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FRACTAL DIMENSION FOR IFS-ATTRACTORS REVISITED

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ABSTRACT. One of the milestones in Fractal Geometry is the so-called Moran's Theorem, which allows the calculation of the similarity dimension of any strict self-similar set under the open set condition. In this paper, we contribute a generalized version of the Moran's theorem, which does not require the OSC to be satisfied by the similitudes that give rise to the corresponding attractor. To deal with, two generalized versions for the classical fractal dimensions, namely, the box dimension and the Hausdorff dimension, are explored and described in terms of fractal structures, which constitute a kind of uniform spaces. Finally, we posit the similarity dimension of any IFS-attractor in terms of irreducible fractal structures.

1. INTRODUCTION

In this paper, we re-explore a classical problem in Fractal Geometry, namely, how to calculate the Hausdorff dimension of the attractor of an iterated function system (IFS in the sequel). It is worth mentioning that a particular solution for such an awkward problem needs the open set condition to be satisfied by the similitudes of the corresponding IFS. The OSC hypothesis allows to control the overlapping among the self-similar copies of the whole IFS-attractor, sometimes called as *pre-fractals*. Equivalently, it is also said that under the OSC, the pieces $f_i(\mathcal{K})$ have only "small overlap", which is also called as "just touching", as pointed out in [18].

We have to trace back to the forties in order to find out the key result that allows the effective calculation of the Hausdorff dimension for strict IFS-attractors from their similarity ratios. It was firstly contributed by the Australian mathematician P.A.P. Moran, who became a Besicovitch pupil at Cambridge (c.f. [14, Theorem II]). It is worth mentioning that this theorem becomes a particular case of stronger [14, Theorem III], though it could be also deduced from [14, Theorem I], which establishes a connection between the Hausdorff dimension and the existence of finite measures having some metric properties. More specifically, it states that whether there exists a finite nonzero measure ϕ such that $\phi(R) \leq \kappa(\mathrm{diam}\,(R))^p$, where R is a $q-$dimensional cube containing a compact subset $E \subseteq \mathbb{R}^q$, for appropriate constants κ and p, then $\dim_H(E) \geq p$. In addition, he also provided an easy formula to calculate the Hausdorff dimension of attractors derived from IFSs whose similitudes have a common similarity ratio. That quantity only depends on the number of similitudes in that IFS and such a common value, as well.

A new point of view regarding *fractals* arises from the concept of *fractal structure*, which derives from asymmetric topology. A fractal structure is a kind of uniformity

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which provides better approaches of a given space as deeper stages in its structure, called *levels*, are explored. In fact, the underlying idea is to endow a fractal structure on a (topological) space, which allows to study fractal patterns therein, in contrast to understand such a space as a fractal itself depending on the self-similar properties it presents at a whole range of scales. It is also worth mentioning that fractal structures do provide a novel context where new Hausdorff type measures could be defined. In other words, the classical fractal dimension models, namely, both the box dimension and the Hausdorff dimension, remain as particular cases from some discrete models of fractal dimension for a fractal structure that are explored along this paper. Thus, the main contribution in this paper is to prove a generalized Moran's theorem for IFS-attractors which are not required to be under the OSC.

The structure of this paper is as follows. In Section 2, we provide all the mathematical background which allows to make this study self-contained. This includes the basics on IFS-attractors, the OSC, fractal structures, and in particular, the *natural* fractal structure which any (strict) self-similar set can be endowed with, as well as a brief description regarding the classical fractal dimension models, namely, both the box dimension and the Hausdorff dimension. Section 3 explains how the discrete models of fractal dimension we explore along this paper lead to generalize the classical fractal dimensions. This has been carried out through both Theorems (3.6) and (3.8). Section 4 contains the main result in this paper, namely, the generalized Moran's Theorem (see upcoming Theorem (4.1)). Finally, Section 5 summarizes the main results contributed along this paper and also provides an interesting open question regarding the OSC in terms of irreducible fractal structures.

2. Preliminaries

2.1. **IFS-attractors.** Let $k \geq 2$. By an IFS, we understand a finite collection of similitudes on $\mathbb{R}^d$, say $\mathcal{F} = \{f_1, \ldots, f_k\}$, where each self-map $f_i : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ fulfills the following equality:

$$d(f_i(x), f_i(y)) = c_i\, d(x, y), \text{ for all } x, y \in \mathbb{R}^d,$$

$c_i \in (0, 1)$ being its similarity ratio, and d denotes the Euclidean distance. There exists a unique compact nonempty subset $\mathcal{K} \subset \mathbb{R}^d$ satisfying the next Hutchinson's equation [12]:

$$\mathcal{K} = \bigcup_{i=1}^{k} f_i(\mathcal{K}). \tag{2.1}$$

$\mathcal{K}$ is called the IFS-attractor (equivalently, the *self-similar set* generated by $\mathcal{F}$) and consists of (smaller) self-similar copies $\mathcal{K}_i$ of the whole attractor $\mathcal{K}$, which are also known as *pre-fractals* of $\mathcal{K}$ [7]. In fact, it holds that $\mathcal{K}_i = f_i(\mathcal{K})$, for all $i = 1, \ldots, k$. We shall also denote $\mathcal{K}_{ij} = f_i(f_j(\mathcal{K}))$, and so on. In general, we will follow the notation used in Bandt's paper [4]: let $n \in \mathbb{N}$ and $S = \{1, \ldots, k\}$ be a finite alphabet. Moreover, let us denote $S^n = \{\mathbf{i} = i_1 \cdots i_n : i_j \in S, j = 1, \ldots, n\}$ the collection of all $n-$length *words* from S. Further, we can also write $f_{\mathbf{i}} = f_{i_1} \circ \cdots \circ f_{i_n}$, $c_{\mathbf{i}} = c_{i_1} \cdots c_{i_n}$, and $\mathcal{K}_{\mathbf{i}} = f_{\mathbf{i}}(\mathcal{K})$, as well. Accordingly, Eq. (2.1) can be rewritten equivalently in the following terms:

$$\mathcal{K} = \bigcup_{i \in S^n} \mathcal{K}_{\mathbf{i}}.$$

Letting $n \to \infty$, the so-called *address map* $\pi : S^\infty \longrightarrow \mathcal{K}$ yields as a continuous map from the set S^∞ of infinite length words (sequences) onto the IFS-attractor $\mathcal{K}$.

