Dynamics of a tethered satellite with variable mass

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Abstract

In this paper, we present an analytical study about the dynamics of the tethered satellite system when the central gravitational field is generated by an object whose variable mass. We prove that the tethered satellite equations of motion in general case and satellite approximation are different from the classical one when the variable mass is considered. We also prove that these equations can be reduced to the classical case under the first Meshcherskii's law for variable mass. Moreover, we show that Meshcherskii's transformation is not valid for the dumbbell satellite system dynamics.

Key words Tethered satellite, Dumbbell satellite, Variable mass, Beletsky's equation, Meshcherskii's transformation.

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1 Introduction

At the end of nineteenth century, the concept of a Tethered Satellite Systems (TSS) was first established by Tsiolkovsky in 1895. He suggested a tools of creating artificial gravity that includes connecting a spacecraft to a counterweight by a long chain and spinning the entire system. In Tsiolkovsky's study, the length of the tether was 0.5 km. The first experimental investigation of a TSS in space was carried out by NASA as a part of Gemini programs (Gemini 11 and Gemini 12) in the 1960's, Beletsky and Levin (1993). On the Gemini 11 flight in 1966, the Gemini spacecraft was tethered to the Agena target vehicle by a 0.03 km. The main purpose of the linking between the two spacecraft was to test the docking maneuvers in space as well as to check the possibility of using a TSS to create artificial gravity, as proposed by Tsiolkovsky. Later in the same year, similar experiments were carried out also on the Gemini 12 flight.

The investigation of the first practical use for space TSS was a class of space sensor. Colombo et al. (1975) proposed using tethered satellites as sensors to measure both the gravitational gradient at different positions around the earth and the magnetic field surrounding the Earth. In general, TSS have been proposed for a number of space applications involving formation control of satellite clusters, orbital maneuvering of satellites, and numerous scientific applications such as observations of Earth's upper atmosphere and magnetic field. TSS are not a new concept, however, and in fact have been studied since well before the dawn of human space flight. In addition to the various theoretical studies of TSS that have been performed in the past, a number of TSS missions have already own in space, providing a solid foundation for the design of future missions and the further development of the theory underlying the behavior of TSS.

TSS are powerful systems to accomplish many distinct types of space missions that cannot be achieved with typical satellites. Therefore, early contributions into the dynamics of tethered satellites in the context of rotating space stations were performed by Austin (1965), Pengelley (1966), Chobotov (1967) and Nixon (1972). The utility of tethered satellites in gathering planetary atmospheric data has also been investigated by Hurlbut and potter (1991), Pasca and Lorenzini (1991). Polzin et al. (2002) studied a tethered array as the basis of the Terrestrial Planet Finder. Moreover, the dynamics of tethered sections of larger, rigid spacecraft have been investigated, see Quadrelli (2003). A variety of other uses are presented in the Tethers in Space, Handbook of Cosmo and Lorenzini (1997).

In recent years, with the increasing of knowledge about TSS some researchers focus their studied on the dynamics of tethered satellite systems. For instance, Wong and Misra (2008) examined the planar dynamics of a wheel-and-spoke configured multi-spacecraft system, connected together by variable length tethers, near the second Sun–Earth Lagrangian point. They also found a closed form solutions of the system under some simple tether length functions and obtained numerical results for the tether pitch librations under more complex tether length functions, along with the control effort required to maintain the desired tether librations. Zhang et al. (2012) presented several criteria on the existence of periodic solutions for a TSS in an elliptical orbit. They determined the uniqueness of periodic solutions for the TSS in a circular orbit on the basis of coincidence degree theory. Furthermore, they also addressed the conditions on the global asymptotic stability of the equilibrium states for the TSS in accordance with the Lyapunov stability theory and Barbashin–Krasovski theory. Gang et al. (2013) established that the dynamics of a rotating TSS in the vicinity of libration points are highly nonlinear and inherently unstable. They designed suboptimal output tracking controller in order to achieve the station-keeping control for the rotating tethered system based on the technique. Furthermore, the authors also presented numerical simulation results demonstrate that the capacity of the proposed controller in terms of tracking. Avanzini and Fedi (2014) discussed the relevance of eccentric reference orbits on the dynamics of a tethered formation, when a massive cable model is included in the analysis of a multi-tethered satellite formation. They stated that the examined formations in their study are hub-and-spoke (HAS) and closed-hub-and-spoke (CHAS) configurations for in-plane and Earth-facing spin planes. The authors also studied the stability of the formations by means of numerical simulation, together with the evaluation of the effects of eccentricity on tether elongation, agents relative position, and formation orientation and shape.

