

# LIBRATION POINTS IN THE RESTRICTED THREE-BODY PROBLEM: EULER ANGLES, EXISTENCE AND STABILITY

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ABSTRACT. The objective of the present paper is to study in an analytical way the existence and the stability of the libration points, in the restricted three-body problem, when the primaries are triaxial rigid bodies in the case of the Euler angles of the rotational motion are equal to  $\theta_i = \pi/2$ ,  $\psi_i = 0$ ,  $\varphi_i = \pi/2$ ,  $i = 1, 2$ . We prove that the locations and the stability of the triangular points change according to the effect of the triaxiality of the primaries. Moreover, the solution of long and short periodic orbits for stable motion is presented.

## 1. INTRODUCTION

In the framework of *three-body problem* there are three objects moving under their mutual gravitational interactions in the space subdued by the Newton's universal law of gravitation. Generally, there are no restrictions neither on the motion of these objects nor their masses. The choreography takes place in three dimensions and a potential resolution of this problem demands that the past and the future motions of the objects be uniquely determined based uniquely on their present locations and velocities.

In the past, many scientists and mathematicians have done vigorous attempts to construct closed form solution to the three-body problem. Unfortunately all these attempts failed. In general there is no closed form solution for the three-body problem similar to the well known model of the two-body one. The reason is because the motion of the three objects is not predictable, thereby this problem is considered one of the most challenging problems in the history of science. The three-body problem in space dynamics is used to find the trajectories of celestial objects in order to predict the behavior of their motion.

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*Key words and phrases.* Restricted three-body problem; triaxial rigid bodies; Euler angles; libration points; stability.

2010 Mathematics Subject Classification: 37N05, 70F15.

In solar system, the planets and asteroids rotate around the Sun, while the moons move around their host planets which in turns also revolves around the Sun. Some of the typical applications of the three-body problem are the Sun-planet-moon, the Sun-planet-planet or the Sun-planet-asteroid systems. The last one has a special significance because it can be simplified attending the mass of the third body (i.e., the asteroid) because it is very small compared with the mass of the Sun or the planet. This implies that the third body does not affect the motions of the Sun and the planet, which shall call *the primaries bodies*. Under the previous condition, i.e. a third body negligible, the three-body problem is reduced to the so called *restricted three-body problem*. In this setting, we shall have the *circular* or the *elliptic restricted three-body problem* when the primaries movement around their center masses is either in circular or in a elliptic periodic orbit, respectively.

The present work is an analytical study of the equilibrium solutions of the planar circular restricted three-body problem when both primaries are triaxial bodies. Our results are an extension of some our previous works and some other publications. The existence and stability of libration points as well as the finding of periodic orbits around these points in restricted three-body problem with the effect the lack of the sphericity of the primaries or the effect of the photogravitational force or a combination of them are studied in [1]–[14], [18] or in [21]–[25].

The *Euler angles* are three angles constructed by Leonhard Euler to determine and describe the exact orientation of a rigid body with respect to the inertial fixed frame. The importance of these angles is because they can be used also to represent the orientation of a mobile reference frame in physics or the orientation of a general basis in 3-dimensional linear algebra. The Euler angles are considered a sufficient and gentle way to reach any references frame.

For example in the case in which the Euler angles of the rotational motion are  $(\theta_i = \psi_i = \varphi_i = 0)$ , [22, 23] study the existence and the stability of motion around the libration points in the restricted three-body problem when both primaries are triaxial rigid bodies. While in the case of the primaries are triaxial bodies and Euler angles of stationary rotational motion are non mutually zero the full construction of the general equations of motion for the restricted three-body problem can be found in [19]. Moreover, in [19] is studied the existence and stability of equilibrium libration points in the special cases of the Euler angles of rotational motion are  $\theta_i = \psi_i = \varphi_i = \pi/2$  and  $\theta_i = \psi_i = \pi/2$ ,  $\varphi_i = 0$ ,  $i = 1, 2$ . It was proved that the three collinear equilibrium points are still unstable

while the two triangular may be stable. The simple symmetric periodic orbits were studied and 31 families of these orbits were determined.

[16] presents necessary and sufficient conditions to find the locations of the three collinear points in five cases in the case of the Euler angles of rotational motion are accordingly  $\theta_i = 0$ ,  $\psi_i + \varphi_i = \pi/2$ ,  $i = 1, 2$ . The linear stability of motion in the vicinity of these points was studied too in this work.

