

# PERIODIC ORBITS FOR THE PERTURBED PLANAR CIRCULAR RESTRICTED 3–BODY PROBLEM

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ABSTRACT. We characterize when the classical first and second kind of periodic orbits of the planar circular restricted 3–body problem obtained by Poincaré, can be extended to perturbed planar circular restricted 3–body problems. We put special emphasis when the perturbed forces are due to zonal harmonic or to a solar sail.

## 1. INTRODUCTION

The circular restricted 3–body problem is considered the simplest non–integrable dynamical system in Celestial Mechanics. However it has many applications not only in the solar system but also in the studies of the stellar dynamics. Therefore many different dynamical astronomical systems can be studied within its frame. Thus, for instance this model is used in some space missions motion of spacecrafts in the Earth–Moon system, or between two planets. Also there are specific applications of the circular restricted 3–body problem in stellar systems for studying exoplanets around either one or both components of a binary star system, (see for instance [13]), and in particular for analyzing the existence of habitable zones in these exoplanets, see [12, 15, 17, 18, 21].

Many researchers pay their attention to present analytical studies on the circular restricted 3–body problem within the frame of some perturbed forces, due to oblateness, zonal harmonic coefficients, triaxial and radiation pressure effects. In this context the existence and linear stability analysis of libration points as well as the periodic orbits around these points are studied by [1, 2, 3, 16, 17, 18, 30]. Furthermore some researchers devoted their work to construct the possible solutions of the same problem with some numerical techniques, if it either includes perturbed forces or not. Some of these techniques are Bulirsch–Stoer method, Runge–Kutta algorithm, Lie series approach, symplectic integrators, hybrid integrators, etc. For more details on these methods see for instance [4, 5, 8, 9, 10, 14, 20].

But within frame of continuation solutions [19] establishes that a symmetric periodic orbit of the circular restricted 3–body problem, when the mass of the third body is small, can be analytically extended into a periodic solution of the full 3–body problem in a synodic reference frame. It also proves that a family of symmetric periodic orbits of this problem can be analytically extended into a family of periodic motion of the full problem, for a fixed value of the small mass of the third body. In [22] the authors develop a new relationship to find a connection between the periodic orbits of the planar 3–body problem and the planar circular restricted 3–body problem. They also state new conditions to prove that the symmetric

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periodic orbits of the elliptic restricted system can be extended into the full planar 3–body problem.

Our objective in the present work is to analyze when the first and second kind of periodic orbits found by Poincaré [26, 27] in the planar circular restricted 3–body problem can be or not extended to periodic orbits of the perturbed planar circular restricted 3–body problem. The orbits of first (respectively second) kind in the planar circular restricted 3–body problem are the ones which come from the circular (respectively elliptic) orbits of the two–body problem or Kepler problem in synodical (or rotating) coordinates.

We recall that in the perturbed circular circular restricted 3–body problem the primaries are rotating in circular orbits around their common center of masses. In this setting the infinitesimal third body is affected by the gravitational forces of the primaries as well as small perturbed force. We shall work in synodical coordinates where the primaries are fixed on the  $x$ –axis. Thus the model of the circular restricted 3–body problem is perturbed by an extra small force, obtaining the so called *the perturbed circular restricted 3–body problem*.

There are two standard proofs for showing that the circular periodic orbits of the rotating Kepler problem can be extended to the first kind of periodic orbits of the circular restricted 3–body problem. The first proof uses the Poincaré variables, see for instance [29] and the proof of Theorem 2 of this paper, and the second proof uses the multipliers associated to the circular periodic orbits, see for instance [24] or the proof of Theorem 1 of this paper. Inspired in both proofs we shall show that, if the circular restricted 3–body problem with a small perturbation has a first integral, then the periodic solutions of first kind can be extended from the circular restricted 3–body problem to the perturbed circular restricted 3–body problem. But we shall see that if such first integral does not exist then both proofs do not work, and it remains as open problem: “to know when the periodic solutions of first kind can be extended or not when the perturbed problem has not a first integral”.

Finally we study in Theorem 3 the continuation of the second kind of periodic solutions from the planar circular restricted 3–body problem to a perturbed planar circular restricted 3–body problem.

## 2. EQUATIONS OF MOTION

We assume, without loss of generality, that the separation distance between the primaries, the sum of their masses and the universal constant of gravitation are equal to one. Then the equations of motion can be written in dimensionless synodic coordinate system  $xyz$  (see [25]) as

$$(1) \quad \ddot{\mathbf{r}} + 2\mathbf{n} \wedge \dot{\mathbf{r}} = \nabla V$$

where

$$\begin{aligned} \mathbf{r} &= [x \ y \ z]^T, \quad \mathbf{n} = [0 \ 0 \ 1]^T, \\ V &= \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}|\mathbf{n} \wedge \mathbf{r}|^2, \\ \mathbf{r}_1 &= [(x-\mu) \ y \ z]^T, \quad \mathbf{r}_2 = [(x-\mu+1) \ y \ z]^T, \end{aligned}$$

Now we assume that the infinitesimal body will receive an additional force, say perturbed force, with the acceleration in synodic frame  $\mathbf{a}$ . Using equation (1) the perturbed restricted

3-body problem can be written as

$$(2) \quad \ddot{\mathbf{r}} + 2\mathbf{n} \wedge \dot{\mathbf{r}} = \nabla V + \mathbf{a},$$

where  $\mathbf{a} = \mathbf{a}(\mathbf{r}, \dot{\mathbf{r}}, t)$  being

$$\mathbf{a} = [a_x(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \ a_y(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \ a_z(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)]^T,$$

and  $\mathbf{n} = [0 \ 0 \ n]^T$  the mean motion vector.

