha} : \mathbf{i} \in \Sigma^m, m \geq n\} = \inf\{\sum c_{\mathbf{i}}^{\alpha} \cdot \mathrm{diam}\,(\mathcal{K})^{\alpha} : \mathbf{i} \in \Sigma^m, m \geq n\}$. Thus, $\dim^3_{\mathbf{\Gamma}}(\mathcal{K}) = \alpha$, since $\mathcal{H}^{\alpha}_{3}(\mathcal{K}) = \mathrm{diam}\,(\mathcal{K})^{\alpha} \notin \{0,\infty\}$.

(ii) In addition, if $\mathcal{F}$ is under the OSC, then Moran's Theorem and Theorem 4.1 (i) lead to $\alpha = \dim^3_{\mathbf{\Gamma}}(\mathcal{K})$, i.e., the fractal dimension III of $\mathcal{K}$ equals its similarity dimension. Moreover, that value of α also equals both the box and the Hausdorff dimensions of $\mathcal{K}$. In fact, the next chain of inequalities is satisfied: $\dim_{\mathrm{H}}(\mathcal{K}) \leq \dim^3_{\mathbf{\Gamma}}(\mathcal{K}) = \dim_{H}(\mathcal{K}) = \dim_{\mathrm{B}}(\mathcal{K}) = \alpha$.
To conclude the proof, just observe that $\mathcal{H}^{\alpha}_{\mathrm{H}}(\mathcal{K}) \leq \mathcal{H}^{\alpha}_{3}(\mathcal{K})$, where $\mathcal{H}^{\alpha}_{3}(\mathcal{K}) < \infty$ by Theorem 4.1 (i) and $\mathcal{H}^{\alpha}_{\mathrm{H}}(\mathcal{K}) > 0$ (due to Moran's Theorem). Accordingly, Theorem 4.1 (ii) follows.

□

Thus, Theorem Theorem 4.1 (i) becomes useful to calculate the fractal dimension of IFS-attractors. Interestingly, it does not require the corresponding IFS to be under the OSC for fractal dimension III calculation purposes.

5. Conclusion

Next, we summarize all the results contained in this paper.

Theorem 5.1. *Let $\mathcal{F}$ be an IFS and $\mathcal{K}$ its attractor. Moreover, let $\mathbf{\Gamma}$ be the natural fractal structure on $\mathcal{K}$ as a self-similar set and c_i be the similarity ratio associated with each similarity $f_i \in \mathcal{F}$. Consider the following statements:*

(i) SOSC.
(ii) OSC.
(iii) $0 < \mathcal{H}^{\alpha}(\mathcal{K}) < \infty$.
(iv) $\dim_{\mathrm{B}}(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^{3}(\mathcal{K}) = \dim_{H}(\mathcal{K}) = \alpha$*, where* α *is the similarity dimension of* $\mathcal{K}$*.*

The next chain of implications and equivalences stands:

$$(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv).$$

It is worth pointing out that the implication $(iv) \Rightarrow (iii)$ is not true, in general, due to a counterexample provided by Mattila (c.f. [14]).

Acknowledgements

The authors acknowledge the partial support of grants No. 19219/PI/14 from Fundación Séneca of Región de Murcia and No. MTM2014-51891-P from Spanish Ministry of Economy and Competitiveness.

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