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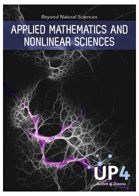

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New and Modified Eccentric Indices of Octagonal Grid O_n^m

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Abstract

The eccentricity ε_u of vertex u in a connected graph G, is the distance between u and a vertex farthermost from u. The aim of the present paper is to introduce new eccentricity based index and eccentricity based polynomial, namely modified augmented eccentric connectivity index and modified augmented eccentric connectivity polynomial respectively. As an application we compute these new indices for octagonal grid O_n^m and we compare the results obtained with the ones obtained by other indices like Ediz eccentric connectivity index, modified eccentric connectivity index and modified eccentric connectivity polynomial $ECP(G,x)$.

Keywords: Degree, Eccentricity, Ediz eccentric connectivity index, modified eccentric connectivity index, modified eccentric connectivity polynomial $ECP(G,x)$, octagonal grid O_n^m
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1 Introduction

In recent years graph theory is extensively used in the branch of mathematical chemistry and some people call it as *chemical graph theory* because this theory is related with the practical applications of graph theory for solving the molecular problems. In mathematics a model of chemical system portrays a chemical graph that deals to explain the relations between its segments such as its atoms, bonds between atoms, cluster of atoms or molecules.

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A *connected simple graph* $G = (V(G) \cup E(G))$ is a graph consisting of n vertices $(V(G))$ and m edges $(E(G))$ in which there is path between any of two its vertices. A *network* is merely a connected graph consisting of no multiple edges and loops. The *degree of a vertex* v in G is the number of edges which are incident to the vertex v and will be represented by d_v. In a graph G, if there is no repetition of vertices in $(u-v)$ walk then such kind of walk is called $(u-v)$ *path*. The number of edges in $(u-v)$ path is called its *length*. The *distance* $d(u,v)$ from vertex u to vertex v is the length of a shortest $(u-v)$ path in a graph G where $u,v \in G$. In a connected graph G, the *eccentricity* ε_v of a vertex v is the distance between v and a vertex furthest from v in G. Thus, $\varepsilon_v = \max_{v \in V(G)} d(v,u)$. Therefore the maximum eccentricity over all vertices of G is the *diameter* of G which is denoted by $D(G)$.

A graph can be recognized by a different type of numeric number, a polynomial, a sequence of numbers or a matrix. A *topological index* is a numeric quantity that is associated with a graph which characterize the topology of graph and is invariant under graph automorphism. Over the years topological indices like Wiener index Balabans index [24–26], Hosoya index [16,17], Randić index [19] and so on, have been studied extensively and recently the research and interest in this area has been increased exponentially. See too for more information [3, 13, 14, 18, 21, 23].

There are some major classes of topological indices such as *distance based topological indices*, *eccentricity based topological indices*, *degree based topological indices* and *counting related polynomials* and indices of graphs. In this article we shall consider the eccentricity based indices. We note that in [5] is introduced the *total eccentricity* of a graph G and is defined as the sum of eccentricities of all vertices of a given graph G and denote by $\zeta(G)$. It is easy to see that for a k–regular graph G is held $\zeta(G) = k\zeta(G)$.

The *Eccentric-connectivity index* $\xi(G)$ which was proposed by Sharma, Goswami and Madan defined as [20]:

$$\xi(G) = \sum_{u \in V(G)} d_u \varepsilon_u, \tag{1}$$

Another very relevant and special eccentricity based topological index is *connective Eccentric index* $C^{\xi}(G)$ that was proposed by Gupta et al. in [11]. The *connective eccentric index* is defined as.

$$C^{\xi}(G) = \sum_{u \in V(G)} \frac{d_u}{\varepsilon_u}, \tag{2}$$

In 2010, A. R. Ashrafi and M. Ghorbani [1] introduces the so called *modified eccentric connectivity index* $\xi_c(G)$ and it is defined as

$$\xi_c(G) = \sum_{v \in V(G)} (S_v \varepsilon_v), \tag{3}$$

where $S_v = \sum_{u \in N(v)} d_u$ that is S_v is the sum of degrees of all vertices adjacent to vertex v.

In 2010, S. Ediz et al., [8], defined *Ediz eccentric connectivity index* of G as

$$^{E}\xi^{c}(G) = \sum_{v \in V(G)} \left(\frac{S_v}{\varepsilon_v}\right), \tag{4}$$

Similar to other topological polynomials, the corresponding polynomial, that is, the *modified eccentric connectivity polynomial* of a graph, is defined as, [6]:

$$\xi_c(G,x) = \sum_{u \in V(G)} S_u x^{\varepsilon_u}, \tag{5}$$

so that the *modified eccentric connectivity index* is the first derivative of this polynomial for $x = 1$.

Motivated by these above eccentricity indices, in this article we introduce what we call *modified augmented eccentric connectivity index* ${}^{MA}\xi(G)$, as

$$ {}^{MA}\xi^c(G) = \sum_{v \in V(G)} (M_v \varepsilon_v), \tag{6} $$

where $M_v = \prod_{u \in N(v)} d_u$ that is denotes the product of degrees of all neighbors of vertex v of G.

In the same way, we define the *modified augmented eccentric connectivity polynomial* ${}^{MA}\xi^c(G,x)$, as

$$ {}^{MA}\xi^c(G,x) = \sum_{v \in V(G)} M_v \, x^{\varepsilon_v} \tag{7} $$

For more information and properties of eccentricity based topological index, see for instance [2,7,9,10,12,15,27].

The aim of this paper is is the introduction of the augmented eccentric connectivity index and modified augmented eccentric connectivity polynomial. As an application we shall compute these new indices for octagonal grid O_n^m and we shall compare the results obtained with the ones obtained by other indices like Ediz eccentric connectivity index, modified eccentric connectivity index and modified eccentric connectivity polynomial $ECP(G,x)$ via their computation too.