In cartesian coordinates equations (2) have the form

$$(3) \quad \begin{aligned} \ddot{x} - 2n\dot{y} &= n^2x - (x - \mu)f(x, y, z) - (x - \mu + 1)g(x, y, z) + a_x, \\ \ddot{y} + 2n\dot{x} &= n^2y - y[f(x, y, z) + g(x, y, z)] + a_y, \\ \ddot{z} &= -z[f(x, y, z) + g(x, y, z)] + a_z, \end{aligned}$$

where

$$f(x, y, z) = \frac{1 - \mu}{r_1^3} \quad \text{and} \quad g(x, y, z) = \frac{\mu}{r_2^3}.$$

We assume that the parameter  $\mu$  and the acceleration  $\mathbf{a}$  are small. Within this frame we can write  $\mu = \varepsilon\mu_1$  and  $\mathbf{a} = \varepsilon\mathbf{a}_1$ , where the parameter  $\varepsilon$  is very small (i.e.  $0 < \varepsilon \ll 1$ ). In what follows for simplicity the subindex 1 in  $\mu_1$  and  $\mathbf{a}_1$  will be dropped. Then equations (3) can be written as

$$(4) \quad \begin{aligned} \ddot{x} &= 2n\dot{y} + n^2x - \frac{(1 - \mu\varepsilon)(x - \mu\varepsilon)}{[(x - \mu\varepsilon)^2 + y^2 + z^2]^{3/2}} - \frac{\mu\varepsilon(x - \mu\varepsilon + 1)}{[(x - \mu\varepsilon + 1)^2 + y^2 + z^2]^{3/2}} + \varepsilon a_x, \\ \ddot{y} &= -2n\dot{x} + n^2y - y \left[ \frac{\mu\varepsilon}{[(x - \mu\varepsilon + 1)^2 + y^2 + z^2]^{3/2}} + \frac{1 - \mu\varepsilon}{[(x - \mu\varepsilon)^2 + y^2 + z^2]^{3/2}} \right] + \varepsilon a_y, \\ \ddot{z} &= -z \left[ \frac{\mu\varepsilon}{[(x - \mu\varepsilon + 1)^2 + y^2 + z^2]^{3/2}} + \frac{1 - \mu\varepsilon}{[(x - \mu\varepsilon)^2 + y^2 + z^2]^{3/2}} \right] + \varepsilon a_z. \end{aligned}$$

After expanding (4) in powers of  $\varepsilon$  this system can be written into the form

$$(5) \quad \begin{aligned} \ddot{x} &= 2n\dot{y} + n^2x - \frac{x}{(x^2 + y^2 + z^2)^{3/2}} + O(\varepsilon), \\ \ddot{y} &= -2n\dot{x} + n^2y - \frac{y}{(x^2 + y^2 + z^2)^{3/2}} + O(\varepsilon), \\ \ddot{z} &= -\frac{z}{(x^2 + y^2 + z^2)^{3/2}} + O(\varepsilon). \end{aligned}$$

If  $z = 0$ , then  $\dot{z} = 0$ . So the plane  $(x, y)$  is invariant, in the sense that if we choose a point in that plane then the solution which passes through this point with  $\dot{z} = 0$  remains always in that plane. We shall study if the periodic orbits of first and second kind persist in the perturbed planar circular restricted 3-body problem (5) with  $z = 0$ , i.e. in the differential

system

$$(6) \quad \begin{aligned} \ddot{x} &= 2n\dot{y} + n^2x - \frac{x}{(x^2 + y^2)^{3/2}} + O(\varepsilon), \\ \ddot{y} &= -2n\dot{x} + n^2y - \frac{y}{(x^2 + y^2)^{3/2}} + O(\varepsilon). \end{aligned}$$

### 3. PERIODIC ORBITS OF FIRST KIND

For  $\varepsilon = 0$  system (6) is the Kepler problem in dimensionless synodic coordinates. We do the usual change to polar coordinates with the corresponding momenta as follows  $(x, y, \dot{x}, \dot{y}) \rightarrow (r, \theta, Q, P)$  where

$$x = r \cos \theta, \quad y = r \sin \theta, \quad \dot{x} = ny + Q \cos \theta - \frac{1}{r}P \sin \theta, \quad \dot{y} = -nx + Q \sin \theta + \frac{1}{r}P \cos \theta.$$

Then the differential system (6) writes

$$(7) \quad \begin{aligned} \dot{r} &= f_1(r, \theta, Q, P) = Q, \\ \dot{\theta} &= f_2(r, \theta, Q, P) = \frac{P}{r^2} - n, \\ \dot{Q} &= f_3(r, \theta, Q, P) = \frac{P^2}{r^3} - \frac{1}{r^2} + O(\varepsilon), \\ \dot{P} &= f_4(r, \theta, Q, P) = O(\varepsilon). \end{aligned}$$

Let  $(r(t), \theta(t), Q(t), P(t))$  be a  $T$ -periodic solution of system (7) for  $\varepsilon = 0$ . A continuation of this periodic solution is a pair of smooth functions,  $(r(t, \varepsilon), \theta(t, \varepsilon), Q(t, \varepsilon), P(t, \varepsilon))$  and  $T(\varepsilon)$ , defined for  $\varepsilon > 0$  near 0 such that  $(r(t, 0), \theta(t, 0), Q(t, 0), P(t, 0)) = (r(t), \theta(t), Q(t), P(t))$ ,  $T(0) = T$ , and  $(r(t, \varepsilon), \theta(t, \varepsilon), Q(t, \varepsilon), P(t, \varepsilon))$  is a  $T(\varepsilon)$ -periodic solution of system (7).

Note that the *variational equation* associated to the  $T$ -periodic solution  $\phi(t, \varepsilon) = (r(t, \varepsilon), \theta(t, \varepsilon), Q(t, \varepsilon), P(t, \varepsilon))$  is

$$(8) \quad \dot{M} = \left( \frac{\partial(f_1, f_2, f_3, f_4)}{\partial(r, \theta, Q, P)} \Big|_{(r, \theta, Q, P) = \phi(t, \varepsilon)} \right) M,$$

where  $M$  is a  $4 \times 4$  matrix. Of course  $\partial(f_1, f_2, f_3, f_4)/\partial(r, \theta, Q, P)$  denotes the Jacobian matrix of  $(f_1, f_2, f_3, f_4)$  with respect to the variables  $(r, \theta, Q, P)$ . The *monodromy matrix* associated to the  $T$ -periodic solution  $\phi(t, \varepsilon)$  is the solution  $M(T, \varepsilon)$  of (8) satisfying that  $M(0, \varepsilon)$  is the identity matrix. The eigenvalues of the monodromy matrix  $M(T, 0)$  associated to the periodic solution  $\phi(t, \varepsilon)$  are called the *multipliers* of the periodic orbit.

