PERIODS OF CONTINUOUS MAPS ON CLOSED SURFACES

JUAN LUIS GARCÍA GUIRAO[1] AND JAUME LLIBRE[2]

Abstract. The objective of the present work is to present what information on the set of periodic points of a continuous self–map on a closed surface can be obtained using the action of this map on the homological groups of the closed surface.

1. Introduction

The periodic orbits play an important role in the description of the dynamics of a map, for studying them we can use topological information. Perhaps the best known result in this direction is the one contained in the seminal paper entitle *Period three implies chaos* for continuous self–maps on the interval, see [8], and of course the famous Sharkovskiĭ's theorem [10] describing the full set of periods of the continuous self–maps on the interval. Also the set of periods of the continuous self-maps on the circle have been characterized, see for instance [1].

The interval and the circle are the unique compact manifolds of dimension one, after studying the set of periods of the self-continuous maps on these manifolds, the following natural step is to start the study of the continuous self-maps on the compact manifolds of dimension two, i.e. on the closed surfaces. This is the main goal of this paper, and we shall use the homological information of these maps for providing information about the periods of their periodic orbits.

Along this work by a *closed surface* we denote a connected compact surface with or without boundary, orientable or not. More precisely, *an orientable connected compact surface without boundary of genus* $g \geq 0$, $\mathbb{M}_g$, is homeomorphic to the sphere if $g = 0$, to the torus if $g = 1$, or to the connected sum of g copies of the torus if $g \geq 2$. *An orientable connected compact surface with boundary of genus* $g \geq 0$, $\mathbb{M}_{g,b}$, is homeomorphic to $\mathbb{M}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows $\mathbb{M}_{g,0} = \mathbb{M}_g$.

A non–orientable connected compact surface without boundary of genus $g \geq 1$, $\mathbb{N}_g$, is homeomorphic to the real projective plane if $g = 1$, or to the connected sum of g copies of the real projective plane if $g > 1$. *A non–orientable connected*

Key words and phrases. Closed surface, continuous self–map, Lefschetz fixed point theory, periodic point, set of periods.

2010 Mathematics Subject Classification: 58F20, 37C05, 37C25, 37C30.

compact surface with boundary of genus $g \geq 1$, $\mathbb{N}_{g,b}$, is homeomorphic to $\mathbb{N}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closure. In what follows $\mathbb{N}_{g,0} = \mathbb{N}_g$.

Let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map on a closed surface $\mathbb{X}$. A point $x \in \mathbb{X}$ is periodic of period n if $f^n(x) = x$ and $f^k(x) \neq x$ for $k = 1, \dots, n-1$. We denote by $\mathrm{Per}(f)$ the set of periods of all periodic points of f. The aim of the present paper is to provide some information on $\mathrm{Per}(f)$.

Let A be an $n \times n$ complex matrix. A $k \times k$ *principal submatrix* of A is a submatrix lying in the same set of k rows and columns, and a $k \times k$ *principal minor* is the determinant of such a principal submatrix. There are $\binom{n}{k}$ different $k \times k$ principal minors of A, and the sum of these is denoted by $E_k(A)$. In particular, $E_1(A)$ is the trace of A, and $E_n(A)$ is the determinant of A, denoted by $\det(A)$.

It is well known that the characteristic polynomial of A is given by

$$\det(tI - A) = t^n - E_1(A)t^{n-1} + E_2(A)t^{n-2} - \dots + (-1)^n E_n(A).$$

Our main result is stated in the following theorem.

Theorem 1. *Let $\mathbb{X}$ be a closed surface and let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map and let A and (d) be the integral matrices of the endomorphisms $f_{*i} : H_i(\mathbb{X}, \mathbb{Q}) \to H_i(\mathbb{X}, \mathbb{Q})$ induced by f on the i–th homology group of $\mathbb{X}$, $i = 1, 2$.*

If $\mathbb{X}$ is either $\mathbb{M}_{g,b}$ with $b > 0$, or $\mathbb{N}_{g,b}$ with $b \geq 0$, then the following statements hold.

(a) *If $E_1(A) \neq 1$, then $1 \in \mathrm{Per}(f)$.*
(b) *If $E_1(A) = 1$ and $E_2(A) \neq 0$, then $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.*

If $\mathbb{X} = \mathbb{M}_{g,b}$ with $b = 0$, then the following statement hold.

