TWO TIGHT INDEPENDENT SET CONDITIONS FOR FRACTIONAL (g, f, m)-DELETED GRAPHS SYSTEMS

Abstract. A graph G is called a fractional (g, f, m)-deleted graph if the resulting graph admits a fractional (g, f)-factor after m edges are removed. An important fact in the characterization of a discrete graph dynamical system is played by the independent sets, i.e., subsets of the vertex set of G in which any two of them are not adjacent because they reflect the sparsity and stability of the graph system in somehow. The neighborhoods union of independent sets characterizes the local density and local clustering characteristics of the graph system. In this paper, we study the relationship between characteristics of independent sets and fractional (g, f, m)-deleted graph systems. The main contributions cover two aspects: first, we present an independent set degree condition for a graph to be fractional (g, f, m)-deleted; later an independent set neighborhood union condition for fractional (g, f, m)-deleted graphs is determined. Furthermore, we show that the results obtained are the best in some sense.

1. Introduction and statement of the main results

The problem of fractional factor can be considered as a relaxation problem of the well-known cardinality matching. It possesses a wide range of applications in fields such as combinatorial polyhedron, scheduling and network design because such real situations are modeled using dynamical systems defined by graphs, see for more information [2]. For instance, in a communication network, several large data packets are sent to various destinations through channels. The effectiveness of such transmission work can be improved if large data packets are allowed to be partitioned into small parcels. The available distribution of data packets can be considered as a fractional flow problem and it can be described as a fractional factor problem if the network has disjoint destinations and sources.

Let $G = (V(G), E(G))$ be a graph (finite, loopless, and without multiple edges) with vertex set $V(G)$ and edge set $E(G)$. We shall say that two vertices x and y are *adjacent* if there is an edge between them, and we shall say that the edge e and the vertex x are *incident* if e connects x. We shall use $x \sim y$ to express that two vertices x and y are adjacent, and $e \sim x$ to denote that the edge e and the vertex x are incident.

For any vertex $x \in V(G)$, $N_G(x) := \{y | y \in V(G), x \sim y\}$ denotes *the neighborhood of* x *in* G. By $d_G(x) := |N_G(x)|$ we denote the *degree of* x *in*

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G where $|\cdot|$ means cardinality. By $\delta(G) := \min_{x\in V(G)}\{d_G(x)\}$ we define *the minimum degree of* G. For an arbitrary $S \subseteq V(G)$, we denote by $G[S]$ *the subgraph of* G *induced by* S, where $V(G[S]) = S$ and $E(G[S])$ are the edges of G linking two elements of S. By $G - S$ we denote *the complementary graph of* S *in* G, i.e., the graph $G[V(G)\backslash S]$. For any $T \subseteq V(G)$, let $N_T(x)$ the restriction of $N_G(x)$ to the elements of T, i.e., $N_T(x) := N_G(x) \cap T$. By $N_\bullet[x] := N_\bullet(x) \cup \{x\}$ where $\bullet \in \{G, T\}$ we denote respectively the closed neighborhoods of vertex x in G and in T. Given S and T two disjoint subsets of $V(G)$, by $e_G(S, T)$ we denote the number of edges with one end in S and the other in T. We shall say that a graph is *complete* if all its vertices are adjacent. Our notation and terminology is inspired by [1] and [3], see them for more details.

Let g and f be two positive integer-valued maps defined on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for any $x \in V(G)$. A *fractional* (g, f)*-factor* is a map $h : E(G) \to [0, 1]$ such that $g(x) \leq d^h_G(x) \leq f(x)$ for each vertex x, where $d^h_G(x) := \sum_{e\sim x} h(e)$ is the so called *fractional degree* of vertex x. If $g(x) = f(x)$ for all $x \in V(G)$, then the fractional (g, f)-factor is a *fractional* f*-factor*. Moreover, if $g(x) = f(x) = k$ for all $x \in V(G)$, then the fractional f-factor is called a *fractional* k*-factor*.