A periodic solution  $\phi(t, \varepsilon)$  always has  $+1$  as an eigenvalue of its associated monodromy matrix  $M(T, \varepsilon)$ , and  $+1$  with multiplicity two for a system with a non-degenerate first integral. Drop one  $+1$  multiplier for a general autonomous differential system, and drop two  $+1$  multipliers from the list of multipliers for an autonomous differential system with a non-degenerate first integral to get the *nontrivial multipliers*. If the nontrivial multipliers are not equal to one then the periodic solution is called *elementary*. It is known that an elementary

periodic solution for  $\varepsilon = 0$  can be extended to  $\varepsilon > 0$  sufficiently small, see Proposition 9.1.1 of [24].

System (7) for  $\varepsilon = 0$  becomes

$$\dot{r} = Q, \quad \dot{\theta} = \frac{P}{r^2} - n, \quad \dot{Q} = \frac{P^2}{r^3} - \frac{1}{r^2}, \quad \dot{P} = 0.$$

Clearly the angular momentum  $P$  is a first integral. Taking  $P = c$  for a fixed constant  $c^3 \neq 1/n$ , we get the circular periodic solution  $r = c^2$ ,  $Q = 0$  of period  $|2\pi c^3/(1 - nc^3)|$ . Computing the multipliers of this circular periodic solution we obtain  $+1$  with multiplicity two and  $\exp(\pm i2\pi/(1 - nc^3)) \neq +1$ . If the perturbed problem has a first integral, by Proposition 9.1.1 of [24], this periodic orbits can be extended, and take the following result:

**Theorem 1.** *Consider a circular restricted 3-body problem with a small perturbation having a first integral. If  $c^3 \neq 1/n$  and  $1/(1 - nc^3)$  is an integer, then the perturbed circular orbits of the rotating Kepler problem with angular momentum  $c$  can be extended to this problem.*

For instance this theorem can be applied to the models of the perturbed circular restricted 3-body problem by zonal harmonic coefficients, oblateness, triaxial, and radiation pressure effects because these models have a first integral. If the circular restricted 3-body problem with a small perturbation has not a first integral the multipliers of the circular periodic orbits cannot be used for knowing if such orbits can be extended, this is the case of planar circular restricted 3-body problem with a solar sail [28].

We introduce the *Delaunay variables*  $(\ell, g, L, G)$  instead of the variables  $(r, \theta, Q, P)$  through

$$Q = \sqrt{-\frac{G^2}{r^2} + \frac{2}{r} - \frac{1}{L^2}}, \quad P = G, \quad \ell = \frac{t}{L^3}, \quad g = \theta - f,$$

where  $t$  is the time and  $f$  the true anomaly. For  $\varepsilon = 0$  we have the Kepler problem in rotating coordinates and the Delaunay variables have the following meaning:  $L = \sqrt{a}$  where  $a$  is the semi-major axis of the elliptic orbits,  $G = L\sqrt{1 - e^2}$  where  $e$  is the eccentricity of the elliptic orbits,  $g$  is the angle of the pericenter of the elliptic orbits, and  $\ell$  is the mean anomaly (which is the angular variable measured from the pericenter). For more details on the Delaunay variables see page 164 of [24] and its neighborhood.

In terms of the Delaunay variables the differential system (7) writes

$$(9) \quad \dot{\ell} = \frac{1}{L^3} + O(\varepsilon), \quad \dot{L} = O(\varepsilon), \quad \dot{g} = -n + O(\varepsilon), \quad \dot{G} = O(\varepsilon).$$

We note that the Delaunay variables are valid only in a neighborhood of the phase space where we have the elliptic orbits of the Kepler problem in rotating coordinates, and now we introduce the *Poincaré variables*  $(\lambda, \Lambda, \xi, \eta)$  given by

$$\lambda = \ell + g, \quad \Lambda = L, \quad \xi = \sqrt{2(L - G)} \cos \ell, \quad \eta = \sqrt{2(L - G)} \sin \ell,$$

for studying the orbits in a neighborhood of circular periodic orbits of the rotating Kepler problem. In the Poincaré variables the differential system (9) becomes

$$(10) \quad \dot{\lambda} = \Gamma - n + O(\varepsilon), \quad \dot{\Lambda} = O(\varepsilon), \quad \dot{\xi} = -\Gamma\eta + O(\varepsilon), \quad \dot{\eta} = \Gamma\xi + O(\varepsilon),$$

where  $\Gamma = 1/\Lambda^3$ .

The periodic solutions of system (10) for  $\varepsilon = 0$  are

$$(11) \quad \lambda = (\Gamma_0 - n)t + \lambda_0, \quad \Lambda = \Lambda_0, \quad \xi = \xi_0 \cos(\Gamma_0 t) - \eta_0 \sin(\Gamma_0 t), \quad \eta = \xi_0 \sin(\Gamma_0 t) + \eta_0 \cos(\Gamma_0 t),$$

where  $\Lambda_0$  is a non-zero constant and  $\Gamma_0 = 1/\Lambda_0^3$ .

The orbit (11) is periodic if and only if  $2\pi l/(\Gamma_0 - n) = 2\pi l/\Gamma_0$ , i.e.  $\Gamma_0$  must be  $k/(k-l)$  with  $l < k$  positive integers. Note that  $k/(k-l)$  can be any positive rational number. The circular periodic solutions of system (10) for  $\varepsilon = 0$  are

$$\lambda = (\Gamma_0 - n)t + \lambda_0, \quad \Lambda = \Lambda_0, \quad \xi = 0, \quad \eta = 0.$$

Note that  $\xi = \eta = 0$  if and only if  $L = G$ , and since  $G = L\sqrt{1-e^2}$ , we have that  $\xi = \eta = 0$  if and only if the eccentricity  $e = 0$ . Now applying the Poincaré continuation method (see for more details on this method [11]), we shall study which of these circular periodic solutions can be extended to periodic solutions of system (10) with  $\varepsilon > 0$  sufficiently small.