We will recall a dumbbell satellite systems (DSS) when the length between two stations or two bodies is a constant. According to this joint, there are many significant contraptions. For instance, Celletti and Sidorenko (2008) investigated the dumbbell satellite's attitude dynamics, when the center of mass moves on a Keplerian trajectory. They found a stable relative equilibrium position in case of circular orbits which disappears as far as elliptic trajectories are considered. They replaced the equilibrium position by planar periodic motions and they proved this motion is unstable with respect to out–of–plane perturbations. They also gave some numerical evidences of the existence of stable spatial periodic motions. Nakanishi et al. (2011) searched periodic solutions for a dumbbell model in elliptic orbits using bifurcation. They projected the trajectories of the dumbbell on the van der Pol planes as well as they used these trajectories as a tool to predict when the control of delayed feedback control will need to act to maintain the periodic motions. Much research into the dynamics of DSS was conducted by Munitsina (2007), Burov and Dugain (2011), Burov et al. (2012), Guirao et al. (2013), Maciejewski et al. (2013) and Abouelmagd et al. (2015).

There are several aspects of the dynamics of the tethered or dumbbell satellites systems are studied by many authors in the framework of both of them moves in a central gravity field generated by an object whose a constant mass. But in this paper, we will present analytical study about the tethered and dumbbell satellites systems dynamics when the central gravity field is created by a body whose variable mass. This paper will be organized as follow, actually a brief history of tethered satellite is presented in Section 1. The model of study and assumptions are designed in Section 2. The tethered satellite equations of motion in general case with variable mass are constructed in Section 3. While these equations under Meshcherskii's transformation are conducted in Section 4. Moreover, the equations of motion in satellite approximation with variable mass are also derived in Section 5. After that, in Section 6 dumbbell satellite equations of motions with variable mass are found. At the end, the conclusion is drawn in Section 7. In addition, the significant results of our work are presented through theorems from one to five.

2 Model descriptions

We assume that the dumbbell satellite is formed by massless rod of length λ varies in time with two constant masses m_1 and m_2 placed at its ends. Let c be the center of two masses moving in a gravity central field generated by body whose mass m is variable with time and located at o where the distance between o and c is ρ such that $\lambda / \rho \ll 1$. Let us consider the orbital reference frame $c$xy with origin at the dumbbell's center, and the polar coordinates of the center are (ρ,θ). While the rotation of the satellite relative to ray oc will be determined by an angle ϕ. Furthermore, we denote the reduced mass by μ and the sum of two masses by m_s where $\mu = m_1 m_2 / m_s$ in which $m_s = m_1 + m_2$, see Fig.1 for details.