Gao and Wang [8] extends the concept of fractional (k, m)-deleted graph to fractional (g, f, m)-deleted graph. A graph G is called a *fractional* (g, f, m)*-deleted graph* if for each edge subset $H \subseteq E(G)$ with cardinality equal to m, there exists a fractional (g, f)-factor h such that $h(e) = 0$ for all $e \in H$. That is, after removing any m edges, the resulting graph still has a fractional (g, f)-factor. If $g(x) = f(x)$ for all $x \in V(G)$, then a fractional (g, f, m)-deleted graph is a fractional (f, m)-deleted graph. Furthermore, if $g(x) = f(x) = k$ for all $x \in V(G)$, then a fractional (g, f, m)-deleted graph becomes a *fractional* (k, m)*-deleted graph*. As extension of fractional factor, the fractional deleted graph describes the existence of fractional factor in communication or transmission networks when certain channels are damaged.

The *order* of G is the cardinality of $V(G)$. In what follows we consider that G is a graph of order n and not complete. There are several advances in determining the graph features to be fractional deleted. Gao et al. [7] proved that G is a fractional (g, f, m)-deleted graph if there exist a, b, m non-negative integers such that one of the following conditions holds:

- $n > \frac{(a+b)(a+b+2m-2)}{a}$, $2 \leq a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$, and $\delta(G) \geq \frac{bn}{a+b}$;
- $n > \frac{(a+b)(a+b+2m-1)}{a}$, $\delta(G) \geq \frac{b^2}{a} + m$, $2 \leq a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$ and

$$\max\{d_G(x), d_G(y)\} \geq \frac{bn}{a+b}$$

  for each pair of non-adjacent vertices x and y of G;

- $n > \frac{(a+b)(a+b+2m-2)}{a}$, $\delta(G) \geq \frac{b^2}{a} + m$, $2 \leq a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$, and
$$\min\{d_G(u) + d_G(v)\} \geq \frac{2bn}{a+b},$$
where u and v are two non-adjacent vertices of G.

Zhou and Bian [13] determine an independent set degree condition for fractional (g, f)-deleted graphs (recall this is a special case of fractional (g, f, m)-deleted graph when $m = 1$) stating that G is a fractional (g, f)-deleted graph if $r \geq 2$, $\beta \geq 0$, $n \geq \frac{(a+b)(a+b+1+(r-2)(b-\beta))}{a+\beta}$, $\delta(G) \geq \frac{(r-1)(b-\beta)(b+1)}{a+\beta}$, $2 \leq a \leq g(x) \leq f(x) - \beta \leq b - \beta$ for each $x \in V(G)$ (here r, β, a, b are all integers) and
$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_r)\} \geq \frac{(b-\beta)n}{a+b}$$
for any independent subset $\{x_1, x_2, \cdots, x_r\}$ of $V(G)$, where an *independent subset* is a subset of vertices in G such that there in no pairs of adjacent vertices inside.

More contributions on fraction deleted graphs and their applications can be found in Farahani [4], Gao and Farahani [5], Gao et al. [6], Gao and Wang [8] and [9], Guirao and Luo [10], Jin [11] or Zhou et al. [12, 14, 15, 16, 17].

Our main contribution in this paper is to present the sharp independent set degree condition and tight independent set neighborhood union condition for fractional (g, f, m)-deleted graphs. The statements of our main results are the following and the proofs of them will be presented in the Section 3.

Theorem 1. *Let G be a non complete graph of order n. Let a, b, m, Δ and i be non-negative integers such that $i \geq 2$, $2 \leq a \leq b-\Delta$, $\delta(G) \geq \frac{(b-\Delta)(b+1)(i-1)}{a+\Delta} + m$ and $n \geq \frac{(a+b)(a+b+2m-1+(i-2)(b-\Delta))}{a+\Delta}$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$ for each $x \in V(G)$. If G satisfies*
$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{(b-\Delta)n}{a+b}$$
for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional (g, f, m)-deleted graph.

Theorem 2. *Let G be a non complete graph of order n. Let a, b, m, Δ and i be non-negative integers with $i \geq 2$, $2 \leq a \leq b - \Delta$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$ for each $x \in V(G)$. If $\delta(G) \geq \frac{(i-1)(b-\Delta)(b+1)}{a+\Delta} + m$, $n \geq \frac{(a+b)(i(a+b)+2m-2)}{a+\Delta}$, and*
$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \geq \frac{(b-\Delta)n}{a+b}$$
for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional (g, f, m)-deleted graph.