We denote by

$$(12) \quad (\xi(t; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0), \eta(t; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0), \lambda(t; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0), \Lambda(t; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0))$$

the solution of system (10) such that

$$\begin{aligned} \xi(0; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0) &= \xi_0, \\ \eta(0; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0) &= \eta_0, \\ \lambda(0; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0) &= \lambda_0, \\ \Lambda(0; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0) &= \Lambda_0. \end{aligned}$$

We want to study the periodic solutions of period  $T = T_0 + \tau$  of system (10) which are continuation of the circular periodic solutions of (10) with  $\varepsilon = 0$ , i.e. the solutions (12) satisfying

$$(13) \quad \begin{aligned} \xi(T; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0) &= \xi(0; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0), \\ \eta(T; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0) &= \eta(0; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0), \\ \lambda(T; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0) &= \lambda(0; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0) + 2\pi l, \\ \Lambda(T; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0) &= \Lambda(0; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0), \end{aligned}$$

and

$$\begin{aligned} \xi(t; 0, \xi_0, \eta_0, \lambda_0, \Lambda_0) &= 0, \\ \eta(t; 0, \xi_0, \eta_0, \lambda_0, \Lambda_0) &= 0, \\ \lambda(t; 0, \xi_0, \eta_0, \lambda_0, \Lambda_0) &= (\Gamma_0 - n)t + \lambda_0, \\ \Lambda(t; 0, \xi_0, \eta_0, \lambda_0, \Lambda_0) &= \Lambda_0. \end{aligned}$$

Thus we expect to find initial conditions  $(\xi_0(\varepsilon), \eta_0(\varepsilon), \lambda_0(\varepsilon), \Lambda_0(\varepsilon))$  for a periodic solution of period  $T_0$  of system (10), such that for  $\varepsilon = 0$  we have

$$\xi_0(0) = \eta_0(0) = 0, \quad \Lambda_0(0) = \Lambda_0 \quad \text{and} \quad (\Gamma_0(0) - n)T_0 = 2\pi l.$$

where  $\Gamma_0(0) = 1/\Lambda_0(0)$ , therefore

$$\Gamma_0(0) = \frac{2\pi l + nT_0}{T_0} \quad \text{and} \quad \Lambda_0(0) = \left( \frac{T_0}{2\pi l + nT_0} \right)^{1/3},$$

with  $\lambda_0(0)$  arbitrary.

We want to solve system (13) with  $T = T_0 + \tau$ , more precisely to find the zeros  $(\xi_0, \eta_0, \Lambda_0, \tau) = (\xi_0(\varepsilon), \eta_0(\varepsilon), \Lambda_0(\varepsilon), \tau(\varepsilon))$  of the map  $(\psi_1, \psi_2, \psi_3, \psi_4)$  given by

$$(14) \quad \begin{aligned} \psi_1(\varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0, \tau) &= \xi(T_0 + \tau; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0) - \xi(0; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0), \\ \psi_2(\varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0, \tau) &= \eta(T_0 + \tau; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0) - \eta(0; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0), \\ \psi_3(\varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0, \tau) &= \lambda(T_0 + \tau; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0) - \lambda(0; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0) - 2\pi l, \\ \psi_4(\varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0, \tau) &= \Lambda(T_0 + \tau; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0) - \Lambda(0; \varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0). \end{aligned}$$

The Jacobian matrix of  $(\psi_1, \psi_2, \psi_3, \psi_4)$  with respect to  $\xi_0, \eta_0, \Lambda_0$  and  $\tau_0$  evaluated at  $(\xi_0, \eta_0, \Lambda_0, \tau) = (0, 0, \bar{\Lambda}_0, 0)$  and  $\varepsilon = 0$  is

$$D = \begin{pmatrix} \frac{\partial \psi_1}{\partial \xi_0} & \frac{\partial \psi_1}{\partial \eta_0} & \frac{\partial \psi_1}{\partial \Lambda_0} & \frac{\partial \psi_1}{\partial \tau} \\ \frac{\partial \psi_2}{\partial \xi_0} & \frac{\partial \psi_2}{\partial \eta_0} & \frac{\partial \psi_2}{\partial \Lambda_0} & \frac{\partial \psi_2}{\partial \tau} \\ \frac{\partial \psi_3}{\partial \xi_0} & \frac{\partial \psi_3}{\partial \eta_0} & \frac{\partial \psi_3}{\partial \Lambda_0} & \frac{\partial \psi_3}{\partial \tau} \\ \frac{\partial \psi_4}{\partial \xi_0} & \frac{\partial \psi_4}{\partial \eta_0} & \frac{\partial \psi_4}{\partial \Lambda_0} & \frac{\partial \psi_4}{\partial \tau} \end{pmatrix} \Bigg|_{(\xi_0, \eta_0, \Lambda_0, \tau, \varepsilon) = (0, 0, \bar{\Lambda}_0, 0, 0)}$$

Since

$$\begin{aligned} \psi_1(\varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0, \tau) &= \xi_0 \cos[\Gamma_0(T_0 + \tau)] - \eta_0 \sin[\Gamma_0(T_0 + \tau)] - \xi_0 + O(\varepsilon), \\ \psi_2(\varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0, \tau) &= \xi_0 \sin[\Gamma_0(T_0 + \tau)] + \eta_0 \cos[\Gamma_0(T_0 + \tau)] - \eta_0 + O(\varepsilon), \\ \psi_3(\varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0, \tau) &= \left(\frac{1}{\Lambda_0^3} - n\right)\tau + O(\varepsilon), \\ \psi_4(\varepsilon, \xi_0, \eta_0, \lambda_0, \Lambda_0, \tau) &= O(\varepsilon). \end{aligned}$$

then

$$D = \begin{pmatrix} \cos(\Gamma_0 T_0) - 1 & -\sin(\Gamma_0 T_0) & * & * \\ \sin(\Gamma_0 T_0) & \cos(\Gamma_0 T_0) - 1 & * & * \\ 0 & 0 & 0 & \frac{2\pi l}{T_0} \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0.$$