2 Octagonal Grid O_n^m

In [4] and [22] Diudea et al. constructed a C_4C_8 net as a trivalent decoration made by alternating squares C_4 and octagons C_8 in two different ways. One is by alternating squares C_4 and octagons C_8 in different ways denoted by $C_4C_8(S)$ and other is by alternating rhombus and octagons in different ways denoted by $C_4C_8(R)$. We denote $C_4C_8(R)$ by O_n^m see Figure 1. In [21] they also called it as *the Octagonal grid.*

For $n,m \geq 2$ the Octagonal grid O_n^m, is the grid with m rows and n columns of octagons. The symbols $V(O_n^m)$ and $E(O_n^m)$ will denote the vertex set and the edge set of O_n^m, respectively.

$$ \begin{aligned} V(O_n^m) = & \{u_s^t : 1 \leq s \leq n,\ 1 \leq t \leq m+1\} \cup \{v_s^t : 1 \leq s \leq n;\ 1 \leq t \leq m+1\} \\ & \cup \{w_s^t : 1 \leq s \leq n+1,\ 1 \leq t \leq m\} \cup \{y_s^t : 1 \leq s \leq n+1,\ 1 \leq t \leq m\}. \end{aligned} $$

$$ \begin{aligned} E(O_n^m) = & \{u_s^t v_s^t :\ 1 \leq s \leq n,\ 1 \leq t \leq m+1\} \cup \{u_s^t w_s^t ; 1 \leq s \leq n,\ 1 \leq t \leq m\} \\ & \cup \{w_s^t y_s^t : 1 \leq s \leq n+1,\ 1 \leq t \leq m\} \cup \{v_s^t w_{s+1}^t : 1 \leq s \leq n,\ 1 \leq t \leq m\} \\ & \cup \{v_s^t y_{s+1}^{t-1} : 1 \leq s \leq n,\ 2 \leq t \leq m+1\} \cup \{u_s^{t+1} y_s^t :\ 1 \leq s \leq n,\ 2 \leq t \leq m\}. \end{aligned} $$

In this paper, we consider O_n^m with $n = m$.

3 Statement of main results

As we have said previously for O_n^m with $n = m$we shall compute modified eccentric connective index, Ediz eccentric connectivity index, modified eccentric connective polynomial, modified augmented eccentric connective index and modified augmented eccentric connectivepolynomial and we shall compare the results obtained. For this we have discussed two cases of n, when $n \equiv 0 (\mod 2)$ and when $n \equiv 1 (\mod 2)$. Also to avoid any ambiguity related to Figure 1 note that the vertices $u_s^t = \mathrm{u}_s^t$.

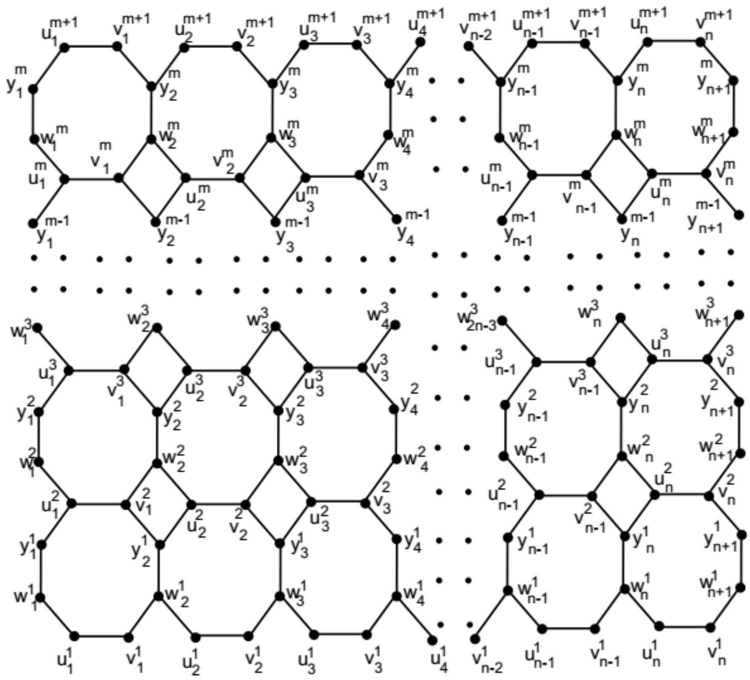

Fig. 1 The Octagonal grid O_n^m.

Theorem 1. For every $n \geq 4$ and $n \equiv 0(\mod 2)$ consider the graph of $G \cong O_n^m$, with $n = m$. Then the modified eccentric connectivity index $\xi_c(G)$ of G is equal to

$$\begin{aligned}
\xi_c(O_n^m) &= 225n^2 - 112n + 28 \\
&+ 36\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=s+1}^{\frac{n}{2}+1}\{4n-3(s-1)-t\}\right] + 36\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=2}^{s}\{4(n+1)-s-3t\}\right] \\
&+ 36\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1}\{3(n-t)+s+2\}\right] + 36\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=n-s+2}^{\frac{n}{2}+1}\{n+3s-t-1\}\right] \\
&+ 36\sum_{s=2}^{\frac{n}{2}-1}\left[\sum_{t=\frac{n}{2}+2}^{n-s+1}\{3(n-s)+t+1\}\right] + 36\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=n+2-s}^{n}\{n-s+3t-2\}\right] \\
&+ 36\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=\frac{n}{2}+2}^{s}\{4s+t-n-3\}\right] + 36\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n}\{s+3t-4\}\right].
\end{aligned}$$

Proof. Let G be the graph of O_n^m. Note that graph of O_n^m is a symmetric about reflection and rotation at right angles. Thus the eccentricities $\varepsilon_{u_s^t} = \varepsilon_{v_{n+1-s}^t}$ and from the symmetry at right angles we can obtain that the eccentricities $\varepsilon_{y_s^t} = \varepsilon_{u_t^s}$, $\varepsilon_{w_s^t} = \varepsilon_{v_t^s}$. Therefore, from Table 1 and formula (3), given below, the modified eccentric connectivity index $\xi_c(G)$ of O_n^m is equal to

$$\xi_c(G) = \sum_{v \in V(G)}\left(S_v \varepsilon_v\right) = 4\sum_{u_s^t \in V(G)}\left(S_{u_s^t}\varepsilon_{u_s^t}\right),$$

$$\xi_c(O_n^m) = 4\left[2\times 4\times 4n + 2\sum_{s=2}^{\frac{n}{2}+1}5\{4n+1-s\} + 2\sum_{s=\frac{n}{2}+2}^{n}5\{3n+s-1\}\right]$$

$$+4\Bigg[\sum_{t=2}^{\frac{n}{2}+1} 7(4n-t)+\sum_{s=2}^{\frac{n}{2}}\Bigg[\sum_{t=s+1}^{\frac{n}{2}+1} 9\{4n-3(s-1)-t\}\Bigg]$$

$$+\sum_{s=2}^{\frac{n}{2}}\Bigg[\sum_{t=2}^{s} 9\{4(n+1)-s-3t\}\Bigg]+\sum_{s=\frac{n}{2}+1}^{n-1}\Bigg[\sum_{t=2}^{n-s+1} 9\{3(n-t)+s+2\}\Bigg]$$

$$+\sum_{s=\frac{n}{2}+2}^{n}\Bigg[\sum_{t=n-s+2}^{\frac{n}{2}+1} 9\{n+3s-t-1\}\Bigg]+\sum_{t=\frac{n}{2}+2}^{n} 7\{3(n-1)+t+1\}$$

$$+\sum_{s=2}^{\frac{n}{2}-1}\Bigg[\sum_{t=\frac{n}{2}+2}^{n-s+1} 9\{3(n-s)+t+1\}\Bigg]+\sum_{s=2}^{\frac{n}{2}}\Bigg[\sum_{t=n+2-s}^{n} 9\{n-s+3t-2\}\Bigg]$$

$$+\sum_{s=\frac{n}{2}+2}^{n}\Bigg[\sum_{t=\frac{n}{2}+2}^{s} 9\{4s+t-n-3\}\Bigg]+\sum_{s=\frac{n}{2}+1}^{n-1}\Bigg[\sum_{t=s+1}^{n} 9\{s+3t-4\}\Bigg]\Bigg].$$