Clearly, Theorem 1 extends the main result of Zhou and Bian [13].

2. Auxiliary results

Given ϕ an integer-valued function defined on $V(G)$ and $S \subseteq V(G)$, we define

$$\phi(S) := \sum_{x \in S} \phi(x).$$

If $H \subseteq E(G)$, $x \in V(G)$ and S, T are disjoint subsets of $V(G)$, we define

$$d_H(x) := |\{e \in H | e \sim x\}|,$$

$$e_H(x, S) := |\{e = xy \in H | y \in S\}|,$$

and

$$e_H(T, S) := |\{e = xy \in H | x \in S, y \in T\}|.$$

Here xy denotes the edge with ends x and y. In light of the previous definitions, we deduce that each edge in H contribution to 2 degrees in $\sum_{x \in T} d_H(x)$ and thus the value of $\sum_{x \in T} d_H(x)$ is at most $2m$ if $H \subseteq E(G)$ contains m edges.

A key tool in the proofs of our main results is the following lemma which determines a necessary and sufficient condition for a graph to be a fractional (g, f, m)-deleted graph.

Lemma 3. (Gao and Wang [8]) *Let G be a non complete graph and g, f be two integer-valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for each $x \in V(G)$. Then G is a fractional (g, f, m)-deleted graph if and only if for any subset S of $V(G)$ and any subset $H \subseteq E(G)$ with m edges,*

$$f(S) + \sum_{x \in T} d_{G-S}(x) - g(T) \geq \sum_{x \in T} d_H(x) - e_H(T, S), \tag{1}$$

where $T = \{x \in V(G) \backslash S \mid d_{G-S}(x) - d_H(x) + e_H(x, S) \leq g(x)\}$.

Note that if we add some vertices from $V(G) \backslash S$ holding $d_{G-S}(x) - d_H(x) + e_H(x, S) > g(x)$ into the set T stated in Lemma 3, then Equation 1 is still held. It implies that Lemma 3 can be easily reformulated in the following equivalent form :

Lemma 4. *Let G be a non complete graph and g, f be two integer-valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for each $x \in V(G)$. Then G is a fractional (g, f, m)-deleted graph if and only if*

$$f(S) - g(T) + d_{G-S}(T) \geq \sum_{x \in T} d_H(x) - e_H(T, S) \tag{2}$$

for any disjoint subsets S, T of $V(G)$ and any $H \subseteq E(G)$ with m edges.

3. Proof of the main results

The aim of this section is to give the detailed proofs of our main results Theorem 1 and Theorem 2. Our tricks to prove these results are inspired by the ones stated in Zhou and Bian [13].

3.1. **Proof of Theorem 1.** Assume that G holds the hypotheses of Theorem 1, but it is not a fractional (g, f, m)-deleted graph. By Lemma 4 and the fact that $\sum_{x\in T} d_H(x) - e_H(T, S) \leq 2m$ for all S, T disjoint subsets of $V(G)$ and $H \subseteq E(G)$, there exist disjoint subsets S and T of $V(G)$ such that

$$f(S) + d_{G-S}(T) - g(T) \leq 2m - 1. \tag{3}$$

We select S and T such that $|T|$ is minimum. If $T = \emptyset$, then $\sum_{x\in T} d_H(x) - e_H(T, S) = 0$ by means of its definition. Using Lemma 4, (2) becomes $f(S) \geq 0$ and thus G is a fractional (g, f, m)-deleted graph, which is a contradiction. Hence, we have $T \neq \emptyset$.

If there exists $x \in T$ satisfying $d_{G-S}(x) \geq g(x)$, then the subsets S and $T \setminus \{x\}$ satisfy (3) as well. This contradicts the selection rule of S and T. It implies that $d_{G-S}(x) \leq g(x) - 1 \leq b - \Delta - 1$ for any $x \in T$.