Since the determinant  $D = 0$ , the Poincaré variables which allow to continue the circular periodic orbits of the rotating Kepler problem to the circular restricted 3-body problem, do not work for continuing periodic orbits to the circular restricted 3-body problem with some small perturbation without first integrals. But if this perturbed problem has a first integral, using this first integral we can compute in each level of the fixed first integral the variable  $\Lambda$

as a function of the variables  $(\xi, \eta, \lambda)$ , and we can omit in system (14) the equation  $\psi_4 = 0$ . Thus the determinant becomes

$$D = \begin{pmatrix} \cos(\Gamma_0 T_0) - 1 & -\sin(\Gamma_0 T_0) & * \\ \sin(\Gamma_0 T_0) & \cos(\Gamma_0 T_0) - 1 & * \\ 0 & 0 & \frac{2\pi l}{T_0} \end{pmatrix} = -\frac{8\pi l}{T_0} \sin^2 \frac{1}{2} \Gamma_0 T_0.$$

and  $D \neq 0$  if  $\Gamma_0 T_0 \neq 2\pi k$  with  $k$  a positive integer. Consequently we have the following result:

**Theorem 2.** *Let  $T_0 = 2\pi r$  with  $r$  an arbitrary positive rational number. Consider a planar circular restricted 3–body problem with a small perturbation having a first integral. If  $T_0 \neq 2\pi k$  with  $k$  being a positive integer and  $\varepsilon > 0$  sufficiently small, then there exists a periodic orbit for this perturbed problem which tends to a circular periodic orbit of the rotating Kepler problem with period  $T_0$  when  $\varepsilon \rightarrow 0$ .*

Clearly Theorem 1 and 2 are related because both provide sufficient conditions in order that the circular periodic orbits of the rotating Kepler problem can be extended to the perturbed planar circular restricted 3–body problem.

#### 4. APPLICATIONS OF THE PERIODIC ORBITS OF FIRST KIND

In this section the previous results of Theorems 1 and 2 will be applied to two perturbed planar circular restricted 3–body problems. In the first application the perturbed problem has a first integral [2], and the second one does not have a first integral [28].

**4.1. Planar circular restricted 3–body problem with a zonal harmonic effect.** The equations of motion of the planar circular restricted 3–body problem under the effect of zonal harmonic coefficients up to  $J_2$ , for the more massive primary, within dimensionless synodic coordinate system  $xyz$  can be written as in [2]

$$(15) \quad \begin{aligned} \ddot{x} - 2\dot{y} &= x - (x - \mu)f(x, y, z) - (x - \mu + 1)g(x, y, z) + a_x, \\ \ddot{y} + 2\dot{x} &= y - y[f(x, y, z) + g(x, y, z)] + a_y, \end{aligned}$$

where

$$a_x = A(x - \mu)h(x, y, z), \quad a_y = Ayh(x, y, z),$$

being  $A$  the zonal harmonic parameter and the function  $h(x, y, z) = 3(1 - \mu)/(2r_1^5)$ . System (15) admits a Hamiltonian formulation with the Hamiltonian

$$(16) \quad \begin{aligned} H(x, y, p_1, p_2) &= \frac{1}{2}(p_1^2 + p_2^2) + n(y p_1 - x p_2) - \frac{1 - \mu}{[(x - \mu)^2 + y^2]^{1/2}} \\ &\quad - \frac{\mu}{[(x - \mu + 1)^2 + y^2]^{1/2}} - \frac{(1 - \mu)A}{[(x - \mu)^2 + y^2]^{3/2}}. \end{aligned}$$

Consider  $\mu = \varepsilon\mu_1$  and  $A = \varepsilon A_1$  where  $\varepsilon$  is a small parameter. In what follows for simplicity the subindex 1 in  $\mu_1$  and  $A_1$  will be dropped. Then

$$H(x, y, p_1, p_2) = H_k(x, y, p_1, p_2) + \varepsilon H_p(x, y, p_1, p_2) + O(\varepsilon^2),$$

where  $H_k$  is the Hamiltonian of the rotating Kepler problem and  $H_p$  comes from the perturbed force, i.e.

$$H_k(x, y, p_1, p_2) = \frac{1}{2}(p_1^2 + p_2^2) + n(y p_1 - x p_2) - \frac{1}{(x^2 + y^2)^{1/2}},$$

$$H_p(x, y, p_1, p_2) = -\frac{\mu}{[(x+1)^2 + y^2]^{1/2}} - \frac{A}{2[(x+1)^2 + y^2]^{3/2}} + \frac{\mu[x(x-1) + y^2]}{(x^2 + y^2)^{3/2}}.$$

Now we change the cartesian variables  $(x, y, p_1, p_2)$  to the polar variables  $(r, \theta, Q, P)$ , and the Hamiltonian becomes

$$H(r, \theta, Q, P) = H_k(r, \theta, Q, P) + \varepsilon H_p(r, \theta, Q, P) + O(\varepsilon^2).$$

where

$$H_k(r, \theta, Q, P) = \frac{1}{2}\left(Q^2 + \frac{P^2}{r^2}\right) - nP - \frac{1}{r},$$

$$H_p(r, \theta, Q, P) = -\frac{\mu}{(r^2 + 2r \cos \theta + 1)^{1/2}} - \frac{A}{2(r^2 + 2r \cos \theta + 1)^{3/2}} + \frac{\mu(r - \cos \theta)}{r^2}.$$

and system (15) takes the form

$$(17)$$

$$\begin{aligned} \dot{r} &= Q, \\ \dot{\theta} &= \frac{P}{r^2} - n, \\ \dot{Q} &= \frac{P^2}{r^3} - \frac{1}{r^2} + \varepsilon \left[ \frac{\mu(r - 2 \cos \theta)}{r^3} - \frac{\mu(r + \cos \theta)}{(r^2 + 2r \cos \theta + 1)^{3/2}} - \frac{6A(r + \cos \theta)}{(r^2 + 2r \cos \theta + 1)^{5/2}} \right] + O(\varepsilon^2), \\ \dot{P} &= -\varepsilon \left[ \frac{\mu}{r^2} \sin \theta - \frac{\mu r \sin \theta}{(r^2 + 2r \cos \theta + 1)^{3/2}} - \frac{3Ar \sin \theta}{(r^2 + 2r \cos \theta + 1)^{5/2}} \right] + O(\varepsilon^2). \end{aligned}$$

Since system (17) is a particular case of the general system (7) having a first integral, then Theorems 1 and 2 hold to the the planar circular restricted 3-body problem with the zonal harmonic coefficient perturbation.