After some easy calculations we get

$$\xi_c(O_n^m) = 225n^2-112n+28$$

$$+36\sum_{s=2}^{\frac{n}{2}}\Bigg[\sum_{t=s+1}^{\frac{n}{2}+1}\{4n-3(s-1)-t\}\Bigg]+36\sum_{s=2}^{\frac{n}{2}}\Bigg[\sum_{t=2}^{s}\{4(n+1)-s-3t\}\Bigg]$$

$$+36\sum_{s=\frac{n}{2}+1}^{n-1}\Bigg[\sum_{t=2}^{n-s+1}\{3(n-t)+s+2\}\Bigg]+36\sum_{s=\frac{n}{2}+2}^{n}\Bigg[\sum_{t=n-s+2}^{\frac{n}{2}+1}\{n+3s-t-1\}\Bigg]$$

$$+36\sum_{s=2}^{\frac{n}{2}-1}\Bigg[\sum_{t=\frac{n}{2}+2}^{n-s+1}\{3(n-s)+t+1\}\Bigg]+36\sum_{s=2}^{\frac{n}{2}}\Bigg[\sum_{t=n+2-s}^{n}\{n-s+3t-2\}\Bigg]$$

$$+36\sum_{s=\frac{n}{2}+2}^{n}\Bigg[\sum_{t=\frac{n}{2}+2}^{s}\{4s+t-n-3\}\Bigg]+36\sum_{s=\frac{n}{2}+1}^{n-1}\Bigg[\sum_{t=s+1}^{n}\{s+3t-4\}\Bigg].$$

□

Table 1 Partition of vertices of the type u_s^t of O_n^m based on degree sum and eccentricity of each vertex when $n \equiv 0($ mod $2)$.

Representative	$S_{u_s^t}$	eccentricity	Range	Frequency
u_s^t	4	$4n-s+1$	$t=1,\ n+1;\ s=1$	2
u_s^t	5	$4n-s+1$	$t=1,\ n+1,$ $2 \leq s \leq \frac{n}{2}+1$	n
u_s^t	5	$3n+s-1$	$t=1,\ n+1,$ $\frac{n}{2}+2 \leq s \leq n$	$n-2$
u_s^t	7	$4n-3(s-1)-t$	$s=1,$ $2 \leq t \leq \frac{n}{2}+1$	$\frac{n}{2}$
u_s^t	9	$4n-3(s-1)-t$	$2 \leq s \leq \frac{n}{2},$ $s+1 \leq t \leq \frac{n}{2}+1$	$\frac{n^2}{8}-\frac{3n}{4}$
u_s^t	9	$4(n+1)-s-3t$	$2 \leq s \leq \frac{n}{2},$ $2 \leq t \leq s$	$\frac{n^2}{8}-\frac{n}{4}$
u_s^t	9	$3n+s-3t+2$	$\frac{n}{2}+1 \leq s \leq n-1,$ $2 \leq t \leq n+1-s$	$\frac{n^2}{8}-\frac{n}{4}$
u_s^t	9	$n+3s-t-1$	$\frac{n}{2}+1 \leq s \leq n,$ $n-s+2 \leq t \leq \frac{n}{2}+1$	$\frac{n^2}{8}+\frac{n}{4}$
u_s^t	7	$3(n-s)+t+1$	$s=1,$ $\frac{n}{2}+2 \leq t \leq n$	$\frac{n}{2}-1$
u_s^t	9	$3(n-s)+t+1$	$2 \leq s \leq \frac{n}{2}-1,$ $\frac{n}{2}+2 \leq t \leq n-s+1$	$\frac{1}{8}(n-4)(n-2)$
u_s^t	9	$n-s+3t-2$	$2 \leq s \leq \frac{n}{2},$ $n-s+2 \leq t \leq n$	$\frac{n^2}{8}-\frac{n}{4}$
u_s^t	9	$4s-n+t-3$	$\frac{n}{2}+2 \leq s \leq n,$ $\frac{n}{2}+2 \leq t \leq s$	$\frac{n^2}{8}-\frac{n}{4}$
u_s^t	9	$s+3t-4$	$\frac{n}{2}+1 \leq s \leq n-1,$ $s+1 \leq t \leq n$	$\frac{n^2}{8}-\frac{n}{4}$

Theorem 2. For every $n \geq 3$ and $n \equiv 1(\mod 2)$ consider the graph of $G \cong O_n^m$, with $n=m$. Then the modified eccentric connectivity index $\xi_c(G)$ of G is equal to

$$\begin{aligned}\xi_c(O_n^m) &= 225n^2-132n+35\\ &+36\sum_{s=2}^{\frac{n+1}{2}-1}\left[\sum_{t=s+1}^{\frac{n+1}{2}}\{4n-3(s-1)-t\}\right]+36\sum_{s=2}^{\frac{n+1}{2}}\left[\sum_{t=2}^{s}\{4(n+1)-s-3t\}\right]\\ &+36\sum_{s=\frac{n+1}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1}\{3(n-t)+s+2\}\right]+36\sum_{s=\frac{n+1}{2}+1}^{n}\left[\sum_{t=n-s+2}^{\frac{n+1}{2}}\{n+3s-t-1\}\right]\end{aligned}$$

$$
\begin{aligned}
&+36\sum_{s=1}^{\frac{n+1}{2}-1}\left[\sum_{t=\frac{n+1}{2}+1}^{n-s+1}\{3(n-s)+t+1\}\right]+36\sum_{s=2}^{\frac{n+1}{2}}\left[\sum_{t=n+2-s}^{n}\{n-s+3t-2\}\right]\\
&+36\sum_{s=\frac{n+1}{2}+1}^{n}\left[\sum_{t=\frac{n+1}{2}+1}^{s}\{4s+t-n-3\}\right]+36\sum_{s=\frac{n+1}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n}\{s+3t-4\}\right].
\end{aligned}
$$