Inspired in previous works, in this paper we shall consider the restricted problem of three bodies being both primaries triaxial rigid bodies and the Euler angles of stationary rotational motion are  $\theta_i = \pi/2$ ,  $\psi_i = 0$ ,  $\varphi_i = \pi/2$  and  $\theta_i = \pi/2$ ,  $\psi_i = \varphi_i = 0$  for  $i = 1, 2$ . This paper is organized as follows. In section 2 the equations of motion of the restricted three-body problem when the primaries are triaxial rigid bodies are stated being the Euler angles of rotational motion equal to  $\theta_i = \pi/2$ ,  $\psi_i = 0$ ,  $\varphi_i = \pi/2$ . In sections 3 and 4) the existence and the stability of the libration points are studied. In section 5 we summarize the results obtained along the work and we state how can be obtained the analogous results in the case of Euler angles of rotational motion are equal to  $\theta_i = \pi/2$ ,  $\psi_i = 0$ ,  $\varphi_i = 0$ .

We underline the importance of this study because this model has a clear application in Celestial Mechanics when the third body moves in gravitational fields of some objects with irregular shapes, such as Jupiter model.

## 2. EQUATIONS OF MOTION

Let us adopt the notation and terminology of [19]. In the case of the Euler angles of rotational motion are equal to  $\theta_i = \pi/2$ ,  $\psi_i = 0$ ,  $\varphi_i = \pi/2$ , the values of directions cosines are zero except for  $a_{3i} = b_{1i} = -1$ ,  $c_{2i} = 1$ . Thus, the equation of motion of the infinitesimal body is governed by

$$(1) \quad \begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_x, \\ \ddot{y} + 2n\dot{x} &= \Omega_y, \end{aligned}$$

where

$$\Omega = \sum_{i=1}^2 \mu_i [T_{1i} + T_{2i} + T_{3i} + T_{4i}],$$

and

$$\begin{aligned} T_{1i} &= \left(\frac{n^2}{2} r_i^2 + \frac{1}{r_i}\right), \\ T_{2i} &= \frac{1}{r_i^3} (A_{1i} + A_{2i} + A_{3i}), \\ T_{3i} &= -\frac{3}{2r_i^5} (A_{1i} + A_{3i}) [(x + (-1)^i \mu_{3-i})]^2 \\ T_{4i} &= -\frac{3}{2r_i^5} (A_{1i} + A_{2i}) y^2, \end{aligned}$$

$$(2) \quad \begin{aligned} r_1^2 &= (x - \mu)^2 + y^2, \\ r_2^2 &= (x - \mu + 1)^2 + y^2, \end{aligned}$$

and the mean motion  $n$  is governed by

$$(3) \quad n^2 = 1 + \frac{3}{2} \sum_{i=1}^2 (2A_{2i} - A_{1i} - A_{3i}),$$

where

$$(4) \quad A_{1i} = \frac{a_i^2}{5R^2}, \quad A_{2i} = \frac{b_i^2}{5R^2}, \quad A_{3i} = \frac{c_i^2}{5R^2},$$

and  $R$  is the separation distance between the primaries,  $\mu_1 = 1 - \mu$ ,  $\mu_2 = \mu$ ,  $a_i$ ,  $b_i$  and  $c_i$   $i = 1, 2$ , are the principal axes of the triaxial rigid bodies. Eq. (3) of the mean motion when the rotational motion of Euler angles are  $\theta_i = \pi/2$ ,  $\psi_i = 0$ ,  $\varphi_i = \pi/2$  is agree with the mean motion when Euler angles are  $\theta_i = 0$ ,  $\psi_i + \varphi_i = \pi/2$ , see [19].

### 3. COLLINEAR POINTS

Since  $\Omega = \Omega(x, y)$ , then

$$\frac{d\Omega}{dt} = \dot{x}\Omega_x + \dot{y}\Omega_y.$$

(1) admits the Jacobi integral in the form

$$\dot{x}^2 + \dot{y}^2 - 2\Omega + C = 0.$$

and the locations of equilibrium points are the solutions of

$$\Omega_x = \Omega_y = 0.$$

where

$$(5) \quad \begin{aligned} \Omega_x &= (1 - \mu)(x - \mu)[f_1(r_1) + q_1(x, y, r_1)] + \mu(x - \mu + 1)[f_2(r_2) + q_2(x, y, r_2)], \\ \Omega_y &= y[(1 - \mu)[g_1(r_1) + q_1(x, y, r_1)] + \mu[g_2(r_2) + q_2(x, y, r_2)]], \end{aligned}$$

and

$$(6) \quad \begin{aligned} f_i(r_i) &= \left(n^2 - \frac{1}{r_i^3}\right) - \frac{3}{2r_i^5} (2A_{2i} - A_{1i} + 4A_{3i}), \\ g_i(r_i) &= \left(n^2 - \frac{1}{r_i^3}\right) - \frac{3}{2r_i^5} (4A_{2i} - A_{1i} + 2A_{3i}), \\ q_i(x, y, r_i) &= \frac{15}{2r_i^7} \left[ A_{3i} [x + (-1)^i \mu_{3-i}]^2 + A_{2i} y^2 \right]. \end{aligned}$$