**4.2. Planar circular restricted 3-body problem with solar sail perturbation.** In this case the equations of motion with reflecting the solar sail reflector (see [23] and [28]) are

$$(18) \quad \ddot{\mathbf{r}} + 2\mathbf{n} \wedge \dot{\mathbf{r}} = \nabla V + \mathbf{a},$$

where

$$\begin{aligned} \mathbf{a} &= \frac{\beta(1-\mu)}{r_1^2} (\hat{\mathbf{r}}_1 \cdot \mathbf{s})^2 \mathbf{s}, \\ \mathbf{s} &= \frac{1}{R} \mathbf{R}, \\ \mathbf{R} &= \mathbf{r}_1 + \mathbf{r}_2, \\ R &= \sqrt{[2(x-\mu)+1]^2 + 4(y^2 + z^2)}. \end{aligned}$$

and

- $\mathbf{r}$  is the position vector of the solar reflector,
- $\mathbf{n}$  is the angular velocity vector of a rotating frame with unity value,
- $\mathbf{a}$  is the acceleration on a perfectly reflecting solar sail,
- $V$  is the effective potential function,
- $\mathbf{r}_1$  is the position vector of the space reflector with respect to the Sun,
- $\hat{\mathbf{r}}_1$  is the unit vector in the direction of  $\mathbf{r}_1$ ,
- $\mathbf{r}_2$  is the position vector of the space reflector with respect to the Earth,
- $\mathbf{s}$  is the attitude unit normal vector of the reflector,
- $\mu$  mass ratio parameter,
- $\beta$  sail lightness number, and
- The dot denotes derivative with respect to the time  $t$ .

In cartesian coordinates equations (18) are

$$(19) \quad \begin{aligned} \ddot{x} - 2\dot{y} &= x - (x - \mu)f(x, y, z) - (x - \mu + 1)g(x, y, z) + a_x, \\ \ddot{y} + 2\dot{x} &= y - y[f(x, y, z) + g(x, y, z)] + a_y, \\ \ddot{z} &= -z[f(x, y, z) + g(x, y, z)] + a_z, \end{aligned}$$

where

$$a_x = \beta[2(x - \mu) + 1]h(x, y, z), \quad a_y = 2\beta y h(x, y, z), \quad a_z = 2\beta z h(x, y, z),$$

being

$$f(x, y, z) = \frac{1 - \mu}{r_1^3}, \quad g(x, y, z) = \frac{\mu}{r_2^3}, \quad h(x, y, z) = \frac{1 - \mu}{r_1^4 R^3} [(x - \mu)[2(x - \mu) + 1] + 2(y^2 + z^2)]^2.$$

We assume that both parameters  $\mu$  and  $\beta$  are small. Within this frame we can write  $\mu = \varepsilon\mu_1$  and  $\beta = \varepsilon\beta_1$ , where the parameter  $\varepsilon$  is very small (i.e.  $0 < \varepsilon \ll 1$ ), and  $\mu_1$  and  $\beta_1$  are positive constants. In what follows for simplicity the subindex 1 in  $\mu_1$  and  $\beta_1$  will be dropped. Then equations (19) can be written as

$$(20) \quad \begin{aligned} \ddot{x} &= 2\dot{y} + x - \frac{(1 - \mu\varepsilon)(x - \mu\varepsilon)}{[(x - \mu\varepsilon)^2 + y^2 + z^2]^{3/2}} - \frac{\mu\varepsilon(x - \mu\varepsilon + 1)}{[(x - \mu\varepsilon + 1)^2 + y^2 + z^2]^{3/2}} + \\ &\quad \frac{\beta\varepsilon(1 - \mu\varepsilon)[2(x + \mu\varepsilon) - 1][(x + \mu\varepsilon)[2(x + \mu\varepsilon) - 1] + 2y^2 + 2z^2]^2}{[(x - \mu\varepsilon)^2 + y^2 + z^2]^2 [(2(x + \mu\varepsilon) - 1)^2 + 4(y^2 + z^2)]^{3/2}}, \\ \ddot{y} &= -2\dot{x} + y - y \left[ \frac{\mu\varepsilon}{[(x - \mu\varepsilon + 1)^2 + y^2 + z^2]^{3/2}} + \frac{1 - \mu\varepsilon}{[(x - \mu\varepsilon)^2 + y^2 + z^2]^{3/2}} \right] + \\ &\quad \frac{2\beta y \varepsilon (1 - \mu\varepsilon) [(x + \mu\varepsilon)[2(x + \mu\varepsilon) - 1] + 2y^2 + 2z^2]^2}{[(x - \mu\varepsilon)^2 + y^2 + z^2]^2 [(2(x + \mu\varepsilon) - 1)^2 + 4(y^2 + z^2)]^{3/2}}, \\ \ddot{z} &= -z \left[ \frac{\mu\varepsilon}{[(x - \mu\varepsilon + 1)^2 + y^2 + z^2]^{3/2}} + \frac{1 - \mu\varepsilon}{[(x - \mu\varepsilon)^2 + y^2 + z^2]^{3/2}} \right] + \\ &\quad \frac{2\beta z \varepsilon (1 - \mu\varepsilon) [(x + \mu\varepsilon)[2(x + \mu\varepsilon) - 1] + 2y^2 + 2z^2]^2}{[(x - \mu\varepsilon)^2 + y^2 + z^2]^2 [(2(x + \mu\varepsilon) - 1)^2 + 4(y^2 + z^2)]^{3/2}}. \end{aligned}$$

After expanding (20) in powers of  $\varepsilon$  this system can be written into the form (5), and we can study its periodic orbits in the plane  $z = 0$  using the system (6). But since now the

planar circular restricted 3-body problem with solar sail perturbation has no first integrals, the results of Theorems 1 and 2 cannot be applied, and it remains an open problem to know if the periodic orbits of first kind can be extended or not to this perturbed problem. But in the case of radial solar sail, that is when the sail is oriented along the Sun-Line (i.e. along the normal vector  $\mathbf{n} = \hat{\mathbf{r}}_1$ ), then system (18) has a first integral see [31], and in this case the results of Theorems 1 and 2 will apply.