Proof. Let G be the graph of O_n^m and $n\geq 3$ is odd. As above note that graph of O_n^m is a symmetric about reflection and rotation at right angles. Thus the eccentricities $\varepsilon_{u_s^t}=\varepsilon_{v^t_{n+1-s}}$ and from the symmetry at right angles we can obtain that the eccentricities $\varepsilon_{y_s^t}=\varepsilon_{u_t^s}$, $\varepsilon_{w_s^t}=\varepsilon_{v_t^s}$. Therefore, by using Table 2 and equation (3) the modified eccentric connectivity index $\xi_c(G)$, we get

$$
\begin{aligned}
\xi_c(O_n^m)=\;&4\left[2\times 4\times 4n+2\sum_{s=2}^{\frac{n+1}{2}}5\{4n+1-s\}+2\sum_{s=\frac{n+1}{2}+1}^{n}5\{3n+s-1\}\right]\\
&+4\left[\sum_{t=2}^{\frac{n+1}{2}}7\{4n-t\}+\sum_{s=2}^{\frac{n+1}{2}-1}\left[\sum_{t=s+1}^{\frac{n+1}{2}}9\{4n-3(s-1)-t\}\right]\right.\\
&+\sum_{s=2}^{\frac{n+1}{2}}\left[\sum_{t=2}^{s}9\{4(n+1)-s-3t\}\right]+\sum_{s=\frac{n+1}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1}9\{3(n-t)+s+2\}\right]\\
&+\sum_{s=\frac{n+1}{2}+1}^{n}\left[\sum_{t=n-s+2}^{\frac{n+1}{2}}9\{n+3s-t-1\}\right]+\sum_{t=\frac{n+1}{2}+1}^{n}7\{3(n-1)+t+1\}\\
&+\sum_{s=2}^{\frac{n+1}{2}-1}\left[\sum_{t=\frac{n+1}{2}+1}^{n-s+1}9\{3(n-s)+t+1\}\right]+\sum_{s=2}^{\frac{n+1}{2}}\left[\sum_{t=n+2-s}^{n}9\{n-s+3t-2\}\right]\\
&\left.+\sum_{s=\frac{n+1}{2}+1}^{n}\left[\sum_{t=\frac{n+1}{2}+1}^{s}9\{4s+t-n-3\}\right]+\sum_{s=\frac{n+1}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n}9\{s+3t-4\}\right]\right].
\end{aligned}
$$

After some easy calculations we get

$$
\begin{aligned}
\xi_c(O_n^m)=\;&225n^2-132n+35\\
&+36\sum_{s=2}^{\frac{n+1}{2}-1}\left[\sum_{t=s+1}^{\frac{n+1}{2}}\{4n-3(s-1)-t\}\right]+36\sum_{s=2}^{\frac{n+1}{2}}\left[\sum_{t=2}^{s}\{4(n+1)-s-3t\}\right]\\
&+36\sum_{s=\frac{n+1}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1}\{3(n-t)+s+2\}\right]+36\sum_{s=\frac{n+1}{2}+1}^{n}\left[\sum_{t=n-s+2}^{\frac{n+1}{2}}\{n+3s-t-1\}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+36\sum_{s=1}^{\frac{n+1}{2}-1}\left[\sum_{t=\frac{n+1}{2}+1}^{n-s+1}\{3(n-s)+t+1\}\right]+36\sum_{s=2}^{\frac{n+1}{2}}\left[\sum_{t=n+2-s}^{n}\{n-s+3t-2\}\right]\\
&+36\sum_{s=\frac{n+1}{2}+1}^{n}\left[\sum_{t=\frac{n+1}{2}+1}^{s}\{4s+t-n-3\}\right]+36\sum_{s=\frac{n+1}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n}\{s+3t-4\}\right].
\end{aligned}
$$

□

Table 2 Partition of vertices of the type u_s^t of O_n^m based on degree sum and eccentricity of each vertex when $n \equiv 1($ mod 2).

Representative	$S_{u_s^t}$	eccentricity	Range	Frequency
u_s^t	4	$4n-s+1$	$t=1,\ n+1;\ s=1$	2
u_s^t	5	$4n-s+1$	$t=1,\ n+1,$ $2 \leq s \leq \frac{n+1}{2}$	$n-1$
u_s^t	5	$3n+s-1$	$t=1,\ n+1,$ $\frac{n+1}{2}+1 \leq s \leq n$	$n-1$
u_s^t	7	$4n-3(s-1)-t$	$1=s,$ $2 \leq t \leq \frac{n+1}{2}$	$(\frac{n+1}{2}-1)$
u_s^t	9	$4n-3(s-1)-t$	$2 \leq s \leq \frac{n+1}{2}-1,$ $s+1 \leq t \leq \frac{n+1}{2}$	$\frac{n-3}{4}(\frac{n+1}{2}-1)$
u_s^t	9	$4(n+1)-s-3t$	$2 \leq s \leq \frac{n+1}{2},$ $2 \leq t \leq s$	$\frac{n+1}{4}(\frac{n+1}{2}-1)$
u_s^t	9	$3n+s-3t+2$	$\frac{n+1}{2}+1 \leq s \leq n-1,$ $2 \leq t \leq n+1-s$	$\frac{n-1}{4}(\frac{n-1}{2}-1)$
u_s^t	9	$n+3s-t-1$	$\frac{n+1}{2}+1 \leq s \leq n,$ $n-s+2 \leq t \leq \frac{n+1}{2}$	$\frac{n-1}{4}(\frac{n-1}{2}+1)$
u_s^t	7	$3(n-s)+t+1$	$s=1,$ $\frac{n+1}{2}+1 \leq t \leq n$	$(\frac{n+1}{2}-1)$
u_s^t	9	$n-s+3t-2$	$2 \leq s \leq \frac{n+1}{2},$ $n-s+2 \leq t \leq n$	$\frac{n-3}{4}(\frac{n+1}{2}-1)$
u_s^t	9	$4s-n+t-3$	$\frac{n+1}{2}+1 \leq s \leq n,$ $\frac{n+1}{2}+1 \leq t \leq s$	$\frac{n-1}{4}(\frac{n+1}{2})$
u_s^t	9	$s+3t-4$	$\frac{n+1}{2}+1 \leq s \leq n-1,$ $s+1 \leq t \leq n$	$\frac{n-1}{4}(\frac{n-1}{2}-1)$

Theorem 3. For every $n \geq 4$ and $n \equiv 0(\mod 2)$ consider the graph of $G \cong O_n^m$, with $n=m$. Then the Ediz eccentric connectivity index of G is equal to

$$
{}^{E}\xi^{c}(O_n^m) = \frac{8}{n} + 40\sum_{s=2}^{\frac{n}{2}+1}\frac{1}{4n+1-s} + 40\sum_{s=\frac{n}{2}+2}^{n}\frac{1}{3n+s-1} + 28\sum_{t=2}^{\frac{n}{2}+1}\frac{1}{4n-t} + 28\sum_{t=\frac{n}{2}+2}^{n}\frac{1}{3(n-1)+t+1}
$$

$$
+ 36\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=s+1}^{\frac{n}{2}+1}\frac{1}{4n-3(s-1)-t}\right] + 36\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=2}^{s}\frac{1}{4(n+1)-s-3t}\right]
+ 36\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1}\frac{1}{3(n-t)+s+2}\right] + 36\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=n-s+2}^{\frac{n}{2}+1}\frac{1}{n+3s-t-1}\right]
$$

$$+36\sum_{s=2}^{\frac{n}{2}-1}\left[\sum_{t=\frac{n}{2}+2}^{n-s+1}\frac{1}{3(n-s)+t+1}\right]+36\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=n+2-s}^{n}\frac{1}{n-s+3t-2}\right]$$
$$+36\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=\frac{n}{2}+2}^{s}\frac{1}{4s+t-n-3}\right]+36\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n}\frac{1}{s+3t-4}\right].$$