2.2. **The open set condition (OSC).** We say that the IFS $\mathcal{F} = \{f_1, \ldots, f_k\}$ (or $\mathcal{K}$, for short) fulfills the open set condition (OSC) if there exists a nonempty open subset $V \subseteq \mathbb{R}^d$ such that $f_i(V)$ are pairwise disjoint for $i = 1, \ldots, k$ and all of them are contained in V. Mathematically,

$$\bigcup_{i=1}^{k} f_i(V) \subseteq V, \text{where } f_i(V) \cap f_j(V) = \emptyset, \text{provided that } i \neq j.$$

As it was stated in [4], the open subset V is named a *feasible open set* of the similitudes $f_i \in \mathcal{F}$ (or of $\mathcal{K}$). The OSC was first contributed by Moran in [14] to show that the canonical Hausdorff measure is positive on the IFS-attractor $\mathcal{K}$. The reciprocal is also true, namely, a positive Hausdorff measure implies the OSC. This fact was contributed by Schief [18], who also provided the following combinatorial condition, which is equivalent to the OSC: there exists an integer N such that at most N incomparable pieces $A_{\mathbf{j}}$ of size $\geq \varepsilon$ can intersect the $\varepsilon-$neighborhood of a piece $A_{\mathbf{i}}$ of diameter ε. On the other hand, Bandt and Graf proved that the OSC can be also formulated in algebraic terms via the so-called *neighbor map condition* [3]. To deal with, they considered the following collection of *neighbor maps*:

$$\mathcal{N} = \left\{h = f_{\mathbf{i}}^{-1} f_{\mathbf{j}} : \mathbf{i}, \mathbf{j} \in S^*, i_1 \neq j_1\right\}, \text{ where } S^* = \bigcup_{n \geq 1} S^n.$$

Hence, that algebraic formulation for the OSC is as follows: there exists a constant $\kappa > 0$ such that $\|h - \mathrm{id}\| > \kappa$, for all neighbor map $h \in \mathcal{N}$. Here, the norm of any affine map g on $\mathbb{R}^n$ is given, as usual, by $\|g\| = \sup\{g(x) : \|x\| \leq 1\}$. In other words, such a condition states that *compared to their size*, two self-similar copies $A_{\mathbf{i}}$ and $A_{\mathbf{j}}$ of the attractor $\mathcal{K}$ cannot be arbitrarily close to each other.

It is worth mentioning that in [4], the OSC was described from a constructive viewpoint in terms of a suitable feasible open set, via the so-called *central open set* condition. More specifically, a point $x \in \mathbb{R}^d$ is said to be a *forbidden point* provided that there is no feasible open set V such that $x \in V$. Thus, since all the points in $H = \bigcup\{h(\mathcal{K}) : h \in \mathcal{N}\}$ are forbidden points for $\mathcal{K}$, then the authors defined therein the *central open set* for $\mathcal{F}$ as follows:

$$V_c = \{x : d(x, \mathcal{K}) < d(x, H)\}, \text{ where } d(x, A) = \inf\{|x - a| : a \in A\}.$$

Hence, it holds that $d(x, H) = \inf\{d(x, h(\mathcal{K})) : h \in \mathcal{N}\}$. The following result allows to characterize the OSC via the central open set V_c.

Theorem 2.1. *(c.f.* [4]*, Theorem 1) If the OSC is satisfied, then the central open set V_c is a valid feasible open set. Otherwise, $V_c = \emptyset$.*

As a consequence of Theorem 2.1, the OSC yields, if and only if, $\mathcal{K} \not\subset \overline{H}$ (see [4], Corollary 2).

On the other hand, since the feasible open set V and the IFS-attractor $\mathcal{K}$ may be disjoint, it holds that the OSC may be too weak to reach theoretical results regarding the fractal dimension of $\mathcal{K}$. In this way, Lalley strengthened the definition of the OSC in the following sense [13]: the strong open set condition (SOSC) is fulfilled iff it is satisfied, additionally, that $\mathcal{K} \cap V \neq \emptyset$. Schief also proved that the OSC and the SOSC are equivalent on Euclidean subspaces (c.f. [18], Theorem 2.2). This result was further extended to the case of conformal IFS in [17], and even for self-conformal random fractals [16]. It is also worth mentioning that Schief explored some conditions to reach the equality between the similarity dimension and the Hausdorff dimension of self-similar sets in the context of complete metric spaces [19]. In fact, in such a context, the OSC no longer implies the just mentioned equality.

2.3. **Fractal structures.** The concept of fractal structure was first introduced by Bandt and Retta in [5], and applied afterwards by Arenas and Sánchez-Granero to characterize non-Archimedeanly quasi-metrizable spaces (c.f. [1]). In fact, they appear naturally in several topics related to asymmetric topology. A family Γ of subsets of (a given nonempty set) X is said to be a covering (of X) provided that $X = \cup\{A : A \in \Gamma\}$. Let Γ_1 and Γ_2 be two coverings of X. By $\Gamma_1 \prec \Gamma_2$, we understand that Γ_1 is a *refinement* of Γ_2, namely, for all $A \in \Gamma_1$, there exists $B \in \Gamma_2 : A \subseteq B$. Further, $\Gamma_1 \prec\prec \Gamma_2$ means that $\Gamma_1 \prec \Gamma_2$, and additionally, that for all $B \in \Gamma_2$, it holds that $B = \cup\{A \in \Gamma_1 : A \subseteq B\}$. A fractal structure on X is a countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$, where $\Gamma_{n+1} \prec\prec \Gamma_n$, for all $n \in \mathbb{N}$. It is worth noting that the covering Γ_n is called *level* n of $\mathbf{\Gamma}$. A fractal structure induces a transitive base of quasi-uniformity (and hence, a topology) given by the transitive family of entourages $U_{\Gamma_n} = \{(x, y) \in X \times X : y \in X \setminus \bigcup_{A \in \Gamma_n, x \notin A} A\}$. To simplify, we shall let that a set can appear twice or more in any level of a fractal structure. A fractal structure is said to be finite provided that all its levels are finite coverings. There always exists a *natural* fractal structure for each IFS-attractor whose description can be stated in the following terms.