(c) *If $E_1(A) \neq 1 + d$, then $1 \in \mathrm{Per}(f)$.*
(d) *If $E_1(A) = 1 + d$ and $E_2(A) \neq d^2 + d + 1$, then $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$.*

If $\mathbb{X} = \mathbb{M}_{g,b}$ with $b > 0$, then the following statement hold.

(e) *If $2g + b - 1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ and k is the smallest integer of the set $\{3, 4, ..., 2g + b - 1\}$ such that $E_k(A) \neq 0$, then $\mathrm{Per}(f)$ has a periodic point of period a divisor of k.*

If $\mathbb{X} = \mathbb{N}_{g,b}$ with $b \geq 0$, then the following statement hold.

(f) *If $g + b - 1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ and k is the smallest integer of the set $\{3, 4, ..., g + b - 1\}$ such that $E_k(A) \neq 0$, then $\mathrm{Per}(f)$ has a periodic point of period a divisor of k.*

Theorem 1 is proven in section 2.

Similar results to the ones obtained in Theorem 1 but for homeomorphisms on closed surfaces where obtained by Franks and Llibre in [5], and by the authors in [6]. Other related results can be found in [5], see also the references quoted there.

2. Proof of Theorem 1

Let $f : \mathbb{X} \to \mathbb{X}$ be a continuous map and let $\mathbb{X}$ be either $\mathbb{M}_{g,b}$ or $\mathbb{N}_{g,b}$. Then the *Lefschetz number* of f is defined by

$$L(f) = \text{trace}(f_{*0}) - \text{trace}(f_{*1}) + \text{trace}(f_{*2}).$$

For continuous self–maps f defined on $\mathbb{X}$ the Lefschetz fixed point theorem states (see for instance [2]).

Theorem 2. *If $L(f) \neq 0$ then f has a fixed point.*

With the objective of studying the periodic points of f we shall use the Lefschetz numbers of the iterates of f, i.e. $L(f^n)$. Note that if $L(f^n) \neq 0$ then f^n has a fixed point, and consequently f has a periodic point of period a divisor of n. In order to study the whole sequence $\{L(f^n)\}_{n\geq 1}$ it is defined the formal *Lefschetz zeta function* of f as

$$Z_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n\right). \tag{1}$$

The Lefschetz zeta function is in fact a generating function for the sequence of the Lefschetz numbers $L(f^n)$.

Let f be a continuous self–map defined on $\mathbb{M}_{g,b}$ or $\mathbb{N}_{g,b}$, respectively. For a closed surface the homological groups with coefficients in $\mathbb{Q}$ are linear vector spaces over $\mathbb{Q}$. We recall the homological spaces of $\mathbb{M}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$, $n_1 = 2g + b - 1$ if $b > 0$, $n_2 = 1$ if $b = 0$, and $n_2 = 0$ if $b > 0$; and the induced linear maps $f_{*k} : H_k(\mathbb{M}_{g,b}, \mathbb{Q}) \to H_k(\mathbb{M}_{g,b}, \mathbb{Q})$ by f on the homological group $H_k(\mathbb{M}_{g,b}, \mathbb{Q})$ are $f_{*0} = (1)$, $f_{*2} = (d)$ where d is the *degree* of the map f if $b = 0$, $f_{*2} = (0)$ if $b > 0$, and $f_{*1} = A$ where A is an $n_1 \times n_1$ integral matrix (see for additional details [9, 11]).

We recall that the homological groups of $\mathbb{N}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{N}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = g + b - 1$ and $n_2 = 0$; and the induced linear maps are $f_{*0} = (1)$ and $f_{*1} = A$ where A is an $n_1 \times n_1$ integral matrix (see again for additional details [9, 11]).