Proof. Let G be the graph of O_n^m and $n \equiv 0(\mod 2)$. By using the arguments in proof of Theorem 1, Table 1 and following formula the Ediz eccentric connectivity index ${}^{E}\xi^{c}(G)$ of O_n^m is equal to

$${}^{E}\xi^{c}(G)=\sum_{v\in V(G)}\left(\frac{S_v}{\varepsilon_v}\right)=4\sum_{u_s^t\in V(G)}\left(\frac{S_{u_s^t}}{\varepsilon_{u_s^t}}\right)$$

$$\begin{aligned}
{}^{E}\xi^{c}(O_n^m)=4\Bigg[&2\times\frac{4}{4n}+2\sum_{s=2}^{\frac{n}{2}+1}\frac{5}{4n+1-s}+2\sum_{s=\frac{n}{2}+2}^{n}\frac{5}{3n+s-1}\Bigg]\\
&+4\Bigg[\sum_{t=2}^{\frac{n}{2}+1}\frac{7}{4n-t}+\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=s+1}^{\frac{n}{2}+1}\frac{9}{4n-3(s-1)-t}\right]\\
&+\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=2}^{s}\frac{9}{4(n+1)-s-3t}\right]+\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1}\frac{9}{3(n-t)+s+2}\right]\\
&+\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=n-s+2}^{\frac{n}{2}+1}\frac{9}{n+3s-t-1}\right]+\sum_{t=\frac{n}{2}+2}^{n}\frac{7}{3(n-1)+t+1}\\
&+\sum_{s=2}^{\frac{n}{2}-1}\left[\sum_{t=\frac{n}{2}+2}^{n-s+1}\frac{9}{3(n-s)+t+1}\right]+\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=n+2-s}^{n}\frac{9}{n-s+3t-2}\right]\\
&+\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=\frac{n}{2}+2}^{s}\frac{9}{4s+t-n-3}\right]+\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n}\frac{9}{s+3t-4}\right]\Bigg].
\end{aligned}$$

After an easy computation, we get

$$\begin{aligned}
{}^{E}\xi^{c}(O_n^m)=\ &\frac{8}{n}+40\sum_{s=2}^{\frac{n}{2}+1}\frac{1}{4n+1-s}+40\sum_{s=\frac{n}{2}+2}^{n}\frac{1}{3n+s-1}\\
&+28\sum_{t=2}^{\frac{n}{2}+1}\frac{1}{4n-t}+28\sum_{t=\frac{n}{2}+2}^{n}\frac{1}{3(n-1)+t+1}\\
&+36\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=s+1}^{\frac{n}{2}+1}\frac{1}{4n-3(s-1)-t}\right]+36\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=2}^{s}\frac{1}{4(n+1)-s-3t}\right]\\
&+36\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1}\frac{1}{3(n-t)+s+2}\right]+36\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=n-s+2}^{\frac{n}{2}+1}\frac{1}{n+3s-t-1}\right]
\end{aligned}$$

$$+36\sum_{s=2}^{\frac{n}{2}-1}\left[\sum_{t=\frac{n}{2}+2}^{n-s+1}\frac{1}{3(n-s)+t+1}\right]+36\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=n+2-s}^{n}\frac{1}{n-s+3t-2}\right]$$
$$+36\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=\frac{n}{2}+2}^{s}\frac{1}{4s+t-n-3}\right]+36\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n}\frac{1}{s+3t-4}\right].$$

□

Theorem 4. *For every $n \geq 3$ and $n \equiv 1(\mod 2)$ consider the graph of $G \cong O_n^m$, with $n = m$. Then the Ediz eccentric connectivity index of G is equal to*

$$\begin{aligned}
{}^E\xi^c(O_n^m) = {} & \frac{8}{n} + 40\sum_{s=2}^{\frac{n+1}{2}} \frac{1}{4n+1-s} + 40\sum_{s=\frac{n+1}{2}+1}^{n} \frac{1}{3n+s-1} \\
& + 28\sum_{t=2}^{\frac{n+1}{2}} \frac{1}{4n-t} + 28\sum_{t=\frac{n+1}{2}+1}^{n} \frac{1}{3(n-1)+t+1} \\
& + 36\sum_{s=2}^{\frac{n+1}{2}-1}\left[\sum_{t=s+1}^{\frac{n+1}{2}} \frac{1}{4n-3(s-1)-t}\right] + 36\sum_{s=2}^{\frac{n+1}{2}}\left[\sum_{t=2}^{s} \frac{1}{4(n+1)-s-3t}\right] \\
& + 36\sum_{s=\frac{n+1}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1} \frac{1}{3(n-t)+s+2}\right] + 36\sum_{s=\frac{n+1}{2}+1}^{n}\left[\sum_{t=n-s+2}^{\frac{n+1}{2}} \frac{1}{n+3s-t-1}\right] \\
& + 36\sum_{s=2}^{\frac{n+1}{2}-1}\left[\sum_{t=\frac{n+1}{2}+1}^{n} \frac{1}{3(n-s)+t+1}\right] + 36\sum_{s=2}^{\frac{n+1}{2}}\left[\sum_{t=n+2-s}^{n} \frac{1}{n-s+3t-2}\right] \\
& + 36\sum_{s=\frac{n+1}{2}+1}^{n}\left[\sum_{t=\frac{n+1}{2}+1}^{s} \frac{1}{4s+t-n-3}\right] + 36\sum_{s=\frac{n+1}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n} \frac{1}{s+3t-4}\right].
\end{aligned}$$

Proof. Let G be the graph of O_n^m and $n \equiv 1(\mod 2)$. By using the arguments in proof the of Theorem 2, Table 2 and from formula (4) the result follows. □