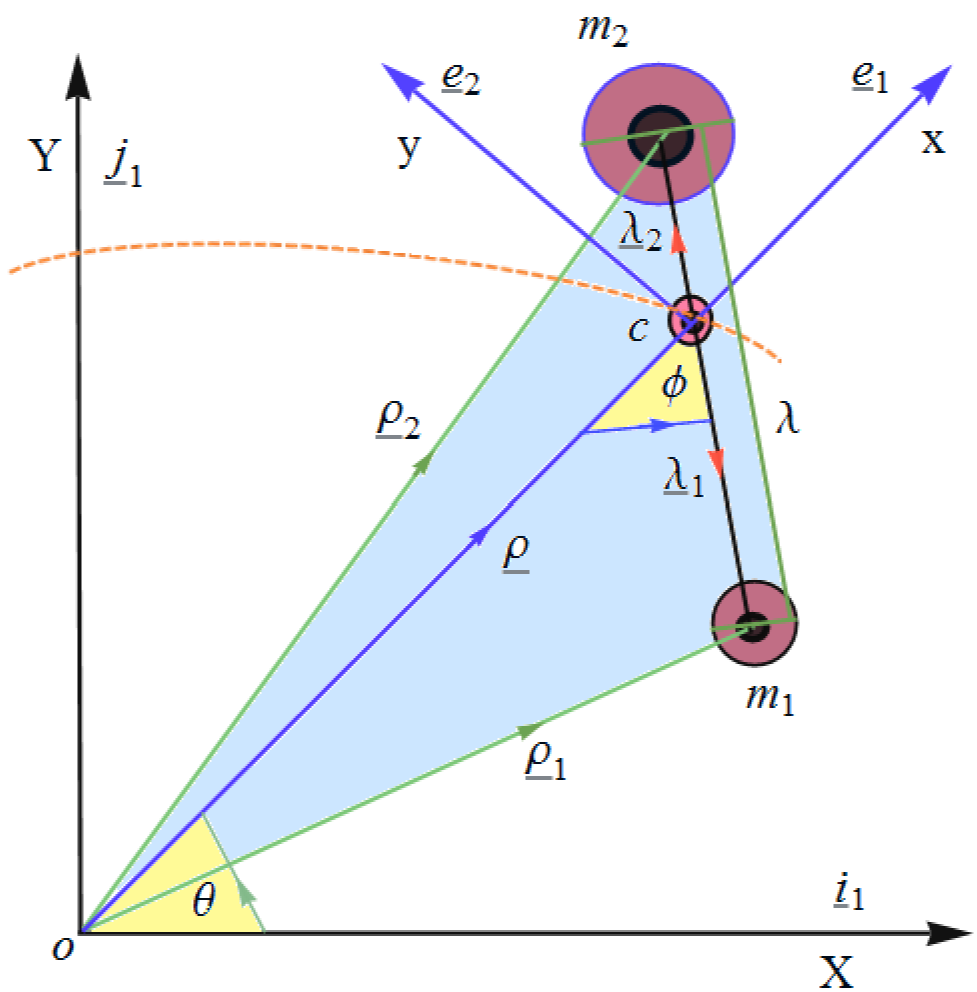

Fig.1 The Diagram of the tethered satellite model

Now, we suppose that $\underline{\rho}_i$ is the positions vectors of m_i, $(i = 1,2)$ with respect to o. While the vector $\underline{\lambda}_i$ denotes the position vector of m_i with respect to the center of mass for the dumbbell satellite.

Therefore, the magnitudes of the position vectors $\underline{\rho}_i$ are controlled by

$$\rho_i^2 = \rho^2 + \lambda_i^2 + 2\left(-1\right)^{2-i} \lambda_i \rho \cos\phi \tag{1}$$

where

$$\lambda_i(t) = m_{3-i} \lambda(t) / m_s \tag{2}$$

2.1 Potential of model

It concerns the interaction of two point masses moving under a mutual gravitational attraction described by Newton's universal law of gravitation, the gravitational potential V of our system can be written in the below form

$$V = -Gm(t)\left(\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}\right) \tag{3}$$

where G is the universal constant of gravitation which its value depends on the unit chosen.