Take $d_1 = \min\{d_{G-S}(x)|x \in T\}$ and choose $x_1 \in T$ such that $d_{G-S}(x_1) = d_1$. If $z \geq 2$ and $T \setminus (\cup_{j=1}^{z-1} N_T[x_j]) \neq \emptyset$, let

$$d_z = \min\{d_{G-S}(x)|x \in T \setminus (\cup_{j=1}^{z-1} N_T[x_j])\}$$

and choose $x_z \in T \setminus (\cup_{j=1}^{z-1} N_T[x_j])$ such that $d_{G-S}(x_z) = d_z$. So, we get a sequence such that

$$0 \leq d_1 \leq d_2 \leq \cdots \leq d_\pi \leq g(x) - 1 \leq b - \Delta - 1 \tag{4}$$

and an independent set

$$\{x_1, x_2, \ldots, x_\pi\} \subseteq T. \tag{5}$$

Lemma 5. *In the previous conditions, we have that*

$$|T| \geq \begin{cases} (i-1)(b+1), & \text{if } d_{G-S}(x) = 1 \text{ for any } x \in T, \\ (i-1)(b+1)+1, & \text{otherwise.} \end{cases}$$

Proof of Lemma 5. Since

$$|S| + d_1 = |S| + d_{G-S}(x_1) \geq d_G(x_1) \geq \delta(G) \geq \frac{(i-1)(b-\Delta)(b+1)}{a+\Delta} + m,$$

we infer

$$|S| \geq \frac{(i-1)(b-\Delta)(b+1)}{a+\Delta} + m - d_1. \tag{6}$$

We first verify that $|T| \geq (i-1)(b+1)$ if $d_{G-S}(x) = 1$ for any $x \in T$. Obviously, we deduce $d_1 = 1$ in this case.

Suppose that $|T| < (i-1)(b+1) - 1$. In view of (3), (6), $d_1 = 1$ and $i \geq 2$, we yield

$$\begin{aligned} 2m-1 &\geq f(S) + d_{G-S}(T) - g(T) \\ &\geq (a+\Delta)(|S|) + |T| - (b-\Delta)|T| = (a+\Delta)|S| - (b-\Delta-1)|T| \\ &\geq (a+\Delta)(\frac{(i-1)(b-\Delta)(b+1)}{a+\Delta} + m - 1) \\ &- (b-\Delta-1)((i-1)(b+1)-1) \\ &= (i-1)(b+1) - 1 - \Delta + (a+\Delta)m \geq b - \Delta + (a+\Delta)m \\ &\geq 2m+2, \end{aligned}$$

which is a contradiction. Therefore, $|T| \geq (i-1)(b+1)$ if $d_{G-S}(x) = 1$ for each $x \in T$.

Next, we shall show that $|T| \geq (i-1)(b+1)+1$ in other cases. Assume that $|T| < (i-1)(b+1)$. The following discussion shall be divided into three parts according to the value of d_1.

Case 1. $d_1 = 0$.

Since $d_{G-S}(T) \geq \sum_{x\in T} d_H(x) - e_H(T,S)$. Using Lemma 4 and (6), we infer

$$\begin{aligned} &\sum_{x\in T} d_H(x) - e_H(T,S) - 1 \geq f(S) + d_{G-S}(T) - g(T) \\ \geq\ & (a+\Delta)|S| + (\sum_{x\in T} d_H(x) - e_H(T,S)) - (b-\Delta)|T| \\ \geq\ & (i-1)(b-\Delta)(b+1) + (\sum_{x\in T} d_H(x) - e_H(T,S)) - (i-1)(b-\Delta)(b+1) \\ =\ & \sum_{x\in T} d_H(x) - e_H(T,S), \end{aligned}$$

which is a contradiction.

Case 2. $d_1 = 1$.

In this situation, there exists $t \in T$ holding $d_{G-S}(t) \geq 2$. By means of (3), (6) and $i \geq 2$, we get

$$\begin{aligned} 2m-1 &\geq f(S) + d_{G-S}(T) - g(T) \\ &\geq (a+\Delta)(|S|) + |T| + 1 - (b-\Delta)|T| = (a+\Delta)|S| - (b-\Delta-1)|T| + 1 \\ &\geq (a+\Delta)(\frac{(i-1)(b-\Delta)(b+1)}{a+\Delta} + m - 1) - (b-\Delta-1)(i-1)(b+1) + 1 \\ &\geq (i-1)(b+1) - (a+\Delta) + 1 + (a+\Delta)m \geq 2m+2, \end{aligned}$$

which leads to contradiction.