Theorem 5. For every $n \geq 4$ and $n \equiv 0(\mod 2)$ consider the graph of $G \cong O_n^m$, with $n = m$. Then the modified eccentric connectivity polynomial of G is equal to

$$\begin{aligned}
\xi_c(O_n^m, x) = {} & \frac{1}{x-1}\Bigg((-28x^{4n-1} + 40x^{3n} - 40x^{4n})(\frac{1}{x})^{(\frac{n}{2})} - 40x^{(3n+1)}(\frac{1}{x})^{n} + 24x^{4n} \\
& - 28x^{(7n/2)} + 56x^{4n-1} + 16x^{4n+1}\Bigg) \\
& + 36\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=s+1}^{\frac{n}{2}+1} x^{(4n-3(s-1)-t)}\right] + 36\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=2}^{s} x^{(4(n+1)-s-3t)}\right] \\
& + 36\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1} x^{(3(n-t)+s+2)}\right] + 36\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=n-s+2}^{\frac{n}{2}+1} x^{(n+3s-t-1)}\right] \\
& + 36\sum_{s=2}^{\frac{n}{2}-1}\left[\sum_{t=\frac{n}{2}+2}^{n-s+1} x^{(3(n-s)+t+1)}\right] + 36\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=n+2-s}^{n} x^{(n-s+3t-2)}\right] \\
& + 36\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=\frac{n}{2}+2}^{s} x^{(4s+t-n-3)}\right] + 36\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n} x^{(s+3t-4)}\right].
\end{aligned}$$

Proof. By using the arguments in the proof of Theorem 1, the values from Table 1 and equation (5) given below we get

$$\xi_c(G,x) = \sum_{u \in V(G)} S_u x^{\varepsilon_u} = 4\sum_{u_s^t \in V(G)} S_{u_s^t} x^{\varepsilon_{u_s^t}}$$

$$\begin{aligned}\xi_c(O_n^m,x) = {} & 4\left[2\times 4x^{4n} + 2\sum_{s=2}^{\frac{n}{2}+1} 5x^{(4n+1-s)} + 2\sum_{s=\frac{n}{2}+2}^{n} 5x^{(3n+s-1)}\right] \\ & + 4\left[\sum_{t=2}^{\frac{n}{2}+1} 7x^{(4n-t)} + \sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=s+1}^{\frac{n}{2}+1} 9x^{(4n-t)}\right]\right. \\ & + \sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=2}^{s} 9x^{(4(n+1)-s-3t)}\right] + \sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1} 9x^{(3(n-t)+s+2)}\right] \\ & + \sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=n-s+2}^{\frac{n}{2}+1} 9x^{(n+3s-t-1)}\right] + \sum_{t=\frac{n}{2}+2}^{n} 7x^{(3(n-1)+t+1)} \\ & + \sum_{s=2}^{\frac{n}{2}-1}\left[\sum_{t=\frac{n}{2}+2}^{n-s+1} 9x^{(3(n-s)+t+1)}\right] + \sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=n+2-s}^{n} 9x^{(n-s+3t-2)}\right] \\ & \left. + \sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=\frac{n}{2}+2}^{s} 9x^{(4s+t-n-3)}\right] + \sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n} 9x^{(s+3t-4)}\right]\right].\end{aligned}$$

After some easy calculations we get

$$\begin{aligned}\xi_c(O_n^m,x) = {} & \frac{1}{x-1}\Bigg((-28x^{4n-1}+40x^{3n}-40x^{4n})(\frac{1}{x})^{(\frac{n}{2})} - 40x^{(3n+1)}(\frac{1}{x})^{n} + 24x^{4n} \\ & - 28x^{(7n/2)} + 56x^{4n-1} + 16x^{4n+1}\Bigg) \\ & + 36\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=s+1}^{\frac{n}{2}+1} x^{(4n-3(s-1)-t)}\right] + 36\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=2}^{s} x^{(4(n+1)-s-3t)}\right] \\ & + 36\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1} x^{(3(n-t)+s+2)}\right] + 36\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=n-s+2}^{\frac{n}{2}+1} x^{(n+3s-t-1)}\right] \\ & + 36\sum_{s=2}^{\frac{n}{2}-1}\left[\sum_{t=\frac{n}{2}+2}^{n-s+1} x^{(3(n-s)+t+1)}\right] + 36\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=n+2-s}^{n} x^{(n-s+3t-2)}\right] \\ & + 36\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=\frac{n}{2}+2}^{s} x^{(4s+t-n-3)}\right] + 36\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n} x^{(s+3t-4)}\right].\end{aligned}$$

□

Theorem 6. *For every $n \geq 3$ and $n \equiv 1(\mod 2)$ consider the graph of $G \cong O_n^m$, with $n = m$. Then the modified eccentric connectivity polynomial of G is equal to*

$$\begin{aligned}\xi_c(O_n^m,x) = {} & \frac{1}{x-1}\Bigg((40x^{3n+2}-28x^{4n+1}-40x^{4n+2})(\frac{1}{x})^{(\frac{n}{2}+\frac{3}{2})} - 40x^{(3n+1)}(\frac{1}{x})^{n} + 24x^{4n} \\ & - 28x^{(\frac{7n}{2})-\frac{1}{2}} + 56x^{4n-1} + 16x^{4n+1}\Bigg)\end{aligned}$$

$$+ 36\sum_{s=2}^{\frac{n+1}{2}-1}\left[\sum_{t=s+1}^{\frac{n+1}{2}} x^{(4n-3(s-1)-t)}\right] + 36\sum_{s=2}^{\frac{n+1}{2}}\left[\sum_{t=2}^{s} x^{(4(n+1)-s-3t)}\right]$$

$$
\begin{aligned}
&+36\sum_{s=\frac{n+1}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1} x^{(3(n-t)+s+2)}\right]+36\sum_{s=\frac{n+1}{2}+1}^{n}\left[\sum_{t=n-s+2}^{\frac{n+1}{2}} x^{(n+3s-t-1)}\right]\\
&+36\sum_{s=1}^{\frac{n+1}{2}-1}\left[\sum_{t=\frac{n+1}{2}+1}^{n-s+1} x^{(3(n-s)+t+1)}\right]+36\sum_{s=2}^{\frac{n+1}{2}}\left[\sum_{t=n+2-s}^{n} x^{(n-s+3t-2)}\right]\\
&+\sum_{s=\frac{n+1}{2}+1}^{n}\left[\sum_{t=\frac{n+1}{2}+1}^{s} x^{(4s+t-n-3)}\right]+36\sum_{s=\frac{n+1}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n} x^{(s+3t-4)}\right].
\end{aligned}
$$

Proof. Let $G\cong O_n^m$, $n\geq 3$ and $n\equiv 1(\mod 2)$. By using the arguments in the proof of Theorem 1, as in Theorem 5, the values from Table 2 and equation (5) the result follows. □

In Table 3 and Table 4 we have partitioned the vertices of the type u_s^t of O_n^m based on degree product and eccentricity of each vertex. This will help us to develop the coming theorems.