Definition 2.2 (c.f. [2], Definition 4.4)**.** Let $\mathcal{F}$ be an IFS whose associated IFS-attractor is $\mathcal{K}$. The natural fractal structure on $\mathcal{K}$ is given as the countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$, where $\Gamma_n = \{f_{\mathbf{i}}(\mathcal{K}) : \mathbf{i} \in S^n\}$.

Remark 2.3. Equivalently, the levels of the natural fractal structure for any IFS-attractor $\mathcal{K}$ could be described as follows: $\Gamma_1 = \{f_i(\mathcal{K}) : i \in S\}$, and $\Gamma_{n+1} = \{f_i(A) : A \in \Gamma_n, i \in S\}$, for all $n \in \mathbb{N}$.

On the other hand, it holds that any Euclidean space $\mathbb{R}^d$ can be always endowed with its so-called *natural fractal structure*, whose levels are given by (c.f. [9, Definition 3.1]):

$$\Gamma_n = \left\{ \left[\frac{k_1}{2^n}, \frac{k_1+1}{2^n}\right] \times \cdots \times \left[\frac{k_d}{2^n}, \frac{k_d+1}{2^n}\right] : k_1, \ldots, k_d \in \mathbb{Z} \right\}.$$

Note that such a fractal structure is a tiling consisting of $2^{-n}-$cubes on $\mathbb{R}^d$.

2.4. **The classical fractal dimensions.** Let (X, ρ) be a metric space. Along this paper, $\operatorname{diam}(A)$ will refer to the diameter of any subset A of X, namely, $\operatorname{diam}(A) = \sup\{\rho(x, y) : x, y \in A\}$, as usual. In addition, let $F \subseteq X$ and $\delta > 0$. By a $\delta-$cover of F, we understand a countable family of subsets $\{U_i : i \in I\}$ such that $F \subseteq \bigcup_{i \in I} U_i$, and $\operatorname{diam}(U_i) \leq \delta$, as well. Moreover, let $\mathcal{C}_\delta(F)$ be the collection of all $\delta-$covers of F and define the following quantity:

$$\mathcal{H}^s_\delta(F) = \inf \left\{ \sum_{i \in I} \operatorname{diam}(U_i)^s : \{U_i : i \in I\} \in \mathcal{C}_\delta(F) \right\}.$$

The expression $\mathcal{H}_H^s(F) = \lim_{\delta\to 0} \mathcal{H}_\delta^s(F)$ always exists and is named the ($s-$dimensional) Hausdorff measure of F. This allows to characterize the Hausdorff dimension of F as the point s where $\mathcal{H}_H^s(F)$ *jumps* from ∞ to zero, namely,

$$\dim_H(F) = \sup\{s : \mathcal{H}_H^s(F) = \infty\} = \inf\{s : \mathcal{H}_H^s(F) = 0\}.$$

In particular, it is worth noting that $\mathcal{H}_H^{\dim_H(F)}(F) \in \{0, d, \infty : d \in (0,\infty)\}$.

Though the Hausdorff dimension is the most accurate model for fractal dimension, the box dimension becomes more appropriate to deal with empirical applications. The (lower/upper) box dimension of $F \subseteq \mathbb{R}^d$ is defined as the (lower/upper) limit that follows:

$$\dim_B(F) = \lim_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta},$$

where $N_\delta(F)$ is the number of $\delta-$cubes that intersect F. Recall that a $\delta-$cube in $\mathbb{R}^d$ is a set of the form $\{[k_1\delta, (k_1+1)\delta] \times \cdots \times [k_d\delta, (k_d+1)\delta] : k_1, \ldots, k_d \in \mathbb{Z}\}$. In particular, δ can be discretized as 2^{-n}. Further, $N_\delta(F)$ could be calculated equivalently as one of the expressions provided in [7, Equivalent definitions 3.1]. In particular, we should mention here that $N_\delta(F)$ could be calculated as the smallest number of sets of diameter at most δ that cover F.

3. Extending the classical fractal dimensions

The main goal in this section is to show that the fractal dimension models that we explore herein from the viewpoint of fractal structures do generalize the classical fractal dimensions. They are denoted as fractal dimension III and fractal dimension IV, respectively. The reason for such a notation lies in the fact that they could be considered as further models from those studied in [9]. More specifically, we shall prove that fractal dimension III generalizes the box dimension on Euclidean subsets, whereas fractal dimension IV extends the classical Hausdorff dimension for any compact Euclidean subspace. Accordingly, the classical fractal dimensions could be calculated equivalently via these discretized models with respect to the natural fractal structure which any Euclidean subset can be always endowed with. To deal with, let $\mathbf{\Gamma}$ be a fractal structure. We define $\mathcal{A}_n(F) = \{A \in \Gamma_n : A \cap F \neq \emptyset\}$, as the collection consisting of all the elements in level n of $\mathbf{\Gamma}$ that intersect a given subset F of X, $\delta(\Gamma_n) = \sup\{\text{diam}\,(A) : A \in \Gamma_n\}$, and $\delta(F, \Gamma_n) = \sup\{\text{diam}\,(A) : A \in \mathcal{A}_n(F)\}$, as well.

Definition 3.1 (Fractal dimension models for a fractal structure)**.** Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X, ρ), F be a subset of X, and let us assume that $\delta(F, \Gamma_n) \to 0$. Additionally, let us define the following expression:

$$\mathcal{H}_{n,k}^s(F) = \inf\left\{\sum_{i\in I} \text{diam}\,(A_i)^s : \{A_i : i \in I\} \in \mathcal{A}_{n,k}(F)\right\},$$

where

(1) $\mathcal{A}_{n,3}(F) = \{\{A : A \in \mathcal{A}_l(F)\} : l \geq n\}$, if $k = 3$.

(2) $\mathcal{A}_{n,4}(F) = \Big\{\{A_i\}_{i\in I} : A_i \in \bigcup_{l\geq n} \Gamma_l, \forall\, i \in I, F \subseteq \bigcup_{i\in I} A_i, \text{Card}\,(I) < \infty\Big\}$, if $k = 4$. Here, $\text{Card}\,(A)$ refers to the number of elements I contains.