Case 3. $2 \leq d_1 \leq b - \Delta - 1$.

In terms of (3), (6) and $i \geq 2$, we derive

$$\begin{aligned}
2m-1 &\geq f(S)+d_{G-S}(T)-g(T) \\
&\geq (a+\Delta)(|S|)+d_1|T|-(b-\Delta)|T| \\
&= (a+\Delta)(|S|)-(b-\Delta-d_1)|T| \\
&\geq (a+\Delta)(\frac{(i-1)(b-\Delta)(b+1)}{a+\Delta}+m-d_1) \\
&- (b-\Delta-d_1)(i-1)(b+1) \\
&\geq d_1((i-1)(b+1)-(a+\Delta))+(a+\Delta)m \\
&\geq d_1((b+1)-(a+\Delta))+(a+\Delta)m \geq 2m+2,
\end{aligned}$$

which is a contradiction. Thus, combining the three cases above, we conclude that $|T| \geq (i-1)(b+1)+1$ ending the proof of Lemma 5. □

Lemma 6. *In the previous conditions, we have that there exists an independent subset $\{x_1, x_2, \cdots, x_i\} \subseteq T$.*

Proof of Lemma 6. If $d_{G-S}(x)=1$ for each $x \in T$, then by Lemma 5 we yield $|T| \geq (i-1)(b+1)$. By means of $b \geq 2$ and $d_{G-S}(x) \leq g(x)-1 \leq b-\Delta-1$ for any $x \in T$, we infer $d_{G-S}(x) \leq b-\Delta-1$ for any $x \in T$. Combining with $|T| \geq (i-1)(b+1) \geq (i-1)b+1$, it is ensured that there exists an independent subset $\{x_1, x_2, \cdots, x_i\} \subseteq T$ for $\pi = i$ in (4) and (5).

In other situation, it holds $|T| \geq (i-1)(b+1)+1$ according to Lemma 5. Combining with the fact $d_{G-S}(x) \leq g(x)-1 \leq b-\Delta-1$ for any $x \in T$, we can obviously take above independent subset $\{x_1, x_2, \cdots, x_i\} \subseteq T$ for $\pi = i$ in (4) and (5). Hence, the proof of Lemma 6 is done. □

In view of Lemma 6 and the hypothesis of Theorem 1, we deduce

$$\frac{(b-\Delta)n}{a+b} \leq \max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \leq |S|+d_i,$$

i.e.,

$$|S| \geq \frac{(b-\Delta)n}{a+b}-d_i. \tag{7}$$

Lemma 7. *In the previous conditions, we have that $|S| < \frac{(b-\Delta)n}{a+b}$.*

Proof of Lemma 7. By means of $d_{G-S}(T) \geq \sum_{x\in T} d_H(x) - e_H(T,S)$, Lemma 4 and $|S| + |T| \leq n$, we derive

$$\begin{aligned}
\sum_{x\in T} d_H(x) - e_H(T,S) - 1 &\geq f(S) + d_{G-S}(T) - g(T) \\
&\geq (a+\Delta)(|S|) + (\sum_{x\in T} d_H(x) - e_H(T,S)) \\
&- (b-\Delta)|T| \\
&\geq (a+\Delta)(|S|) + (\sum_{x\in T} d_H(x) - e_H(T,S)) \\
&- (b-\Delta)(n-|S|) \\
&= (a+b)|S| + (\sum_{x\in T} d_H(x) - e_H(T,S)) \\
&- (b-\Delta)n,
\end{aligned}$$

which implies $|S| < \frac{(b-\Delta)n}{a+b}$. □

Combining (7) and Lemma 7, we get $d_i > 0$. Moreover, since d_i is an integer, we obtain

$$d_i \geq 1. \tag{8}$$

Notice that

$$|N_T[x_j]| - |N_T[x_j] \cap (\cup_{z=1}^{j-1} N_T[x_z])| \geq 1 \tag{9}$$

for any $j = 2, 3, \ldots, i-1$, and

$$|\cup_{z=1}^{j} N_T[x_z]| \leq \sum_{z=1}^{j} |N_T[x_z]| \leq \sum_{z=1}^{j} (d_{G-S}(x_z) + 1) = \sum_{z=1}^{j} (d_z + 1) \tag{10}$$

for any $j = 1, 2, \ldots, i$.