2.2 Kinetic energy of model

To construct the kinetic energy of our system, let the vectors $\underline{e}_1$ and $\underline{e}_2$ are an orthogonal set of unit vectors with $\underline{e}_1$ corresponding to the direction from o to c. While $\underline{i}$ and $\underline{j}$ are another orthogonal set of vectors such that $\underline{i}$ is a unit vector in the direction of $X-$ axis. Hence the position vector $\underline{\rho}_i$ for the mass m_i is

$$\underline{\rho}_i = \underline{\rho} + \underline{\lambda}_i \tag{4}$$

where

$$\underline{\rho} = \rho\left(\cos\theta \underline{i} + \sin\theta\, \underline{j}\right) \tag{5.1}$$

$$\underline{\lambda}_i = (-1)^i \lambda_i \left(\cos\phi \underline{e}_1 + \sin\phi \underline{e}_2\right) \tag{5.2}$$

$$\underline{e}_i = (-1)^i \left(\cos\left(\theta + \frac{\pi}{i}\right)\underline{i} + \sin\left(\theta + \frac{\pi}{i}\right)\underline{j}\right) \tag{5.3}$$

substituting Equations (5) in to (4) we obtain

$$\underline{\rho}_i = \left[\rho\cos\theta + (-1)^i \lambda_i \cos(\theta+\phi)\right]\underline{i} + \left[\rho\sin\theta + (-1)^i \lambda_i \sin(\theta+\phi)\right]\underline{j} \tag{6}$$

since the velocity vector $\underline{v}_i$ of mass m_i is

$$\underline{v}_i = \frac{d\underline{\rho}_i}{dt} \tag{7}$$

after substituting Equation (6) into (7) with some simple calculations, the value square of velocity v_i will be governed by

$$v_i^2 = \begin{pmatrix} \dot{\rho}^2 + \rho^2\dot{\theta}^2\lambda_i^2\left(\dot{\theta}+\dot{\phi}\right)^2 - 2(-1)^i \dot{\rho}\lambda_i\left(\dot{\theta}+\dot{\phi}\right)\sin\phi \\ +2(-1)^i \rho\lambda_i\dot{\theta}\left(\dot{\theta}+\dot{\phi}\right)\cos\phi \\ +\dot{\lambda}_i^2 + 2(-1)^i \dot{\rho}\dot{\lambda}_i \cos\sin\phi + 2(-1)^i \rho\dot{\lambda}_i\dot{\theta}\sin\phi \end{pmatrix} \tag{8}$$

since the kinetic energy of the dumbbell satellite system is

$$T = \frac{1}{2}\sum_{i=1}^{2} m_i v_i^2 \tag{9}$$

substituting Equation (8) into (9), the kinetic energy can be written in the form

$$T = T_s + T_r + T_\lambda \tag{10}$$

where the first term T_s is the kinetic energy of the center's motion and the second term T_ρ is the kinetic energy of the rotation of the dumbbell around its center of mass, while the third term T_λ is the kinetic energy of the vibration of the dumbbell. The value of these terms can be written as the following

$$T_s = \frac{1}{2} m_s \left(\dot{\rho}^2 + \rho^2 \dot{\theta}^2 \right) \tag{11.1}$$

$$T_r = \frac{1}{2} \mu \lambda^2 \left(\dot{\theta} + \dot{\phi} \right)^2 \tag{11.2}$$

$$T_\lambda = \frac{1}{2} \mu \dot{\lambda}^2 \tag{11.3}$$

hence

$$T = \frac{1}{2} m_s \left(\dot{\rho}^2 + \rho^2 \dot{\theta}^2 \right) + \frac{1}{2} \mu \lambda^2 \left(\dot{\theta} + \dot{\phi} \right)^2 + \frac{1}{2} \mu \dot{\lambda}^2 \tag{12}$$

Note that if the joined rod between two masses does not change with time, the third term in the kinetic energy must be omitted.