Moreover, let $\mathcal{H}_k^s(F)=\lim_{n\to\infty}\mathcal{H}_{n,k}^s(F)$, for $k=3,4$. Then the fractal dimension III (resp. IV) of F is given as the non-negative real number satisfying the following equality:

$$\dim_{\mathbf{\Gamma}}^k(F)=\sup\{s:\mathcal{H}_k^s(F)=\infty\}=\inf\{s:\mathcal{H}_k^s(F)=0\}:k=3,4.$$

It is worth noting that fractal dimension III always exists, since the sequence $\{\mathcal{H}_{n,3}^s(F):n\in\mathbb{N}\}$ is monotonic in $n\in\mathbb{N}$. Moreover, it has been assumed that $\inf\emptyset=\infty$ in Definition 3.1. For instance, if there exists a subset F of X for which $\mathcal{A}_{n,4}(F)=\emptyset$, then it holds that $\dim_{\mathbf{\Gamma}}^4(F)=\infty$.

We would like to point out that the condition $\delta(F,\Gamma_n)\to 0$, though necessary in previous Definition 3.1, is not too restrictive, as the following remark states.

Remark 3.2. Let $\mathcal{K}$ be any IFS-attractor. Then it is satisfied that $\delta(\mathcal{K},\Gamma_n)\to 0$, since the sequence of diameters $\{\delta(\Gamma_n):n\in\mathbb{N}\}$ decreases geometrically in the case of self-similar sets.

On the other hand, the next result provides a handier expression for fractal dimension III calculation purposes.

Theorem 3.3 (c.f. [8], Theorem 4.7)**.** *Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ), F be a subset of X, and let us assume that there exists $\mathcal{H}^s(F)=\lim_{n\to\infty}\mathcal{H}_n^s(F)$, where $\mathcal{H}_n^s(F)=\sum\{\mathrm{diam}\,(A)^s:A\in\mathcal{A}_n(F)\}$. Then*

$$\dim_{\mathbf{\Gamma}}^3(F)=\sup\{s:\mathcal{H}^s(F)=\infty\}=\inf\{s:\mathcal{H}^s(F)=0\}.$$

The next two results provide connections between fractal dimensions III and IV.

Lemma 3.4 (c.f. [10], Proposition 3.5 (3))**.** *Let $\mathbf{\Gamma}$ be a finite fractal structure on a metric space (X,ρ), and F be a subset of X. If $\delta(F,\Gamma_n)\to 0$, then*

$$\dim_H(F)\le\dim_{\mathbf{\Gamma}}^4(F)\le\dim_{\mathbf{\Gamma}}^3(F).$$

Corollary 3.5. *Let $\mathcal{F}$ be an IFS whose associated IFS-attractor is $\mathcal{K}$ and Γ be the natural fractal structure on $\mathcal{K}$ as a self-similar set. Then*

$$\dim_H(\mathcal{K})\le\dim_{\mathbf{\Gamma}}^4(\mathcal{K})\le\dim_{\mathbf{\Gamma}}^3(\mathcal{K}).$$

Proof. It follows as a consequence of both Remark 3.2 and Lemma 3.4, since the natural fractal structure which any IFS-attractor can be endowed with is finite. □

The main results in this section have been proved in detail for the sake of completeness. Firstly, we show that fractal dimension III generalizes the classical box dimension. To deal with, we will prove even a more general result, aimed by the next Euclidean property: for each $\delta>0$ and all subset $F\subseteq\mathbb{R}^d:\mathrm{diam}\,(F)\le\delta$, there are at most 3^d $\delta-$cubes in $\mathbb{R}^d$ that are intersected by F.

Theorem 3.6. *Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ), and F be a subset of X. Let us assume that there exist the box dimension of F as well as a natural number γ such that for all $n\in\mathbb{N}$, each subset $A\subseteq X:\mathrm{diam}\,(A)\le\delta(F,\Gamma_n)$ intersects at most to γ elements in level n of $\mathbf{\Gamma}$. In addition, if $\delta(F,\Gamma_n)\to 0$ and $\mathrm{diam}\,(A)=\delta(F,\Gamma_n)$, for all $A\in\mathcal{A}_n(F)$, then the fractal dimension III of F equals the box dimension of F, namely,*

$$\dim_B(F)=\dim_{\mathbf{\Gamma}}^3(F).$$

Proof. First of all, we affirm that

$$\dim_{\mathbf{\Gamma}}^{3}(F) = \underline{\lim}_{n\to\infty} \frac{\log N_n(F)}{-\log \delta(F,\Gamma_n)}. \tag{3.1}$$

In fact, let $\beta = \underline{\lim}_{n\to\infty} \frac{\log N_n(F)}{-\log \delta(F,\Gamma_n)}$. By definition of lower limit, there exists a subsequence

$$\left\{ \frac{\log N_{n_k}(F)}{-\log \delta(F,\Gamma_{n_k})} : n_k \in \mathbb{N} \right\} \subseteq \left\{ \frac{\log N_n(F)}{-\log \delta(F,\Gamma_n)} : n \in \mathbb{N} \right\}, \text{ such that}$$

$$\beta = \lim_{k\to\infty} \frac{\log N_{n_k}(F)}{-\log \delta(F,\Gamma_{n_k})}.$$

Let $\varepsilon > 0$ be fixed but arbitrarily chosen. Thus, there exists $n_1 \in \mathbb{N}$ such that

$$\delta(F,\Gamma_{n_k})^{-(\beta-\varepsilon)} \leq N_{n_k}(F) \leq \delta(F,\Gamma_{n_k})^{-(\beta+\varepsilon)}, \text{ for all } k \geq n_1. \tag{3.2}$$