Using (3), (9), (10), $|S|+|T| \le n$ and $d_i \le b-\Delta-1$, we get

$$
\begin{aligned}
2m-1 &\ge f(S)+d_{G-S}(T)-g(T)\\
&\ge (a+\Delta)(|S|)+d_{G-S}(T)-(b-\Delta)|T|\\
&\ge (a+\Delta)(|S|)+d_1|N_T[x_1]|+d_2(|N_T[x_2]|-|N_T[x_2]\cap N_T[x_1]|)\\
&+ \cdots+d_{i-1}(|N_T[x_{i-1}]|-|N_T[x_{i-1}]\cap \cup_{j=1}^{i-2}N_T[x_j]|)\\
&+ d_i(|T|-|\cup_{j=1}^{i-1}N_T[x_j]|)-(b-\Delta)|T|\\
&\ge (a+\Delta)|S|+(d_1-d_i)|N_T[x_1]|+\sum_{j=2}^{i-1}d_j\\
&+ (d_i-(b-\Delta))|T|-d_i\sum_{j=2}^{i-1}|N_T[x_j]|\\
&\ge (a+\Delta)|S|+(d_1-d_i)(d_1+1)+\sum_{j=2}^{i-1}d_j\\
&+ (d_i-(b-\Delta))|T|-d_i\sum_{j=2}^{i-1}(d_j+1)\\
&= (a+\Delta)|S|+d_1^2+\sum_{j=1}^{i-1}d_j-d_i\sum_{j=1}^{i-1}(d_j+1)+(d_i-(b-\Delta))|T|\\
&\ge (a+\Delta)|S|+d_1^2+\sum_{j=1}^{i-1}d_j-d_i\sum_{j=1}^{i-1}(d_j+1)+(d_i-(b-\Delta))(n-|S|)\\
&= (a+b-d_i)|S|+d_1^2+\sum_{j=1}^{i-1}d_j-d_i\sum_{j=1}^{i-1}(d_j+1)+(d_i-(b-\Delta))n,
\end{aligned}
$$

which implies

$$
2m-1 \ge (a+b-d_i)|S|+d_1^2+\sum_{j=1}^{i-1}d_j-d_i\sum_{j=1}^{i-1}(d_j+1)+(d_i-(b-\Delta))n. \tag{11}
$$

It follows from (7), (8), (11), $0 \le d_1 \le d_2 \le \cdots \le d_i \le b - \Delta - 1$ and $n \ge \frac{(a+b)(a+b+2m-1+(i-2)(b-\Delta))}{a+\Delta}$ that

$$\begin{aligned}
2m-1 &\ge (a+b-d_i)|S| + d_1^2 + \sum_{j=1}^{i-1} d_j - d_i \sum_{j=1}^{i-1}(d_j+1) + (d_i - (b-\Delta))n \\
&\ge (a+b-d_i)(\frac{(b-\Delta)n}{a+b} - d_i) + d_1^2 - (d_i - 1)\sum_{j=1}^{i-1} d_j - d_i(i-1) \\
&+ (d_i - (b-\Delta))n \\
&\ge (a+b-d_i)(\frac{(b-\Delta)n}{a+b} - d_i) + d_1^2 - (d_i-1)(i-1)d_i - d_i(i-1) \\
&+ (d_i - (b-\Delta))n \\
&\ge (a+b-d_i)(\frac{(b-\Delta)n}{a+b} - d_i) - (d_i-1)(i-1)d_i - d_i(i-1) \\
&+ (d_i - (b-\Delta))n \\
&= d_i(\frac{(a+\Delta)n}{a+b} - (a+b) - (i-2)d_i) \\
&\ge d_i(a+b+(i-2)(b-\Delta) - (a+b) - (i-2)(b-\Delta) + 2m - 1) \\
&\ge d_i(2m-1) \ge 2m-1.
\end{aligned}$$