## 5. PERIODIC ORBITS OF SECOND KIND

Birkhoff [7] observed that if the equations of system (6) are invariant with respect to the change variables  $(x, y, t) \rightarrow (x, -y, -t)$ , then an orbit which crosses the  $x$ -axis perpendicular at  $t = 0$  and again at  $t = T/2$  must be periodic of period  $T$ . So a way of finding periodic orbits is to look for orbits having two perpendicular crossings with the  $x$ -axis. Barrar [6] used ideas of Birkhoff for extending the elliptical orbits of the rotating Kepler problem to the planar circular restricted 3-body problem. Now we will see that the work of Barrar goes beyond the planar circular restricted 3-body problem, i.e. that it extends to the perturbed planar circular restricted 3-body problems.

In Delaunay variables two perpendicular crossing with the  $x$ -axis correspond to

$$g(t) = m\pi, \quad l(t) = n\pi.$$

Indeed  $g(t) = m\pi$  says that the major axis of the ellipse coincides with the  $x$ -axis, and  $l(t) = n\pi$  says that the infinitesimal body is on the  $x$ -axis, either at the apocenter or at the pericenter. Since instantaneous velocity of the infinitesimal body is the same than its motion along the ellipse and rotation of the ellipse, and both motions are perpendicular to the  $x$ -axis this implies a perpendicular crossing with the  $x$ -axis.

For  $\varepsilon = 0$  we consider the elliptic orbit with its apocenter at the positive  $x$ -axis when  $t = 0$ , i.e.

$$(21) \quad g(0) = -\pi, \quad l(0) = \pi,$$

from system (9) the equations of motion of this ellipse are

$$\dot{\ell} = \frac{1}{L^3}, \quad \dot{L} = 0, \quad \dot{g} = -n, \quad \dot{G} = 0.$$

So its solution satisfying conditions (21) is given by

$$(22) \quad l(t) = \frac{t}{L_0^3} + \pi, \quad L(t) = L_0, \quad g(t) = -nt - \pi, \quad G(t) = G_0,$$

where  $L_0$  and  $G_0$  are constants.

The period of this elliptic orbit is  $2\pi L_0^3$  and we want to consider the case when  $2\pi L_0^3 = 2\pi p/q$  with  $p$  and  $q$  are coprime. Thus we want to continue this periodic orbit to  $\varepsilon > 0$  sufficiently small with a period  $T = 2\pi p = 2\pi q L_0^3$ . Therefore

$$g(T/2) = -(1+p)\pi, \quad l(T/2) = (1+q)\pi,$$

and a such orbit will have two perpendicular crossings with the  $x$ -axis. Of course we must avoid the collision with the small body localized at the point  $(1, 0)$ .

For  $\varepsilon > 0$  sufficiently small we look for solutions

$$(l(t, \varepsilon, L_0, G_0), g(t, \varepsilon, L_0, G_0), L(t, \varepsilon, L_0, G_0), G(t, \varepsilon, L_0, G_0))$$

of system (9) with initial conditions

$$g(0, \varepsilon, L_0, G_0) = -\pi, \quad l(0, \varepsilon, L_0, G_0) = \pi,$$

such that

$$\begin{aligned} \psi_1(t, \varepsilon, L_0, G_0) &= g(t, \varepsilon, L_0, G_0) + (1 + p)\pi, \\ \psi_2(t, \varepsilon, L_0, G_0) &= l(t, \varepsilon, L_0, G_0) - (1 + q)\pi, \end{aligned}$$

with  $L_0$  near  $(p/q)^{1/3}$  and  $t$  near  $T/2$ .

From the Implicit Function Theorem such a solution will exist if

$$\det \left( \frac{\partial(\psi_1, \psi_2)}{\partial(t, L_0)} \right) \Big|_{\varepsilon=0} \neq 0.$$

on the elliptic orbit. Since

$$\left| \frac{\partial(\psi_1, \psi_2)}{\partial(t, L_0)} \right|_{\varepsilon=0} = \begin{vmatrix} \frac{\partial g}{\partial t} & \frac{\partial l}{\partial t} \\ \frac{\partial g}{\partial L_0} & \frac{\partial l}{\partial L_0} \end{vmatrix}_{\varepsilon=0}$$

then, from (22), we get

$$\left| \frac{\partial(\psi_1, \psi_2)}{\partial(t, L_0)} \right|_{\varepsilon=0} = \frac{3nT}{2L_0^4} \neq 0.$$

So we have the following result:

**Theorem 3.** *Let  $p$  and  $q$  be relative prime positive integers and  $T = 2\pi p$ . Then the elliptical periodic solution of period  $T$  of the Kepler rotating problem satisfying*

$$g(0) = -\pi, \quad l(0) = \pi, \quad L^3(0) = p/q,$$

*and do not going through the point  $(1, 0)$  can be extended to the perturbed planar circular restricted 3-body problem with  $\mu > 0$  and  $\varepsilon > 0$  small such that its equations of motion (9) come from the equations (6) invariant by the symmetry  $(x, y, t) \rightarrow (x, -y, -t)$ .*

## 6. CONCLUSION

We have proved that the periodic orbits of first kind can be extended from the planar circular restricted 3-body problem to a perturbed planar circular restricted 3-body problem if this perturbed problem has a first integral. It is an open problem to know when these periodic orbits can be extended if the perturbed problem has not a first integral.

We also have shown that the periodic orbits of second kind can be extended from the planar circular restricted 3-body problem to a perturbed planar circular restricted 3-body problem if this perturbed problem in cartesian coordinates is invariant by the symmetry  $(x, y, t) \rightarrow (x, -y, -t)$ .