2.4 The Lagrangian function of model

Since Lagrange's function L is defined by $L = T - V$ from Equations (3) and (12) we obtain

$$L = \frac{1}{2} m_s \left(\dot{\rho}^2 + \rho^2 \dot{\theta}^2 \right) + \frac{1}{2} \mu \lambda^2 \left(\dot{\theta} + \dot{\phi} \right)^2 + \frac{1}{2} \mu \dot{\lambda}^2 + Gm \left(\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2} \right) \tag{13}$$

Therefore, the equations of motion will be governed by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\chi}}\right)-\frac{\partial L}{\partial \chi}=0 \ , \quad \chi \in \{\rho,\theta,\phi\} \tag{14}$$

3 Equation of motion

3.1 The general equations of motion

Substituting Equation (13) into (14) when $\chi \in \{\rho,\theta,\phi\}$ the equations of motion can be written as in the following form

$$\left(m_s\rho^2+\mu\lambda^2\right)\dot{\theta}+\mu\lambda^2\dot{\phi}=p_\theta \tag{15.1}$$

$$m_s\left(\ddot{\rho}-\rho\dot{\theta}^2\right)+Gm\left(\frac{m_1\left(\rho-\lambda_1\cos\phi\right)}{\rho_1^3}+\frac{m_2\left(\rho+\lambda_2\cos\phi\right)}{\rho_2^3}\right)=0 \tag{15.2}$$

$$\mu\lambda^2\left(\ddot{\theta}+\ddot{\phi}\right)+2\mu\lambda\dot{\lambda}\left(\dot{\theta}+\dot{\phi}\right)+Gm\mu\lambda\rho\sin\phi\left(\frac{1}{\rho_1^3}-\frac{1}{\rho_2^3}\right)=0 \tag{15.3}$$

where p_θ is a constant expresses the angular momentum conservation, λ_1 and λ_2 are given as in Equation (2), while $m \equiv m(t)$ and $\lambda \equiv \lambda(t)$ are functions of time.

3.2 The center of tethered motion

Since (ρ,θ) is the coordinate of the tethered center, therefore the kinetic energy T_s is given as in Equation (11.1), while the potential of the center of mass V_s is

$$V_s=-\frac{Gmm_s}{\rho} \tag{16}$$

Consequently, the Lagrange function L_s for the center of mass can be represented in the form

$$L_s = \frac{1}{2} m_s \left(\dot{\rho}^2 + \rho^2 \dot{\theta}^2 \right) + \frac{Gmm_s}{\rho} \tag{17}$$

Substituting Equation (17) into (14) with $L = L_s$, $\chi \in \{\rho, \theta\}$ and taking account that the equations of motion can be written on the form

$$\frac{d}{dt}\left(\frac{\partial L_s}{\partial \dot{\rho}} \right) - \frac{\partial L_s}{\partial \rho} = 0 \tag{18.1}$$

$$\frac{d}{dt}\left(\frac{\partial L_s}{\partial \dot{\theta}} \right) - \frac{\partial L_s}{\partial \theta} = 0 \tag{18.2}$$

hence, the equations of motion of tethered's center will be controlled by

$$\ddot{\rho} - \rho \dot{\theta}^2 = -\frac{Gm}{\rho^2} \tag{19.1}$$

$$m_s \rho^2 \dot{\theta} = F \quad \text{or} \quad \rho^2 \dot{\theta} = h \tag{19.2}$$

where F is a constant and h is the angular momentum which is a constant too, that can be evaluated by the initial conditions.

4 Meshcherskii's transformation

We assume that the mass of the central gravitational field vary according to the Meshcherskii's law

$$m(t) = m_0 / R(t) \ , \ \dot{\theta} = d\theta / R^2 d\tau \ , \ \dot{\phi} = d\phi / R^2 d\tau \, , dt = R^2(t) d\tau$$

where $R=\sqrt{\alpha t^2+2\beta t+\gamma}$, m_0 , α , β and γ are constants, i.e. R is a functions of time.

Now we introduce a new coordinates by using Meshcherskii's transformation (Meshcherskii, 1952)

$\rho=rR(t)$, $\rho_i=r_iR(t)$, $\lambda=lR(t)$, $\lambda_i=l_iR(t)$, $(i=1,2)$ i.e. we need transform the coordinate (ρ,θ,ϕ,t) to (r,θ,ϕ,τ).