On the other hand, by definition of the set function $\mathcal{H}^s_{n,3}$, it holds that

$$\mathcal{H}^s_{n_k,3}(F) \leq \delta(F,\Gamma_m)^s N_m(F) \leq \delta(F,\Gamma_m)^{s-(\beta+\varepsilon)}, \text{ for all } m \geq k \geq n_1, \tag{3.3}$$

since $\operatorname{diam}(A) = \delta(F,\Gamma_n)$, for all $A \in \mathcal{A}_n(F)$, and also due to Eq. (3.2). If k goes to ∞ in Eq. (3.3), then

$$\mathcal{H}^s_3(F) = \lim_{k\to\infty} \mathcal{H}^s_{n_k,3}(F) \leq \lim_{m\to\infty} \delta(F,\Gamma_m)^{s-(\beta+\varepsilon)} = \begin{cases} \infty & \text{if } s < \beta+\varepsilon \\ 0 & \text{if } s > \beta+\varepsilon \end{cases},$$

where the condition $\delta(F,\Gamma_n) \to 0$ has been applied to get the last equality. Hence,

$$\dim_{\mathbf{\Gamma}}^{3}(F) \leq \underline{\lim}_{n\to\infty} \frac{\log N_n(F)}{-\log \delta(F,\Gamma_n)} + \varepsilon. \tag{3.4}$$

Next, we shall focus on the opposite inequality. To deal with, let $\delta > 0$ be fixed but arbitrarily chosen. For all $k \in \mathbb{N}$, there exists a natural number $m_k \geq n_k$ satisfying the following expression:

$$\delta(F,\Gamma_{m_k})^{s-(\beta-\varepsilon)} \leq \delta(F,\Gamma_{m_k})^s \, N_{m_k}(F) \leq \delta + \mathcal{H}^s_{n_k,3}(F), \tag{3.5}$$

where Eq. (3.2) has been applied again. Letting $k \to \infty$, one gets that

$$\lim_{k\to\infty} \delta(F,\Gamma_{m_k})^{s-(\beta-\varepsilon)} \leq \lim_{k\to\infty} \delta(F,\Gamma_{m_k})^s N_{m_k}(F) \leq \delta + \mathcal{H}^s_3(F).$$

It is worth noting that

$$\lim_{k\to\infty} \delta(F,\Gamma_{m_k})^{s-(\beta-\varepsilon)} = \begin{cases} \infty & \text{if } s < \beta-\varepsilon \\ 0 & \text{if } s > \beta-\varepsilon \end{cases}.$$

Thus, for all $s < \beta - \varepsilon$, it holds that $\delta + \mathcal{H}^s_3(F) = \infty$. The arbitrariness of δ leads to $\mathcal{H}^s_3(F) = \infty$, for all $s < \beta - \varepsilon$. In other words,

$$\underline{\lim}_{n\to\infty} \frac{\log N_n(F)}{-\log \delta(F,\Gamma_n)} - \varepsilon \leq \dim_{\mathbf{\Gamma}}^{3}(F). \tag{3.6}$$

Therefore, Eq. (3.1) follows from both Eqs. (3.4) and (3.6). To end the proof, let $N_\delta(F)$ be the smallest number of sets of diameter at most δ that cover F. This quantity will be applied for box dimension calculation purposes. Since

$$F \subseteq \cup\{A \in \Gamma_n : A \cap F \neq \emptyset\},$$

then F can be covered by $N_n(F)$ sets with diameter at most $\delta(F,\Gamma_n)$, so

$$\underline{\dim}_B(F) \leq \underline{\lim}_{n\to\infty} \frac{\log N_n(F)}{-\log \delta(F,\Gamma_n)}.$$

Additionally, since

$$N_n(K) \leq \gamma N_{\delta_n}(F),$$

then

$$\overline{\lim}_{n\to\infty} \frac{\log N_n(F)}{-\log \delta(F,\Gamma_n)} \leq \overline{\lim}_{n\to\infty} \frac{\log N_{\delta_n}(F)}{-\log \delta_n} = \overline{\dim}_B(F).$$

The existence of the box dimension of F gives the proof. □

The next corollary follows immediately from Theorem 3.6.

Corollary 3.7 (c.f. [8], Theorem 4.15)**.** *Let $\mathbf{\Gamma}$ be the natural fractal structure on the Euclidean space $\mathbb{R}^d$, and F be a subset of $\mathbb{R}^d$. Then*

$$\underline{\dim}_B(F) = \dim_{\mathbf{\Gamma}}^3(F).$$

To justify that, just observe that the natural fractal structure on any Euclidean subspace consists of elements with diameter equal to 2^{-n} (in each level n), and also that the main hypothesis in Theorem 3.6 is satisfied by such a natural fractal structure.

Next, we state the main result in this section. It establishes that the Hausdorff dimension of any compact Euclidean subset could be calculated equivalently via a fractal dimension model for which finite coverings play a relevant role. Interestingly, such a theoretical result enables the calculation of the Hausdorff dimension in computational applications [11]. In the upcoming section, it will applied to calculate the fractal dimension of IFS-attractors which are not required to be under the OSC. In other words, both Theorems 3.6 and 3.8 will lead to generalized versions of the classical Moran's Theorem.

Theorem 3.8. *Let F be a compact subset of any Euclidean space $\mathbb{R}^d$, and $\mathbf{\Gamma}$ be the natural fractal structure on $\mathbb{R}^d$. Then the fractal dimension IV of F equals the Hausdorff dimension of F, namely:*

$$\dim_H(F) = \dim_{\mathbf{\Gamma}}^4(F).$$

Proof. Firstly, it is clear by definition that $\mathcal{A}_{n,4}(F) \subseteq \mathcal{C}_\delta(F)$, so $\dim_H(F) \leq \dim_{\mathbf{\Gamma}}^4(F)$. Accordingly, we will be focused on the opposite inequality. To deal with, let $s \geq 0$ be such that $\mathcal{H}_H^s(F) = 0$, and $\varepsilon > 0$ be fixed but arbitrarily chosen. Then there exists $\xi > 0$ such that $\mathcal{H}_\delta^s(F) < \lambda$, for all $\delta < \xi$. More specifically, λ can be chosen to be equal to $\frac{\varepsilon}{3^d d^{s/2}}$. Moreover, let $m \in \mathbb{N}$ be such that $\xi > 2^{-m}$. Thus, it becomes clear, by definition, that $\mathcal{H}_{2^{-m}}^s(F) < \lambda$. Hence, for I countable, there exists a $2^{-m}-$covering of F, say $\mathcal{D} = \{D_i : i \in I\}$, via open balls on $\mathbb{R}^d$ (see [7, Section 2.4]). The three following hold:

(i) $\operatorname{diam}(D_i) < 2^{-m}$, for all $i \in I$.
(ii) $F \subseteq \cup\{D_i : i \in I\}$.
(iii) $\sum\{\operatorname{diam}(D_i)^s : i \in I\} < \lambda$.