Theorem 7. For every $n\geq 4$ and $n\equiv 0(\mod 2)$ consider the graph of $G\cong O_n^m$, with $n=m$. Then the modified augmented eccentric connectivity index ${}^{MA}\xi_c^c(G)$ of G is equal to

$$
\begin{aligned}
{}^{MA}\xi_c^c(O_n^m) &= 324n^2-232n+48\\
&+108\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=s+1}^{\frac{n}{2}+1}\{4n-3(s-1)-t\}\right]+108\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=2}^{s}\{4(n+1)-s-3t\}\right]\\
&+108\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1}\{3(n-t)+s+2\}\right]+108\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=n-s+2}^{\frac{n}{2}+1}\{n+3s-t-1\}\right]\\
&+108\sum_{s=2}^{\frac{n}{2}-1}\left[\sum_{t=\frac{n}{2}+2}^{n-s+1}\{3(n-s)+t+1\}\right]+108\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=n+2-s}^{n}\{n-s+3t-2\}\right]\\
&+108\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=\frac{n}{2}+2}^{s}\{4s+t-n-3\}\right]+108\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n}\{s+3t-4\}\right].
\end{aligned}
$$

Proof. Let G be the graph of O_n^m. Therefore, from Table 3 and formula (8), given below, the modified augmented eccentric connectivity index ${}^{MA}\xi_c^c(O_n^m)$ of O_n^m can be calculated. Hence the result.

$$
{}^{MA}\xi^c(G,x)=\sum_{v\in V(G)} M_v\,\varepsilon_v \tag{8}
$$

$$
{}^{MA}\xi^c(O_n^m)=4\sum_{u_s^t\in V(G)}\left(M_{u_s^t}\varepsilon_{u_s^t}\right)
$$

$$
\begin{aligned}
{}^{MA}\xi^c(O_n^m) &= 4\left[2\times 4\times 4n+2\sum_{s=2}^{\frac{n}{2}+1}6\{4n+1-s\}+2\sum_{s=\frac{n}{2}+2}^{n}6\{3n+s-1\}\right]\\
&+4\left[\sum_{t=2}^{\frac{n}{2}+1}12(4n-t)+\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=s+1}^{\frac{n}{2}+1}27\{4n-3(s-1)-t\}\right]\right.
\end{aligned}
$$

$$
+\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=2}^{s}27\{4(n+1)-s-3t\}\right]+\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1}27\{3(n-t)+s+2\}\right]
$$

$$+\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=n-s+2}^{\frac{n}{2}+1}27\{n+3s-t-1\}\right]+\sum_{t=\frac{n}{2}+2}^{n}12\{3(n-1)+t+1\}$$
$$+\sum_{s=2}^{\frac{n}{2}-1}\left[\sum_{t=\frac{n}{2}+2}^{n-s+1}27\{3(n-s)+t+1\}\right]+\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=n+2-s}^{n}27\{n-s+3t-2\}\right]$$
$$+\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=\frac{n}{2}+2}^{s}27\{4s+t-n-3\}\right]+\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n}27\{s+3t-4\}\right]\Bigg].$$

After some easy calculations we get

$$^{MA}\xi_c^c(O_n^m)=324n^2-232n+48$$
$$+108\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=s+1}^{\frac{n}{2}+1}\{4n-3(s-1)-t\}\right]+108\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=2}^{s}\{4(n+1)-s-3t\}\right]$$
$$+108\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1}\{3(n-t)+s+2\}\right]+108\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=n-s+2}^{\frac{n}{2}+1}\{n+3s-t-1\}\right]$$
$$+108\sum_{s=2}^{\frac{n}{2}-1}\left[\sum_{t=\frac{n}{2}+2}^{n-s+1}\{3(n-s)+t+1\}\right]+108\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=n+2-s}^{n}\{n-s+3t-2\}\right]$$
$$+108\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=\frac{n}{2}+2}^{s}\{4s+t-n-3\}\right]+108\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n}\{s+3t-4\}\right].$$

□

Theorem 8. For every $n\geq 3$ and $n\equiv 1(\mod 2)$ consider the graph of $G\cong O_n^m$, with $n=m$. Then the modified augmented eccentric connectivity index $\xi_c(G)$ of G is equal to

$$^{MA}\xi_c^c(O_n^m)=324n^2-256n+60$$
$$+108\sum_{s=2}^{\frac{n+1}{2}-1}\left[\sum_{t=s+1}^{\frac{n+1}{2}}\{4n-3(s-1)-t\}\right]+108\sum_{s=2}^{\frac{n+1}{2}}\left[\sum_{t=2}^{s}\{4(n+1)-s-3t\}\right]$$
$$+108\sum_{s=\frac{n+1}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1}\{3(n-t)+s+2\}\right]+108\sum_{s=\frac{n+1}{2}+1}^{n}\left[\sum_{t=n-s+2}^{\frac{n+1}{2}}\{n+3s-t-1\}\right]$$
$$+108\sum_{s=1}^{\frac{n+1}{2}-1}\left[\sum_{t=\frac{n+1}{2}+1}^{n-s+1}\{3(n-s)+t+1\}\right]+108\sum_{s=2}^{\frac{n+1}{2}}\left[\sum_{t=n+2-s}^{n}\{n-s+3t-2\}\right]$$
$$+108\sum_{s=\frac{n+1}{2}+1}^{n}\left[\sum_{t=\frac{n+1}{2}+1}^{s}\{4s+t-n-3\}\right]+108\sum_{s=\frac{n+1}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n}\{s+3t-4\}\right].$$

Proof. As in Theorem 7, by using Table 4 and equation (8) the result follows.

□

Theorem 9. For every $n\geq 4$ and $n\equiv 0(\mod 2)$ consider the graph of $G\cong O_n^m$, with $n=m$. Then the modified augmented eccentric connectivity polynomial of G is equal to

$$^{MA}\xi_c^c(O_n^m,x)=\frac{1}{x-1}\Bigg((48x^{4n-1}+48x^{3n}-48x^{4n})(\frac{1}{x})^{(\frac{n}{2})}-48x^{(3n+1)}(\frac{1}{x})^{n}+32x^{4n}$$

Table 3 Partition of vertices of the type u_s^t of O_n^m based on degree product and eccentricity of each vertex when $n \equiv 0($ mod 2).