Here we refer to if $\beta^2=\alpha\gamma$ the function R becomes a linear in time ($R(t)=\sqrt{\alpha}t+\sqrt{\gamma}$) and the first law of Meshcherskii is satisfied, while the case of $\alpha=0$ leads to the second law (Jha and Shrivastava, 2001).

4.1 The equations of motion for the general case under Meshcherskii's transformation

Theorem 1: If a tethered satellite moves in a gravity field generated by a body whose variable mass, the equations of motion in general case is different from classical one and they can be reduced to the classical case under applying the first Meshcherskii's law of variable mass.

- **Proof theorem 1**

Applying Meshcherskii's transformation on Equations (15) we obtain

$$\left(m_s r^2+\mu l^2\right)\theta'+\mu l^2\phi'=p_\theta \tag{20.1}$$

$$m_s\left[r''-\left[\theta'^2+\left(\beta^2-\alpha\gamma\right)\right]r\right]$$

$$+k\left(\frac{m_1\left(r-l_1\cos\phi\right)}{r_1^3}+\frac{m_2\left(r+l_2\cos\phi\right)}{r_2^3}\right)=0 \quad (20.2)$$

$$\mu l^2\left(\theta''+\phi''\right)+2\mu l l'\left(\theta'+\phi'\right)+k\mu l r\sin\phi\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right)=0 \quad (20.3)$$

where $\left(\ \right)'\equiv d/d\tau$, $k=Gm_0$ and p_θ is a constant

these equations can be written in the form

$$\left(m_s r^2+\mu l^2\right)\theta'+\mu l^2\phi'=p_\theta \quad (21.1)$$

$$r''-\left[\left(\frac{p_\theta-\mu l^2\phi'^2}{m_s r^2+\mu l^2}\right)^2+\left(\beta^2-\alpha\gamma\right)\right]r$$

$$+\frac{k}{m_s}\left(\frac{m_1\left(r-l_1\cos\phi\right)}{r_1^3}+\frac{m_2\left(r+l_2\cos\phi\right)}{r_2^3}\right)=0 \quad (21.2)$$

$$\phi''+\frac{2\left(\mu l^3 r'+m_s r^3 l'\right)}{lr\left(m_s r^2+\mu l^2\right)}\phi'+\frac{2p_\theta}{lr\left(m_s r^2+\mu l^2\right)}\left(rl'-lr'\right)$$

$$+\frac{k\left(m_s r^2+\mu l^2\right)\sin\phi}{m_s l r}\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right)=0 \quad (21.3)$$

Equations (21) represent the equation of motion for tethered satellite with variable mass (the central gravitational field generated by a body whose variable mass). It is different from the classical case (the central gravitational field generated by a body whose constant mass) of Burov and Dugain (2011). But these equations can be reduced to the classical one if $\beta^2 = \alpha\gamma$, i.e. $R(t) = \sqrt{\alpha}t + \sqrt{\gamma}$ is a linear functions of t, ending the proof.

4.2 Tethered's center motion under Meshcherskii's transformation

Theorem 2: If a tethered satellite moves in a gravity field generated by a body whose variable mass, the trajectory of a tethered satellite mass center is different from the classical one. While if we use the first Meshcherskii's law of variable mass the solution is periodic and it coincides with the elliptical classical case.

- **Proof theorem 2**

Applying Meshcherskii's transformation on Equations (19) we obtain

$$r'' + \left(\beta^2 - \alpha\beta\right) r - r\theta'^2 = -\frac{k}{r^2} \tag{22.1}$$

$$m_s r^2 \theta' = F \quad \text{or} \quad r^2\theta' = h \tag{22.2}$$

The above equations is different from the classical one, while these equations is coincident with the classical case if $\beta^2 = \alpha\gamma$. Therefore, they can be rewritten in the following form

$$r'' - r\theta'^2 = -\frac{k}{r^2} \tag{23.1}$$

$$m_s r^2 \theta' = F \quad \text{or} \quad r^2\theta' = h \tag{23.2}$$

Equations (23) represents tethered's center motion, which will be followed a Kepler's orbit. Hence the solution is periodic and it can be written in the form

$$r = \frac{h^2 / k}{(1 + e\cos\theta)} \tag{24}$$

where e is the orbit eccentricity such that $0 < e < 1$ in the framework of elliptic orbits and θ is a true anomaly of the center of mass, which ends the proof.