Assume that the triaxial rigid body of mass  $m_i$ ,  $i = 1, 2$ , be nearly a sphere with radius  $R_{0i}$ , thereby the principal axes of the triaxial rigid bodies can be written as

$$(7) \quad \begin{aligned} a_i &= R_{0i} + \sigma_{1i}, \\ b_i &= R_{0i} + \sigma_{2i}, \\ c_i &= R_{0i} + \sigma_{3i}, \end{aligned}$$

where  $\sigma_{1i}, \sigma_{2i}, \sigma_{3i} \ll 1$  such that  $\sigma_{1i} \neq \sigma_{2i} \neq \sigma_{3i}$ , see [19] for more details.

Substituting (7) into (4), we get

$$(8) \quad \begin{aligned} A_{1i} &= \lambda_i + \delta_i \sigma_{1i}, \\ A_{2i} &= \lambda_i + \delta_i \sigma_{2i}, \\ A_{3i} &= \lambda_i + \delta_i \sigma_{3i}, \end{aligned}$$

where  $\lambda_i = \frac{R_{0i}^2}{5R^2}$ ,  $\delta_i = \frac{2R_{0i}}{5R^2}$ .

After substituting (8) into (5), (6) and (3), we get

$$(9) \quad \begin{aligned} \Omega_x &= (1 - \mu)(x - \mu)[F_1(r_1) + Q_1(x, y, r_1)] + \mu(x - \mu + 1)[F_2(r_2) + Q_2(x, y, r_2)], \\ \Omega_y &= y[(1 - \mu)[G_1(r_1) + Q_1(x, y, r_1)] + \mu[G_2(r_2) + Q_2(x, y, r_2)]], \end{aligned}$$

where

$$(10) \quad \begin{aligned} F_i(r_i) &= \left(n^2 - \frac{1}{r_i^3}\right) - \frac{3\delta_i}{2r_i^5}(2\sigma_{2i} - \sigma_{1i} + 4\sigma_{3i}), \\ G_i(r_i) &= \left(n^2 - \frac{1}{r_i^3}\right) - \frac{3\delta_i}{2r_i^5}(4\sigma_{2i} - \sigma_{1i} + 2\sigma_{3i}), \\ Q_i(x, y, r_i) &= \frac{15\delta_i}{2r_i^7} \left( \sigma_{3i} [x + (-1)^i \mu_{3-i}]^2 + \sigma_{2i} y^2 \right), \end{aligned}$$

and the mean motion will take the form

$$(11) \quad n^2 = 1 + \frac{3}{2} \sum_{i=1}^2 \delta_i (2\sigma_{2i} - \sigma_{1i} - \sigma_{3i}).$$

The location of the collinear points  $L_i$ ,  $i = 1, 2, 3$  are the solution  $\Omega_x = \Omega_y = 0$  with  $y = 0$ . Using (9), (10) and (11), we obtain

$$(12) \quad f(x) = \left\{ \begin{array}{l} x \left[ 1 + \frac{3\delta_1}{2} (2\sigma_{21} - \sigma_{11} - \sigma_{31}) + \frac{3\delta_2}{2} (2\sigma_{22} - \sigma_{12} - \sigma_{32}) \right] \\ - \frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{3\delta_1(1-\mu)(x-\mu)}{r_1^5} [(2\sigma_{21} - \sigma_{11} - \sigma_{31})] \\ - \frac{\mu(x-\mu+1)}{r_2^3} - \frac{3\delta_2\mu(x-\mu+1)}{r_2^5} [(2\sigma_{22} - \sigma_{12} - \sigma_{32})] \end{array} \right\} = 0,$$

where  $r_1 = |x - \mu|$  and  $r_2 = |x - \mu + 1|$ .

The zeros of (12) represent the locations of the collinear points in the case of Euler angles of rotational motion be equal to  $\theta_i = \pi/2$ ,  $\psi_i = 0$ ,  $\varphi_i = \pi/2$ . We note that this result agrees with the result obtained when the Euler angles are equal to  $\theta_i = 0$ ,  $\psi_i + \varphi_i = \pi/2$ , see [16].