It is also worth mentioning that Eq. (ii) can be rewritten in the following terms, due to the compactness of F: there exists $J \subseteq I$, with J being finite, such that $F \subseteq \cup\{D_j : j \in J\}$. On the other hand, for all $D_j \in \mathcal{D}$, let $n_j \in \mathbb{N}$ be such that

$$2^{-n_j} \leq \operatorname{diam}(D_j) \leq 2^{1-n_j}. \tag{3.7}$$

By Eq. (i), it holds that $m < n_j$, for all $j \in J$. Thus, for each level n_j of $\mathbf{\Gamma}$, we can consider a covering of F via all the elements in that level which meet D_j. In other words, let $\mathcal{C}_j = \{A \in \Gamma_{n_j} : A \cap D_j \neq \emptyset\}$, and let us denote $\mathcal{C} = \cup\{\mathcal{C}_j : j \in J\}$. Further, since $\operatorname{diam}(D_j) < 2^{1-n_j}$, then there are at most 3^d elements in each covering $\mathcal{C}_j$ of D_j. The three following can be stated.

(1) For each $A \in \mathcal{C}$, there exists $n_j \in \mathbb{N} : m < n_j$, such that $A \in \Gamma_{n_j}$. This becomes clear from the definition of each family $\mathcal{C}$.

(2) $\mathcal{C}$ is a covering of F. In fact,

$$\begin{aligned} F &\subseteq \cup\{D_j : j \in J\} \subseteq \cup_{j\in J} \cup \{A : A \in \mathcal{C}_j\} \\ &= \cup\{A : A \in \cup_{j\in J}\mathcal{C}_j\} = \cup\{A : A \in \mathcal{C}\}. \end{aligned}$$

(3) $\sum\{\operatorname{diam}(A)^s : A \in \mathcal{C}\} < \varepsilon$. To prove that, recall that $\operatorname{diam}(A) = 2^{-n_j}\sqrt{d}$, for all $A \in \Gamma_{n_j}$. Hence,

$$\begin{aligned} \sum\{\operatorname{diam}(A)^s : A \in \mathcal{C}\} &= \sum\{\operatorname{diam}(A)^s : A \in \cup_{j\in J}\mathcal{C}_j\} \\ &= \sum_{j\in J}\sum_{A\in\mathcal{C}_j} \operatorname{diam}(A)^s = \sum_{j\in J}\sum_{A\in\mathcal{C}_j} 2^{-n_j s} d^{s/2} \\ &\leq 3^d d^{s/2} \sum\{\operatorname{diam}(D_j)^s : j \in J\} < \varepsilon, \end{aligned}$$

since each covering $\mathcal{C}_j$ contains 3^d elements at most.

In short, it is satisfied that $\mathcal{H}_4^s(F) = 0$, for all $s > \dim_H(F)$. This leads to $\dim_{\mathbf{\Gamma}}^4(F) \leq s$, for all $s > \dim_H(F)$, and hence, the desired inequality yields. □

4. The Theorem

The Moran's Theorem constitutes one of the milestones in Fractal Geometry. It was first contributed by P.A.P. Moran (1946), who required the pre-fractals $\mathcal{K}_i$ of an IFS-attractor not to overlap among them, in order to show that the Hausdorff dimension of the attractor $\mathcal{K}$ can be calculated by the unique solution of an equation involving only the similarity ratios. In other words, the OSC was applied therein in order to control the overlap of the pieces $\mathcal{K}_{\mathbf{i}}$. However, the result still remains quite powerful, since without a wide amount of effort, the Hausdorff dimension of a wide class of self-similar sets follows immediately. For instance, both the box dimension and the Hausdorff dimension of the standard middle third Cantor set equals $\log 2/\log 3$, since two similarities can be applied to construct it, each of them having a similarity ratio of a half. Next, we recall such a classical result.

Moran's Theorem (1946). *Let $\mathcal{F}$ be an Euclidean IFS whose associated IFS-attractor is $\mathcal{K}$. Let c_i be the similarity ratio associated with each similarity $f_i \in \mathcal{F}$, and let us assume that $\mathcal{F}$ is under the OSC. If s is the solution of the equation $\sum_{i\in I} c_i^s = 1$, then*

$$\dim_B(\mathcal{K}) = \dim_H(\mathcal{K}) = s, \text{ and it also holds that } \mathcal{H}_H^s(F) \in (0, \infty).$$

The unique (positive) solution s of the equation $\sum_{i\in I} c_i^s = 1$ is usually called as *similarity dimension.* Thus, if s is the similarity dimension, then it follows that $\mathcal{H}_H^s(\mathcal{K}) \in (0, \infty)$. A proof for Moran's Theorem can be found in Falconer's book (see [7, Subsection 9.2]), though the reader may check that the proof for a lower bound for the Hausdorff dimension, namely, $s \leq \dim_H(\mathcal{K})$, becomes quite awkward. Moreover, whether the OSC is not fulfilled by $\mathcal{F}$, then the calculation of the Hausdorff dimension of $\mathcal{K}$ becomes harder and only some partial results are known (see, for instance, [6, 15]). Nevertheless, even in that situation, it holds that both the box dimension and the Hausdorff dimension of $\mathcal{K}$ can be approximated via fractal dimension III, which still equals the similarity dimension. Next, we provide the main theoretical result in this paper, which provides a generalized version of the classical Moran's Theorem.

Theorem 4.1. *Let $\mathcal{F}$ be an Euclidean IFS whose associated IFS-attractor is $\mathcal{K}$. Let us assume that c_i is the similarity factor associated with each similarity $f_i \in \mathcal{F}$, and let $\mathbf{\Gamma}$ be the natural fractal structure on $\mathcal{K}$ as a self-similar set. If s is the similarity dimension, then*

(1) *[c.f. [8], Theorem 4.20] $\dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = s$, and it holds that $\mathcal{H}_3^s(\mathcal{K}) \in (0, \infty)$.*
(2) *In addition, if $\mathcal{F}$ is under the OSC, then*
$$\dim_B(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = \dim_H(\mathcal{K}) = s.$$
Moreover, for that s, it holds that $\mathcal{H}^s(\mathcal{K}), \mathcal{H}_3^s(\mathcal{K}), \mathcal{H}_4^s(\mathcal{K}) \in (0, \infty)$.