5 The equations of motion in the satellite approximation

Theorem 3: If a tethered satellite moves in a gravity field generated by a body whose variable mass, the equations of motion in the satellite approximation is different from the classical one and they can be reduced to the classical equations, if the first Meshcherskii's law of variable mass is applied.

- **Proof theorem 3**

In order to construct the equations of motion in satellite approximation, firstly we will substitute Equations (1) into (3) to obtain the potential in satellite approximation. Since $\lambda \ll \rho$, we will take in our consideration neglecting all terms that contain coefficients with $o(\lambda / \rho)^3$ or more. The approximation of the potential energy is

$$V = -\frac{Gm}{\rho}\left(m_s + \frac{\mu}{2}\left(\frac{\lambda}{\rho}\right)^2 \left(3\cos^2\phi - 1\right)\right) \tag{25}$$

therefore, the Lagrangian function will be governed by

$$\begin{aligned} L = {} & \frac{1}{2} m_s \left(\dot{\rho}^2 + \rho^2 \dot{\theta}^2\right) + \frac{1}{2}\mu\lambda^2 \left(\dot{\theta} + \dot{\phi}\right)^2 + \frac{1}{2}\mu\dot{\lambda}^2 \\ & + Gm\left(\frac{m_s}{\rho} + \frac{\mu\lambda^2}{2\rho^3}\left(3\cos^2\phi - 1\right)\right) \end{aligned} \tag{26}$$

Substituting Equation (26) into (14) we obtain

$$\left(m_s\rho^2+\mu\lambda^2\right)\dot{\theta}+\mu\lambda^2\dot{\phi}=p_\theta \tag{27.1}$$

$$m_s\left[\ddot{\rho}-\rho\dot{\theta}^2\right]+k\left(\frac{m_s}{\rho^2}+\frac{3\mu\lambda^2}{2\rho^4}\left(3\cos^2\phi-1\right)\right)=0 \tag{27.2}$$

$$\mu\lambda^2\left(\ddot{\theta}+\ddot{\phi}\right)+2\mu\lambda\dot{\lambda}\left(\dot{\theta}+\dot{\phi}\right)+\frac{3k\mu\lambda^2}{\rho^3}\cos\phi\sin\phi=0 \tag{27.3}$$

Using the Meshcherskii's law of variable mass with some simple calculations, the equations of motion in satellite approximation can be written in the form

$$\left(m_s r^2+\mu l^2\right)\theta'+\mu l^2\phi'=p_\theta \tag{28.1}$$

$$m_s\left[r''-\left[\left(\beta^2-\alpha\gamma\right)+\theta'^2\right]r\right]+k\left(\frac{m_s}{r^2}+\frac{3\mu l^2}{2r^4}\left(3\cos^2\phi-1\right)\right)=0 \tag{28.2}$$

$$\mu l^2\left(\theta''+\phi''\right)+2\mu l l'\left(\theta'+\phi'\right)+\frac{3k\mu l^2}{r^3}\cos\phi\sin\phi=0 \tag{28.3}$$

The above equations is different from the classical one and represent the equations of motions of tethered satellite system in satellite approximation, while these equations are coincident with the classical case if $\beta^2=\alpha\gamma$.

Theorem 4: If a tethered satellite moves in a gravity field generated by a body whose variable mass, the equations of motion in the satellite approximation, in some special cases, can be reduced to Beletsky's equation if the first Meshcherskii's law of variable mass is applied.