#### 4. TRIANGULAR POINTS

**4.1. Locations of the triangular points.** To determine the locations of the triangular points, we have to find the solution of  $\Omega_x = \Omega_y = 0$  when  $y \neq 0$ . Since the solutions of (9) when the primaries are spherical bodies are  $r_1 = r_2 = 1$ , thereby we can write the solution of (9) when both primary bodies are triaxial rigid bodies in the form

$$(13) \quad \begin{aligned} r_1 &= 1 + \alpha, \\ r_2 &= 1 + \beta, \end{aligned}$$

where  $\alpha, \beta \ll 1$ . Substituting (13) into (2) removing the high order terms of  $\alpha$  and  $\beta$ , we get

$$(14) \quad \begin{aligned} x_{4,5} &= -\frac{1}{2}[1 - 2\mu + 2(\alpha - \beta)], \\ y_{4,5} &= \pm \frac{\sqrt{3}}{2}[1 + \frac{2}{3}(\alpha + \beta)]. \end{aligned}$$

Now substituting (13), (14) into (9) and (10) using (11) we are able to obtain the square of the value of the mean motion  $n^2$ . Hence the values of  $\alpha$  and  $\beta$  are governed by

$$(15) \quad \begin{aligned} \alpha &= -\frac{1}{8}[11\delta_1(\sigma_{21} - \sigma_{31}) + 4\delta_2(2\sigma_{22} - \sigma_{12} - \sigma_{32}) + \frac{4\delta_2\mu}{(1-\mu)}(\sigma_{32} - \sigma_{22})], \\ \beta &= -\frac{1}{8}[4\delta_1(2\sigma_{21} - \sigma_{11} - \sigma_{32}) + 11\delta_2(\sigma_{22} - \sigma_{32}) + \frac{4\delta_1(1-\mu)}{\mu}(\sigma_{31} - \sigma_{21})]. \end{aligned}$$

Now we emphasize that we are previous development the locations of the triangular points may change depending on the values of the quantities  $\alpha$  and  $\beta$  which characterize the triaxiality of the primary bodies.

**4.2. Stability of the triangular points.** In this subsection we shall study the linear stability of the possible motion in the vicinity of the triangular points. Thereby let  $(x_0, y_0)$  be the coordinates of the triangular points  $L_4$  or  $L_5$ , and we assume that the variation  $\xi$  and  $\eta$  describes the possible motion of the infinitesimal body in the neighborhood of one the triangular points where this variation is defined by

$$(16) \quad \begin{aligned} \xi &= x - x_0, \\ \eta &= y - y_0. \end{aligned}$$

Substituting (16) into (1) and since we are studying the the linear stability of the motion, we do not consider the effect of the terms of high order of  $\xi$  and  $\eta$ . Then, considering only the linear terms the variational equations are equal to

$$\begin{aligned} \ddot{\xi} - 2n\dot{\eta} &= \Omega_{xx}^0 \xi + \Omega_{xy}^0 \eta, \\ \ddot{\eta} + 2n\dot{\xi} &= \Omega_{xy}^0 \xi + \Omega_{yy}^0 \eta. \end{aligned}$$

Here the subscripts  $x, y$  denote the partial derivatives of order two for  $\Omega$  and the superscript 0 indicates that such derivative is evaluated at one of the triangular points. Thus, the characteristic equation associated to the dynamical system given by (16) is equal to

$$(17) \quad \omega^4 + (4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0) \omega^2 + \Omega_{xx}^0 \Omega_{yy}^0 - (\Omega_{xy}^0)^2 = 0.$$

The stability of the variational dynamical system depends on the values of  $\omega$ . If the quadratic part of (17) has two not equal negative roots for  $\omega^2$ , then solution is stable otherwise it is not, see [1, 3, 5, 7, 15] for more details.