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## REFERENCES

- [1] E. I. ABOUELMAGD , M. S. ALHOTHUALI, J. L. G. GUIRAO AND H. M. MALAIKAH, *The effect of zonal harmonic coefficients in the framework of the restricted three-body problem*. Adv. Space Res. **55** (2015), 1660–1672.
- [2] E. I. ABOUELMAGD, *Existence and stability of triangular points in the restricted three-body problem with numerical applications*. Apstrophys Space Sci. **342**(2012), 45–53.
- [3] E. I. ABOUELMAGD AND J. L.G. GUIRAO, *On the perturbed restricted three-body problem*. Applied Mathematics and Nonlinear Sciences **1**(1) (2016), 123–144.
- [4] E. I. ABOUELMAGD, J. L.G. GUIRAO AND A. MOSTAFA, *Numerical integration of the restricted three-body problem with Lie series*. Astrophys Space Sci. (2014) **354** (2014), 369–378.
- [5] D. Bancelin, D. Hestroffer, and W. Thuillot, *Numerical integration of dynamical systems with Lie series relativistic acceleration and nongravitational forces*. Celest. Mech. Dyn. Astron. **112** (2012), 221–234.
- [6] R. B. BARRAR, *Existence of periodic orbits of second kind in the restricted problem of three bodies*. Astro. J. **70** (1) (1965), 3–6.
- [7] G. D. BIRKHOFF, *Dynamical System*. American Math. Society ColloquimPublication, vol. IX, New York, 1927.
- [8] J. CANDY AND W. ROZMUS, *A symplectic integration algorithm for separable Hamiltonian functions*. J. Comp. Physiol. **92** (1991), 230–256.
- [9] J. R. CASH AND A. H. KARP, *A variable order RungeKutta method for initial value problems with rapidly varying right-hand sides*. ACM Trans. Math. Softw. **16**(3) (1990), 201–222.
- [10] J. E. CHAMBERS, *A hybrid symplectic integrator that permits close encounters between massive bodies*. Mon. Not. R. Astron. Soc. **304** (1999), 793–799.
- [11] M. CORBERA AND J. LLIBRE, *Symmetric periodic orbits for the elliptical 3-body problem via continuation method*. Developments in Mathematical and Experimental Physics, Klauaer, Acod., 2003, pp113–137.
- [12] M. CUNTZ, *S-type and p-type habitability in stellar binary systems: A comprehensive approach. i. method and applications*, Astrophysical Journal **780**(1) (2014), 14.
- [13] R. DVORAK, C. FROESCHLE, AND C. FROESCHLE, *Stability of outer planetary orbits (p-types) in binaries*, Astronomy & Astrophysics **226** (1989), 335–342.
- [14] S. EGGL AND R. DVORAK, *An introduction to common numerical integration codes used in dynamical astronomy*. In: Souchay, J., Dvorak, R. (eds.) Lecture Notes in Physics, vol. 790, pp. 431480. Springer, Berlin 2010.
- [15] S. EGGL, E. PILAT-LOHINGER, N. GEORGAKARAKOS, M. GYERGYOVITS, AND B. FUNK, *An analytic method to determine habitable zones for s-type planetary orbits in binary star systems*, Astrophysical Journal **752** (2012), 74.
- [16] S.M. ELSHABOURY, E.I. ABOUELMAGD, V.S. KALANTONIS, E.A. PERDIOS, *The planar restricted three-body problem when both primaries are triaxial rigid bodies: Equilibrium points and periodic orbits*, Astrophys. Space Sci. **361**(9) (2016), 315.
- [17] G. GÓMEZ, J. LLIBRE, R. MARTÍNEZ AND C. SIMÓ, *Dynamics and Mission Design Near Libration Points. Vol. I Fundamentals: The case of collinear libration points*, World Scientific Monograph Series in Mathematics, Vol. 2, World Scientific, Singapore, 2001.
- [18] G. GÓMEZ, J. LLIBRE, R. MARTÍNEZ AND C. SIMÓ, *Dynamics and Mission Design Near Libration Points. Vol. II Fundamentals: The case of triangular libration points*, World Scientific Monograph Series in Mathematics, Vol. 3, World Scientific, Singapore, 2001.

- [19] J. D. HADJIDEMETRIOU, *The continuation of Periodic orbits from the restricted to the general three-body problem*, *Celest. Mech. Dyn- Astr.* **12** (1975), 155–174.
- [20] A. HANSLMEIER AND R. DVORAK, *Numerical integration with Lie series*. *Astron. Astrophys.* **132** (1984), 203–207.
- [21] L. KALTENEGGER AND N. HAGHIGHIPOUR, *Calculating the habitable zone of binary star systems. i. s-type binaries*, *Astrophysical Journal* **777** (2013), 165.
- [22] J. LLIBRE AND D. G. SAARI, *Periodic orbits for the planar Newtonian three-body problem coming from the elliptical restricted three-body problems*, *Trans. Amer. Math. Soc.* **347** (1995), 3017–3030.
- [23] C. R. MCINNES, A. J. C. McDONALD, J.F. L. SIMMONS AND E. W. MACDONALDG, *Solar Sail Parking in Restricted Three-Body Systems*, *J. Guid. Control Dyn.* **17** (1994), 399–406.
- [24] K.R. MEYER, G.R. HALL AND D. OFFIN, *Introduction to Hamiltonian dynamical systems and the N-body problem*, Second edition. *Applied Mathematical Sciences* **90**, Springer, New York, 2009.
- [25] C. D. MURRAY AND S. F. DERMOTT, *Solar system dynamics*. Cambridge University Press, (1999).
- [26] H. POINCARÉ, *Leçons de Mécanique Céleste*, Vol. 1, 1905; Vol. 2, Part I 1907; Vol. 2, Part II, 1909; Vol. 3, 1910; Gauthier-Villars, Paris.
- [27] H. POINCARÉ, *Les Méthodes Nouvelles de la Mécanique Céleste*, Vol. 1, 1892; Vol. 2, 1983; Vol. 3, 1899; Gauthier-Villars, Paris. Reprinted by Dover, New York, 1957.
- [28] F. J. T. SALAZAR, C.R. MCINNES AND O.C. WINTER, *Periodic orbits for space-based reflectors in the circular restricted three-body problem*, *Celest Mech Dyn Astr* **128(2)**: (2017), 95–113.
- [29] S. STERNBERG, *Celestial mechanics. Part II*. W. A. Benjamin, Inc, New York. (1969)
- [30] R. K. SHARMA, *The linear stability of libration points of the photogravitational restricted three-body problem when the smaller primary is an oblate spheroid*, *Astrophys. Space Sci.* **135** (1987), 271–281.
- [31] P. VERRIER AND T. WATERS, *Families of periodic orbits for solar sails in the CRBTP*, M. Macdonald(ed), *Advances in Solar Sailing*, Springer Praxis Books, Springer-Verlag Berlin Heidelberg, 2014, pp. 871–884.

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