- **Proof theorem 4**

The first step, we prove Theorem 3. After that we recall Equation (28.3) and rewrite it in the form

$$\left(\theta'' + \phi''\right) + 2\frac{l'}{l}\left(\theta' + \phi'\right) + \frac{3k}{r^3}\cos\phi\sin\phi = 0 \tag{29}$$

It is possible to introduce the true anomaly θ as an independent variable in Equation (29) instead of τ. Now replacing the variable τ with the variable θ in sense that $\theta' = \Omega(\theta)$ where $r^2\theta' = h$. Therefore, we obtain

$$\Omega(\theta) = \frac{k^2}{h^3}\left(1 + e\cos\theta\right)^2 \tag{30}$$

In this case, the rules of derivatives can be written in the form

$$\frac{d}{d\tau} = \Omega\frac{d}{d\theta} \tag{31.1}$$

$$\frac{d^2}{d\tau^2} = \Omega^2\frac{d^2}{d\theta^2} + \Omega\Omega_\theta\frac{d}{d\theta} \tag{31.2}$$

where $\chi_\theta, \chi_{\theta\theta}$ means that $\dfrac{d\chi}{d\theta}$ and $\dfrac{d^2\chi}{d\theta^2}$ respectively

Insert Equations (31) into (29) and using Equation (30), we obtain

$$\begin{aligned}&\left(1 + e\cos\theta\right)\phi_{\theta\theta} + 2\left[\frac{l_\theta}{l}\left(1 + e\cos\theta\right) - e\sin\theta\right]\left(1 + \phi_\theta\right)\\ &+3\cos\phi\sin\phi = 0\end{aligned} \tag{32}$$

If l is a constant ($l_\theta = 0$) or the change rate of l is very small such that $l_\theta \ll 1$ in which $l_\theta / l \approx 0$ consequently, Equation (32) can be reduced to Beletsky equation, Beletsky (1966). Therefore Equation (32) can be rewritten in the form of Beletsky equation

$$\left(1 + e\cos\theta\right)\phi_{\theta\theta} - 2e\sin\theta\phi_\theta + 3\cos\phi\sin\phi = 2e\sin\theta \qquad (33)$$

which ends the proof.

6 Dumbbell satellite equation of motion

In this section we study dumbbell satellite equations when it moves in a gravity field generated by a body whose variable mass. For a dumbbell satellite the length of weightless rod λ is a constant. Therefore the equations of motion under Meshcherskii's transformation will be controlled by

$$\left(m_s r^2 + \frac{\mu\lambda^2}{R^2}\right)\theta' + \frac{\mu\lambda^2}{R^2}\phi' = p_\theta \qquad (34.1)$$

$$m_s\left[r'' - \left[\theta'^2 + \left(\beta^2 - \alpha\gamma\right)\right]r\right] + k\left(\frac{m_1\left(r - \frac{\lambda_1}{R}\cos\phi\right)}{r_1^3} + \frac{m_2\left(r + \frac{\lambda_2}{R}\cos\phi\right)}{r_2^3}\right) = 0 \qquad (34.2)$$

$$\mu\lambda^2\left[\left(\theta'' + \phi''\right) - 2\left(\alpha t + \beta\right)\left(\theta' + \phi'\right)\right] + k\mu\lambda r R\sin\phi\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) = 0 \qquad (34.3)$$

Equations (34) is still depend on R (i.e. depends on the variable t) under Meshcherskii's transformation where $R \equiv R(t)$ is a function of time, hence this transformation is not valid for the dynamics of the dumbbell satellite.

7 Conclusion

The tethered satellite equations of motions with variable mass are found in the general case and satellite approximation as well as the equations which represent the tethered center of motion. We use Meshcherskii's transformation to reduce the aforementioned equations to the classical one. We also prove that the pass of the mass center is periodic in the framework of elliptic orbits. Moreover, we investigate that the equations of motion in the satellite approximation under Meshcherskii's transformation can be reduced to Beletsky equation when $l_\theta = 0$ or $l_\theta / l \ll 1$. At the end, we derive the dumbbell's equations with variable mass and we show that the Meshcherskii's transformation is not valid for the dynamics of dumbbell satellite system.

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