Now we deduce stability conditions for the motion in the proximity of the triangular points  $L_{4,5}$  from the characteristic equations. The motion is stable if the four roots of (17) are pure imaginary and the condition which guarantees this is

$$(18) \quad 4n^2 > (\Omega_{xx}^0 + \Omega_{yy}^0)$$

where  $\Omega_{xx}^0$ ,  $\Omega_{xy}^0$  and  $\Omega_{yy}^0$  are given at one of triangular points  $L_{4,5}$  being the Euler angles of the rotational motion equal to  $\theta_i = \pi/2$ ,  $\psi_i = 0$ ,  $\varphi_i = \pi/2$ ,  $i = 1, 2$ , and

$$\Omega_{xx}^0 = \left\{ \begin{array}{l} n^2 - \frac{1}{4}(1 - \mu) [1 - 9\alpha + 12\beta] \\ -\frac{1}{4}\mu [1 + 12\alpha - 9\beta] \\ +\frac{3\delta_1}{32}(1 - \mu) [41\sigma_{31} - 37\sigma_{21} - 4\sigma_{11}] \\ +\frac{3\delta_2}{32}\mu [41\sigma_{32} - 37\sigma_{22} - 4\sigma_{12}] \end{array} \right\}$$

$$\Omega_{xy}^0 = \left\{ \begin{array}{l} -(1-\mu) \left[ (1-3\alpha) \mp \frac{\sqrt{3}}{4}(7\alpha+4\beta-3) \right] \\ -\mu \left[ (1-3\beta) \mp \frac{\sqrt{3}}{4}(-4\alpha-7\beta+3) \right] \\ -\frac{3\delta_1}{2}(1-\mu) \left( 1 \pm \frac{5\sqrt{3}}{4} \right) [2\sigma_{21} + 4\sigma_{31} - \sigma_{11}] \\ +\frac{15\mu}{8}(1-\delta_1) \left[ \left( 1 \pm \frac{7\sqrt{3}}{4} \right) (3\sigma_{21} + \sigma_{31}) \mp 2\sqrt{3}\sigma_{21} \right] \\ -\frac{3\delta_2}{2}\mu \left( 1 \mp \frac{5\sqrt{3}}{4} \right) [2\sigma_{22} + 4\sigma_{32} - \sigma_{12}] \\ +\frac{15\delta_2}{8}\mu \left[ \left( 1 \mp \frac{7\sqrt{3}}{4} \right) (3\sigma_{22} + \sigma_{32}) \pm 2\sqrt{3}\sigma_{22} \right] \end{array} \right\}$$

$$\Omega_{yy}^0 = \left\{ \begin{array}{l} n^2 + \frac{1}{4}(1-\mu) [5 - 21\alpha + 12\beta] \\ +\frac{1}{4}\mu [5 + 12\alpha - 21\beta] \\ +\frac{3\delta_1}{32}(1-\mu) [3\sigma_{31} + 41\sigma_{21} - 44\sigma_{11}] \\ +\frac{3\delta_2}{32}\mu [3\sigma_{32} + 41\sigma_{22} - 44\sigma_{12}] \end{array} \right\}$$

where the upper sign denotes the value for  $L_4$  while the lower sign is for  $L_5$ .

Thus (18) provides necessary and sufficient condition for the stability of motion around the triangular points. Hence the triangular points are stable if this condition is satisfied otherwise they are unstable. Furthermore the solution with long and short periodic terms for stable motion

can be written in the following form

$$\xi(t) = \sum_{i=1}^2 A_i \cos(\omega_i t + \theta_0),$$

$$\eta(t) = \sum_{i=1}^2 B_i \cos(\omega_i t + \theta_0),$$

where  $A_i$ ,  $B_i$  and  $\theta_0$ ,  $i = 1, 2$ , are constants that will be evaluated from the initial conditions and  $\omega_1$  and  $\omega_2$  are the angular frequencies or mean motion with respect to long and short periodic orbits respectively.

## 5. CONCLUSIONS

In the framework of the primaries being triaxial bodies, when the Euler angles of rotational motion are equal to  $\theta_i = \pi/2$ ,  $\psi_i = 0$  and  $\varphi_i = \pi/2$ , we found that the existence and linear stability of the collinear points are agree with the results stated [16] when the Euler angles are equal to  $\theta_i = 0$ ,  $\psi_i + \varphi_i = \pi/2$  for the restricted three-body problem. The locations of the triangular points and the conditions of stable motion around these points are explicitly found.

Moreover we show that the locations of triangular points and the region of stable motion around these points may be change depending on the parameters of the triaxiality. Furthermore we emphasize that in the case of Euler angles of the rotational motion are  $\theta_i = \pi/2$  and  $\psi_i = \varphi_i = 0$ ,  $i = 1, 2$ , the corresponding results can be obtained by interchanging the parameters  $\sigma_{21}$  by  $\sigma_{31}$  as well as  $\sigma_{22}$  by  $\sigma_{32}$ .

## ACKNOWLEDGEMENTS

This work was partially supported by MINECO grant number MTM2014-51891-P and Fundación Séneca de la Región de Murcia grant number 19219/PI/14.

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