Thus equality holds throughout, which reveals that $d_1 = d_2 = \cdots = d_i$ and $d_1 = 0$. Therefore, we have $d_i = 0$. But it contradicts (8). Thus, we complete the proof of Theorem 1. □

3.2. **Proof of Theorem 2.** We are going to use similar techniques to the ones stated in the proof of Theorem 1.

Assume that G meets the hypothesis of Theorem 2, but it is not a fractional (g, f, m)-deleted graph. Obviously, $T \neq \emptyset$. It is not hard to verify that (3),(6), Lemma 5, Lemma 6 and Lemma 7 are hold as well. Hence, there exists an independent subset $\{x_1, x_2, \cdots, x_i\} \subseteq T$ and a sequence such that $0 \le d_1 \le d_2 \le \cdots \le d_i \le g(x) - 1 \le b - \Delta - 1$, where $d_G(x_j) = d_j$ for $j \in \{1, \cdots, i\}$.

In light of the condition of the Theorem 2, we get

$$\frac{(b-\Delta)n}{a+b} \le |N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \le |S| + \sum_{j=1}^{i} d_j,$$

i.e.,

$$|S| \ge \frac{(b-\Delta)n}{a+b} - \sum_{j=1}^{i} d_j. \tag{12}$$

From these facts, we obtain

$$\begin{array}{rcl} 2m-1 & \geq & f(S)+d_{G-S}(T)-g(T) \\ & \geq & (a+\Delta)|S|-(b-\Delta)|T|+d_1|N_T[x_1]|+d_2(|N_T[x_2]| \\ & - & |N_T[x_2]\cap N_T[x_1]|)+\cdots+d_{i-1}(|N_T[x_{i-1}]| \\ & - & |N_T[x_{i-1}]\cap(\cup_{j=1}^{i-2}N_T[x_j])|)+d_i(|T|-|\cup_{j=1}^{i-1}N_T[x_j])||) \\ & \geq & (a+\Delta)|S|+(d_1-d_i)|N_T[x_1]|+\sum_{j=2}^{i-1}d_j \\ & + & (d_i-b+\Delta)|T|-d_i\sum_{j=2}^{i-1}|N_T[x_j]| \\ & = & (a+\Delta)|S|+(d_1-d_i)(d_1+1)+\sum_{j=2}^{i-1}d_j \\ & + & (d_i-b+\Delta)|T|-d_i\sum_{j=2}^{i-1}(d_j+1) \\ & = & (a+\Delta)|S|+d_1^2+\sum_{j=1}^{i-1}d_j+(d_i-b+\Delta)|T|-d_i\sum_{j=1}^{i-1}(d_j+1) \\ & \geq & (a+\Delta)|S|+d_1^2+\sum_{j=1}^{i-1}d_j+(d_i-b+\Delta)(n-|S|)-d_i\sum_{j=1}^{i-1}(d_j+1) \\ & = & (a+b-d_i)|S|+d_1^2+\sum_{j=1}^{i-1}d_j-d_i\sum_{j=1}^{i-1}(d_j+1)+(d_i-b+\Delta)n, \end{array}$$

which implies

$$2m-1\geq(a+b-d_i)|S|+d_1^2+\sum_{j=1}^{i-1}d_j-d_i\sum_{j=1}^{i-1}(d_j+1)+(d_i-b+\Delta)n. \tag{13}$$