From the work of Franks in [3] we have for a continuous self–map of a closed surface that its Lefschetz zeta function is the rational function

$$Z_f(t) = \frac{\det(I - tf_{*1})}{\det(I - tf_{*0})\det(I - tf_{*2})},$$

where in $I - tf_{*k}$ the I denotes the $n_k \times n_k$ identity matrix, and $\det(I - tf_{*2}) = 1$ if $f_{*2} = (0)$. Then for a continuous map $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ we have

$$Z_f(t) = \begin{cases} \dfrac{\det(I - tA)}{(1-t)(1-dt)} & \text{if } b = 0, \\ \dfrac{\det(I - tA)}{1-t} & \text{if } b > 0, \end{cases} \tag{2}$$

and for a continuous map $f : \mathbb{N}_{g,b} \to \mathbb{N}_{g,b}$ we have

$$Z_f(t) = \frac{\det(I - tA)}{1-t}. \tag{3}$$

Proof of Theorem 1. Combining the expressions (1) and (2) if $\mathbb{X} = \mathbb{M}_{g,b}$ and $b > 0$, and the expressions (1) and (3) if $\mathbb{X} = \mathbb{N}_{g,b}$ with $b \geq 0$, we obtain the following equalities

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n &= \log(Z_f(t)) \\ &= \log\left(\frac{\det(I - tA)}{1-t}\right) \\ &= \log\left(\frac{1 - E_1(A)t + E_2(A)t^2 - ... + (-1)^m E_m(A)t^m}{1-t}\right) \\ &= \log(1 - E_1(A)t + E_2(A)t^2 - ...) - \log(1-t) \\ &= \left(-E_1(A)t + \left(E_2(A) - \frac{E_1(A)^2}{2}\right)t^2 - ...\right) - \left(-t - \frac{t^2}{2} - ...\right) \\ &= (1 - E_1(A))t + \left(\frac{1}{2} - \frac{E_1(A)^2}{2} + E_2(A)\right)t^2 + O(t^3). \end{aligned}$$

Here $n_1 = 2g + b - 1$ if $\mathbb{X} = \mathbb{M}_{g,b}$ with $b > 0$, or $n_1 = g + b - 1$ if $\mathbb{X} = \mathbb{N}_{g,b}$ with $b \geq 0$. Therefore we have

$$L(f) = 1 - E_1(A) \quad \text{and} \quad L(f^2) = 1 - E_1(A)^2 + 2E_2(A).$$

Hence, if $E_1(A) \neq 1$ then $L(f) \neq 0$, and by Theorem 2 statement (a) follows.

If $E_1(A) = 1$ and $E_2(A) \neq 0$, then $L(f^2) = 2E_2(A) \neq 0$, and again by Theorem 2 we get that $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$. So statement (b) is proved.

Let $\mathbb{X} = \mathbb{M}_{g,b}$ with $b = 0$. By (1) and (2) with $b = 0$ we obtain the following equalities

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n &= \log(Z_f(t)) \\
&= \log\left(\frac{\det(I - tA)}{(1-t)(1-dt)}\right) \\
&= \log\left(\frac{1 - E_1(A)t + E_2(A)t^2 - ... + (-1)^m E_m(A)t^m}{(1-t)(1-dt)}\right) \\
&= \log(1 - E_1(A)t + E_2(A)t^2 - ...) - \log((1-t)(1-dt)) \\
&= \left(-E_1(A)t + \left(E_2(A) - \frac{E_1(A)^2}{2}\right)t^2 - ...\right) \\
&\quad - \left(-(1+d)t - \left(\frac{d^2+1}{2}\right)t^2 - ...\right) \\
&= (1 + d - E_1(A))t + \left(E_2(A) - \frac{E_1(A)^2}{2} - \frac{d^2+1}{2}\right)t^2 + O(t^3).
\end{aligned}
$$

Here $n_1 = 2g$. Therefore we have

$$L(f) = 1 + d - E_1(A), \quad \text{and} \quad L(f^2) = 2E_2(A) - E_1(A)^2 - (d^2+1).$$

Hence, if $E_1(A) \neq 1 + d$ then $L(f) \neq 0$, and by Theorem 2 statement (c) follows.

If $E_1(A) = 1+d$ and $E_2(A) \neq d^2+d+1$, then $L(f^2) = 2E_2(A) - 2(d^2+d+1) \neq 0$, and again by Theorem 2 we get that $\mathrm{Per}(f) \cap \{1, 2\} \neq \emptyset$. So statement (d) is proved.