Proof. First of all, it becomes clear that the IFS-attractor $\mathcal{K}$ is the unique non-empty compact subset of $\mathbb{R}^d$ satisfying the following Hutchinson's equation:
$$\mathcal{K} = \bigcup \{\mathcal{K}_{\mathbf{i}} : \mathbf{i} \in S^n\}.$$
Hence,

(1) Observe that $\mathcal{A}_{n,3}(\mathcal{K}) = \{\Gamma_m : m \geq n\}$. Further, let s be a non-negative real number such that $\sum_{i \in I} c_i^s = 1$. In addition to that, it holds that $c_{\mathbf{i}}$ is the similarity factor associated with $f_{\mathbf{i}}$. Thus, it follows that $\text{diam}\,(\mathcal{K}_{\mathbf{i}}) = c_{\mathbf{i}}\,\text{diam}\,(\mathcal{K})$, for all $\mathbf{i} \in I^l$. It is also worth mentioning that
$$\sum \{c_{\mathbf{i}}^s : \mathbf{i} \in I^l\} = \sum_{i_1 \in I} c_{i_1}^s \cdots \sum_{i_l \in I} c_{i_l}^s = \sum_{\mathbf{i} \in I^l} c_{\mathbf{i}}^s = 1,$$
for all $\mathbf{i} = i_1 \cdots i_l \in S^l$. Accordingly, for all natural number n, the following calculations hold:
$$\begin{aligned} \mathcal{H}_{n,3}^s(\mathcal{K}) &= \inf \left\{ \sum \text{diam}\,(A)^s : A \in \Gamma_m, m \geq n \right\} \\ &= \inf \left\{ \sum \text{diam}\,(\mathcal{K}_{\mathbf{i}})^s : \mathbf{i} = i_1 \ldots i_m \in I^m, m \geq n \right\} \\ &= \inf \left\{ \sum c_{\mathbf{i}}^s \,\text{diam}\,(\mathcal{K})^s : \mathbf{i} \in I^m, m \geq n \right\}. \end{aligned}$$
Thus, since $\mathcal{H}_3^s(\mathcal{K}) = \text{diam}\,(\mathcal{K})^s \notin \{0, \infty\}$, then $\dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = s$.

(2) Firstly, since the natural fractal structure which any IFS-attractor can be always equipped with is finite, then it holds that
$$\mathcal{A}_{n,3}(\mathcal{K}) \subseteq \mathcal{A}_{n,4}(\mathcal{K}),$$
for all natural number n. Hence, it becomes clear that $\mathcal{H}_{n,4}^s(\mathcal{K}) \leq \mathcal{H}_{n,3}^s(\mathcal{K})$. Letting $n \to \infty$, it follows that
$$\dim_{\mathbf{\Gamma}}^4(\mathcal{K}) \leq \dim_{\mathbf{\Gamma}}^3(\mathcal{K}). \tag{4.1}$$
Moreover, since the natural fractal structure for any IFS-attractor also satisfies that $\delta(\mathcal{K}, \Gamma_n) \to 0$, then we can state that
$$\dim_H(\mathcal{K}) \leq \dim_{\mathbf{\Gamma}}^4(\mathcal{K}), \tag{4.2}$$

since all the coverings in each family $\mathcal{A}_{n,4}(\mathcal{K})$ are $\delta-$coverings for appropiately chosen scales δ. Hence, both Eqs. (4.1) and (4.2) lead to

$$\dim_H(\mathcal{K}) \leq \dim_{\mathbf{\Gamma}}^4(\mathcal{K}) \leq \dim_{\mathbf{\Gamma}}^3(\mathcal{K}). \tag{4.3}$$

In addition, if $\mathcal{F}$ is under the OSC, then the Moran's Theorem as well as Theorem (4.1)(1) allows to affirm that $s = \dim_{\mathbf{\Gamma}}^3(\mathcal{K})$, which becomes the unique solution of the equation $\sum_{i\in I} c_i^s = 1$, equals both the box dimension and the Hausdorff dimension, as well. Hence, the result follows, since the next chain of inequalities holds:

$$\dim_H(\mathcal{K}) \leq \dim_{\mathbf{\Gamma}}^4(\mathcal{K}) \leq \dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \dim_B(\mathcal{K}) = \dim_H(\mathcal{K}).$$

To conclude this proof, just note that

$$\mathcal{H}^s(\mathcal{K}) \leq \mathcal{H}_4^s(\mathcal{K}) \leq \mathcal{H}_3^s(\mathcal{K}).$$

Further, recall that $\mathcal{H}_3^s(\mathcal{K}) < \infty$ by Theorem (4.1)(1), and $\mathcal{H}^s(\mathcal{K}) > 0$, due to Moran's Theorem. The result follows.

□

We would like also to point out that Theorem 4.1 (1) can be expressed in more general terms. Indeed, it could be weakened with $\mathcal{F}$ being an IFS on a complete metric space.

It is worth mentioning that Theorem 4.1 becomes quite useful for the explicit calculation regarding the fractal dimension of a given subset. In this way, the following example highlights that it is possible to calculate the fractal dimension III for IFS-attractors whose Hausdorff dimension becomes hard to calculate or to estimate.