By (12), (13), $d_1 \leq d_2 \leq \cdots \leq d_i \leq b-\Delta-1$ and $n \geq \frac{(a+b)(i(a+b)+2m-2)}{a+\Delta}$, we have the following:

$$\begin{aligned}
2m-1 &\geq (a+b-d_i)|S| + d_1^2 + \sum_{j=1}^{i-1} d_j - d_i \sum_{j=1}^{i-1}(d_j+1) + (d_i-b+\Delta)n \\
&\geq (a+b-d_i)(\frac{(b-\Delta)n}{a+b} - \sum_{j=1}^{i} d_j) + d_1^2 + \sum_{j=1}^{i-1} d_j - d_i \sum_{j=1}^{i-1}(d_j+1) \\
&+ (d_i-b+\Delta)n \\
&\geq (a+b-d_i)(\frac{(b-\Delta)n}{a+b} - id_i) + d_1^2 + (i-1)d_i - d_i(i-1)(d_i+1) \\
&+ (d_i-b+\Delta)n \\
&\geq (a+b-d_i)(\frac{(b-\Delta)n}{a+b} - id_i) - d_i^2(i-1) + (d_i-b+\Delta)n \\
&= d_i(\frac{(a+\Delta)n}{a+b} - i(a+b) + d_i) \\
&\geq d_i(i(a+b) + 2m - 2 - i(a+b) + d_i) \\
&\geq 2m-1.
\end{aligned}$$

Similarly, equality holds throughout which implies $d_1 = d_2 = \cdots = d_i$ and $d_1 = 0$. A contradiction. Therefore, the proof of Theorem 2 is completed. □

4. Sharpness

In this section, we shall show that the independent set degree condition in Theorem 1 and the independent set neighborhood union condition in Theorem 2 are both tight.

The lower bound on the independent set degree condition

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{(b-\Delta)n}{a+b}$$

in Theorem 1 is the best possible in the sense that we can not replace $\frac{(b-\Delta)n}{a+b}$ by $\frac{(b-\Delta)n}{a+b} - 1$. Also, the lower bound on the independent neighborhood union condition $|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \geq \frac{(b-\Delta)n}{a+b}$ in Theorem 2 is the best possible in the sense that we can not replace $\frac{(b-\Delta)n}{a+b}$ by $\frac{(b-\Delta)n}{a+b} - 1$. We construct a graph to state the sharpness.

Let K_{iat} and K_1 be the complete graphs of order iat and 1, respectively. Indeed, let $G = K_{iat} \vee (ibt+1)K_1$ with $i \geq 2$, $\Delta \geq 0$ (here the operation $\vee$ means adding edges between all the vertices in K_{iat} and all the vertices in $(ibt+1)K_1$, and $(ibt+1)K_1$ denotes the subgraph contain $ibt+1$ isolated vertices and no edge), and $2 \leq a = b - \Delta$ be integers and $t \in \mathbb{N}$ is a large number. Let $g(x)$ and $f(x)$ be integer-valued functions defined on $V(G)$ holding the condition that $g(x) = a$ and $f(x) = b = a + \Delta$ for any $x \in V(G)$. We infer that $\delta(G) = iat \geq \frac{(i-1)(b-\Delta)(b+1)}{a+\Delta}$ and

$$\frac{(b-\Delta)n}{a+b} > \max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} = iat > \frac{(b-\Delta)n}{a+b} - 1,$$

$$\frac{(b-\Delta)n}{a+b} > |N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| = iat > \frac{(b-\Delta)n}{a+b} - 1$$

for any independent subset $\{x_1, x_2, \cdots, x_i\} \subseteq V(G)$. Set $S = V(K_{iat})$ and $T = V((ibt+1)K_1)$. Then $|S| = iat$, $|T| = ibt + 1$, $d_{G-S}(T) = 0$ and $\sum_{x\in T} d_H(x) - e_H(T,S) = 0$ for any $H \subset E(G)$ with $|H| = m$. Thus, we obtain

$$\begin{aligned} f(S) + d_{G-S}(T) - g(T) &= b|S| + d_{G-S}(T) - a|T| \\ &= biat - a(ibt+1) = -a < 0 \\ &= \sum_{x\in T} d_H(x) - e_H(T,S). \end{aligned}$$

According to Lemma 4, G is not a fractional (g, f, m)-deleted graph. From this point of view, the lower bounds on the condition $\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{(b-\Delta)n}{a+b}$ in Theorem 1 and $|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_i)| \geq \frac{(b-\Delta)n}{a+b}$ in Theorem 2 are optimal.

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