Assume now that $\mathbb{X} = \mathbb{M}_{g,b}$ with $b > 0$, $2g+b-1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ and k is the smallest integer of the set $\{3, 4, ..., 2g+b-1\}$ such that $E_k(A) \neq 0$. Therefore

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{L(f^n)}{n} t^n &= \log\left(\frac{1 - t + (-1)^k E_k(A)t^k + ... + (-1)^{b-1} E_{2g+b-1}(A)t^{2g+b-1}}{1-t}\right) \\
&= \log\left(1 + \frac{(-1)^k E_k(A)t^k + ... + (-1)^{b-1} E_{2g+b-1}(A)t^{2g+b-1}}{1-t}\right) \\
&= (-1)^k E_k(A)t^k + O(t^{k+1}).
\end{aligned}
$$

Hence, $L(f) = ... = L(f^{k-1}) = 0$ and $L(f^k) = (-1)^k k E_k(A) \neq 0$. So, from Theorem 2, it follows the statement (e).

Suppose that $\mathbb{X} = \mathbb{N}_{g,b}$ with $b \geq 0$, $g+b-1 \geq 3$, $E_1(A) = 1$, $E_2(A) = 0$ and k is the smallest integer of the set $\{3, 4, ..., g+b-1\}$ such that $E_k(A) \neq 0$.

Therefore

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{L(f^n)}{n}t^n = & \log\left(\frac{1-t+(-1)^kE_k(A)t^k+...+(-1)^{g+b-1}E_{g+b-1}(A)t^{g+b-1}}{1-t}\right)\\ = & \log\left(1+\frac{(-1)^kE_k(A)t^k+...+(-1)^{g+b-1}E_{g+b-1}(A)t^{g+b-1}}{1-t}\right)\\ = & (-1)^kE_k(A)t^k+O(t^{k+1}).\end{aligned}$$

Again $L(f) = ... = L(f^{k-1}) = 0$ and $L(f^k) = (-1)^k kE_k(A) \neq 0$. Therefore, from Theorem 2, it follows the statement (f). $\square$

Acknowledgements

The first author of this work was partially supported by MICINN/FEDER grant number MTM2011–22587. The second author is partially supported by a MINECO MTM2013-40998-P, an AGAUR grant number 2014SGR568, and the grants FP7-PEOPLE-2012-IRSES 318999 and 316338.

References

[1] L. Alsedà, J. Llibre and M. Misiurewicz, *Combinatorial Dynamics and Entropy in dimension one* (Second Edition), Advanced Series in Nonlinear Dynamics **5**, World Scientific, Singapore, 2000.

[2] R.F. Brown, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Company, Glenview, IL, 1971.

[3] J. Franks, *Homology and Dynamical Systems*, CBMS Regional Conf. Series, vol. **49**, Amer. Math. Soc., Providence R.I., 1982.

[4] J. Franks and J. Llibre, *Periods of surface homeomorphisms*, Contemporary Mathematics **117** (1991), 63–77.

[5] J.M. Gambaudo and J. Llibre, *A note on the set of periods of surface homeomorphisms*, J. Math. Anal. and Appl. **177** (1993), 627–632.

[6] J.L. García Guirao and J. Llibre, *Periods of homeomorphisms on surfaces*, to appear in the Proceeding of the conference ICDEA2012.

[7] B. Halpern, *Fixed point for iterates*, Pacific J. Math. **25** (1968), 255–275.

[8] T.Y. Li and J. Yorke, *Period three implies chaos*, Amer. Math. Monthly **82** (1975), 985–992.

[9] J.R. Munkres, *Elements of Algebraic Topology*, Addison–Wesley, 1984.

[10] A.N. Sharkovskiĭ, *Coexistence of cycles of a continuous map of the line into itself* (Russian), Ukraine Math. Z. **16** (1964), 61–71.

[11] J.W. Vicks, *Homology theory. An introduction to algebraic topology*, Springer–Verlag, New York, 1994. Academic Press, New York, 1973.

[1] Departamento de Matemática Aplicada y Estadística. Universidad Politécnica de Cartagena, Hospital de Marina, 30203-Cartagena, Región de Murcia, Spain.

E-mail address: `juan.garcia@upct.es`

[2] Departament de Matemàtiques. Universitat Autònoma de Barcelona, Bellaterra, 08193-Barcelona, Catalonia, Spain

E-mail address: `jllibre@mat.uab.cat`