Example 4.2. Let $\mathcal{K}$ be the unique IFS-attractor on the closed unit interval $[0,1]$ satisfying the next Hutchinson's equation:

$$\mathcal{K} = \bigcup_{i\in I} \mathcal{K}_i,$$

where the IFS $\mathcal{F}$ which gives rise to $\mathcal{K}$ consists of the three following similitudes $f_i : [0,1] \longrightarrow [0,1]$:

$$f_i(x) = \begin{cases} \frac{3}{10}x & \text{if } i = 1; \\ \frac{1}{10}(1+3x) & \text{if } i = 2; \\ \frac{1}{10}(7+3x) & \text{if } i = 3. \end{cases} \tag{4.4}$$

It is worth noting that the OSC is not satisfied by $\mathcal{F}$, since the three pre-fractals $\mathcal{K}_i$ of $\mathcal{K}$ do overlap among them. Thus, the Moran's Theorem cannot be applied to calculate the Hausdorff dimension of $\mathcal{K}$. However, Theorem 4.1 can still be applied for fractal dimension calculation purposes. In fact, $\dim_{\mathbf{\Gamma}}^3(\mathcal{K})$ yields as the solution of the equation $3\left(\frac{3}{10}\right)^s = 1$, namely,

$$\dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \frac{\log \frac{1}{3}}{\log \frac{3}{10}} \simeq 0.912.$$

5. Conclusion

In this section, we summarize all the results contributed along this paper and also state an open question which may lead, interestingly, to a weaker version of the OSC in terms of irreducible fractal structures.

Theorem 5.1. *Let $\mathcal{F} = \{f_1, \ldots, f_k\}$ be an Euclidean IFS whose associated IFS-attractor is $\mathcal{K}$, $\mathbf{\Gamma}$ be the natural fractal structure on $\mathcal{K}$ as a self-similar set, and c_i be the similarity ratio associated with each similarity $f_i \in \mathcal{F}$. In addition, let us consider the following statements:*

(i) SOSC.
(ii) OSC.
(iii) $\mathcal{H}^s(\mathcal{K}) > 0$.
(iv) *s is the similarity dimension, namely, $P(s) = 0$, where $P(\alpha) = \sum_{i=1}^k c_i^\alpha - 1$.*
(v) $\dim_B(\mathcal{K}) = s$.
(vi) $\dim_H(\mathcal{K}) = s$.
(vii) $\dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = s$.

Then the next chain of implications and equivalences yields:

$$(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi) \Leftrightarrow (vii).$$

To end the paper, let us be focused on the condition (vii) in previous Theorem 5.1. Let Γ be a covering of X. We say that Γ is irreducible provided that it has no proper subcoverings (see c.f. [20, Problem 20D]). Thus, a fractal structure $\mathbf{\Gamma}$ is said to be irreducible provided that all its levels are irreducible coverings.

Lemma 5.2. *Let $\mathcal{F} = \{f_1, \ldots, f_k\}$ be an IFS whose associated IFS-attractor is $\mathcal{K}$, $\mathbf{\Gamma}$ be the natural fractal structure on $\mathcal{K}$ as a self-similar set, and s be the similarity dimension. Moreover, let us assume that $\mathbf{\Gamma}$ is an irreducible fractal structure. Then $\dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = s$.*

Proof. By Corollary 3.5, $\dim_{\mathbf{\Gamma}}^4(\mathcal{K}) \leq \dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = s$, where s is the similarity dimension. Thus, if $\dim_{\mathbf{\Gamma}}^4(\mathcal{K}) < s$, then $\mathcal{H}_4^s(\mathcal{K}) = 0$. Moreover, it holds that $\mathcal{H}_{n,4}^s(\mathcal{K}) = \mathcal{H}_{1,4}^s(\mathcal{K})$, for each $n \in \mathbb{N}$, since any element $f_{\mathbf{i}}(\mathcal{K}) \in \Gamma_k : \mathbf{i} \in S^k$, for some $k \in \mathbb{N}$, could be replaced by $\{f_{\mathbf{i}j}(\mathcal{K}) : j \in S\}$, where $\mathrm{diam}\,(f_{\mathbf{i}}(\mathcal{K}))^s = \sum_{j \in S} \mathrm{diam}\,(f_{\mathbf{i}j}(\mathcal{K}))^s$, as many times as needed. Letting $n \to \infty$, $0 = \mathcal{H}_4^s(\mathcal{K}) = \mathcal{H}_{1,4}^s(\mathcal{K})$ yields.

On the other hand, let $\varepsilon = \frac{1}{2} \sum_{i \in S} \mathrm{diam}\,(f_i(\mathcal{K}))^s$, and $\mathcal{A}$ be a finite covering of $\mathcal{K}$ by elements of $\bigcup_{n \in \mathbb{N}} \Gamma_n$, where $\sum_{A \in \mathcal{A}} \mathrm{diam}\,(A)^s < \varepsilon$. In addition, let n be the greatest level containing (at least) one element of $\mathcal{A}$. Thus, for each $f_{\mathbf{i}}(\mathcal{K}) \in \mathcal{A} \cap \Gamma_m$, we can replace $f_{\mathbf{i}}(\mathcal{K})$ by $\{f_{\mathbf{j}}(\mathcal{K}) : \mathbf{j} \in S^n, \mathbf{i} \sqsubseteq \mathbf{j}\}$, where $\mathrm{diam}\,(f_{\mathbf{i}}(\mathcal{K}))^s = \sum_{\mathbf{j} \in S^n, \mathbf{i} \sqsubseteq \mathbf{j}} \mathrm{diam}\,(f_{\mathbf{j}}(\mathcal{K}))^s$, and $\mathbf{i} \sqsubseteq \mathbf{j}$ denotes that $\mathbf{i}$ is a prefix of $\mathbf{j}$. Accordingly, from $\mathcal{A}$ we can construct a covering $\mathcal{A}' \subseteq \Gamma_n$, such that $\sum_{A \in \mathcal{A}'} \mathrm{diam}\,(A)^s = \sum_{A \in \mathcal{A}} \mathrm{diam}\,(A)^s < \varepsilon$. Since Γ_n is an irreducible covering, then $\mathcal{A}' = \Gamma_n$, but $\sum_{A \in \Gamma_n} \mathrm{diam}\,(A)^s = 2\varepsilon$, which becomes a contradiction. □

Open question 5.1. *Under the same hypothesis as in Lemma 5.2, and s being the similarity dimension, is it true, in general, that $\mathbf{\Gamma}$ is an irreducible fractal structure $\Longleftrightarrow \dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^4(K) = s$?*

In afirmative case, since the OSC implies that $\dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^4(K) = s$, we could even provide a weaker (at least at a first glance) version of the OSC via irreducible fractal structures.

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