Representative	$M_{u_s^t}$	eccentricity	Range	Frequency
u_s^t	4	$4n-s+1$	$t=1,\ n+1;\ s=1$	2
u_s^t	6	$4n-s+1$	$t=1,\ n+1,$ $2 \le s \le \frac{n}{2}+1$	n
u_s^t	6	$3n+s-1$	$t=1,\ n+1,$ $\frac{n}{2}+2 \le s \le n$	$n-2$
u_s^t	12	$4n-3(s-1)-t$	$s=1,$ $2 \le t \le \frac{n}{2}+1$	$\frac{n}{2}$
u_s^t	27	$4n-3(s-1)-t$	$2 \le s \le \frac{n}{2},$ $s+1 \le t \le \frac{n}{2}+1$	$\frac{n^2}{8}-\frac{3n}{4}$
u_s^t	27	$4(n+1)-s-3t$	$2 \le s \le \frac{n}{2},$ $2 \le t \le s$	$\frac{n}{4}(\frac{n}{2}-1)$
u_s^t	27	$3n+s-3t+2$	$\frac{n}{2}+1 \le s \le n-1,$ $2 \le t \le n+1-s$	$\frac{n}{4}(\frac{n}{2}-1)$
u_s^t	27	$n+3s-t-1$	$\frac{n}{2}+1 \le s \le n,$ $n-s+2 \le t \le \frac{n}{2}+1$	$\frac{n}{4}(\frac{n}{2}+1)$
u_s^t	12	$3(n-s)+t+1$	$s=1,$ $\frac{n}{2}+2 \le t \le n$	$\frac{n}{2}-1$
u_s^t	27	$3(n-s)+t+1$	$2 \le s \le \frac{n}{2}-1,$ $\frac{n}{2}+2 \le t \le n-s+1$	$\frac{1}{8}(n-4)(n-2)$
u_s^t	27	$n-s+3t-2$	$2 \le s \le \frac{n}{2},$ $n-s+2 \le t \le n$	$\frac{n}{4}(\frac{n}{2}-1)$
u_s^t	27	$4s-n+t-3$	$\frac{n}{2}+2 \le s \le n,$ $\frac{n}{2}+2 \le t \le s$	$\frac{n}{4}(\frac{n}{2}-1)$
u_s^t	27	$s+3t-4$	$\frac{n}{2}+1 \le s \le n-1,$ $s+1 \le t \le n$	$\frac{n}{4}(\frac{n}{2}-1)$

$$- 48x^{(7n/2)} + 96x^{4n-1} + 16x^{4n+1}\Big)$$

$$+ 48\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=s+1}^{\frac{n}{2}+1} x^{(4n-3(s-1)-t)}\right] + 48\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=2}^{s} x^{(4(n+1)-s-3t)}\right]$$

$$+ 48\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1} x^{(3(n-t)+s+2)}\right] + 48\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=n-s+2}^{\frac{n}{2}+1} x^{(n+3s-t-1)}\right]$$

$$+ 48\sum_{s=2}^{\frac{n}{2}-1}\left[\sum_{t=\frac{n}{2}+2}^{n-s+1} x^{(3(n-s)+t+1)}\right] + 48\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=n+2-s}^{n} x^{(n-s+3t-2)}\right]$$

Table 4 Partition of vertices of the type u_s^t of O_n^m based on degree product and eccentricity of each vertex when $n \equiv 1($ mod $2)$.

Representative	$M_{u_s^t}$	eccentricity	Range	Frequency
u_s^t	4	$4n-s+1$	$t=1,\ n+1;\ s=1$	2
u_s^t	6	$4n-s+1$	$t=1,\ n+1,$ $2 \le s \le \frac{n+1}{2}$	$n-1$
u_s^t	6	$3n+s-1$	$t=1,\ n+1,$ $\frac{n+1}{2}+1 \le s \le n$	$n-1$
u_s^t	12	$4n-3(s-1)-t$	$1=s,$ $2 \le t \le \frac{n+1}{2}$	$(\frac{n+1}{2}-1)$
u_s^t	27	$4n-3(s-1)-t$	$2 \le s \le \frac{n+1}{2}-1,$ $s+1 \le t \le \frac{n+1}{2}$	$\frac{n-3}{4}(\frac{n+1}{2}-1)$
u_s^t	27	$4(n+1)-s-3t$	$2 \le s \le \frac{n+1}{2},$ $2 \le t \le s$	$\frac{n+1}{4}(\frac{n+1}{2}-1)$
u_s^t	27	$3n+s-3t+2$	$\frac{n+1}{2}+1 \le s \le n-1,$ $2 \le t \le n+1-s$	$\frac{n-1}{4}(\frac{n-1}{2}-1)$
u_s^t	27	$n+3s-t-1$	$\frac{n+1}{2}+1 \le s \le n,$ $n-s+2 \le t \le \frac{n+1}{2}$	$\frac{n-1}{4}(\frac{n-1}{2}+1)$
u_s^t	12	$3(n-s)+t+1$	$s=1,$ $\frac{n+1}{2}+1 \le t \le n$	$(\frac{n+1}{2}-1)$
u_s^t	27	$n-s+3t-2$	$2 \le s \le \frac{n+1}{2},$ $n-s+2 \le t \le n$	$\frac{n-3}{4}(\frac{n+1}{2}-1)$
u_s^t	27	$4s-n+t-3$	$\frac{n+1}{2}+1 \le s \le n,$ $\frac{n+1}{2}+1 \le t \le s$	$\frac{n-1}{4}(\frac{n+1}{2})$
u_s^t	27	$s+3t-4$	$\frac{n+1}{2}+1 \le s \le n-1,$ $s+1 \le t \le n$	$\frac{n-1}{4}(\frac{n-1}{2}-1)$

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$$+\,48 \sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=\frac{n}{2}+2}^{s} x^{(4s+t-n-3)}\right] + 48 \sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n} x^{(s+3t-4)}\right].$$

Proof. By using the arguments in the proof of Theorem 1, the values from Table 3 and equation (8) given below we get

$$^{MA}\xi_c^{\,c}(G,x) = \sum_{u\in V(G)} M_u x^{\varepsilon_u} = 4 \sum_{u_s^t\in V(G)} M_{u_s^t} x^{\varepsilon_{u_s^t}}$$

$$^{MA}\xi_c(O_n^m,x) = 4\left[2\times 4x^{4n} + 2\sum_{s=2}^{\frac{n}{2}+1} 6x^{(4n+1-s)} + 2\sum_{s=\frac{n}{2}+2}^{n} 6x^{(3n+s-1)}\right]$$
$$+\,4\left[\sum_{t=2}^{\frac{n}{2}+1} 12x^{(4n-t)} + \sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=s+1}^{\frac{n}{2}+1} 27x^{(4n-t)}\right]\right]$$

$$
\begin{aligned}
&+\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=2}^{s}27x^{(4(n+1)-s-3t)}\right]+\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1}27x^{(3(n-t)+s+2)}\right]\\
&+\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=n-s+2}^{\frac{n}{2}+1}27x^{(n+3s-t-1)}\right]+\sum_{t=\frac{n}{2}+2}^{n}12x^{(3(n-1)+t+1)}\\
&+\sum_{s=2}^{\frac{n}{2}-1}\left[\sum_{t=\frac{n}{2}+2}^{n-s+1}27x^{(3(n-s)+t+1)}\right]+\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=n+2-s}^{n}27x^{(n-s+3t-2)}\right]\\
&+\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=\frac{n}{2}+2}^{s}27x^{(4s+t-n-3)}\right]+\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n}27x^{(s+3t-4)}\right]\Bigg].
\end{aligned}
$$

After some easy calculations we get

$$
\begin{aligned}
{}^{MA}\xi_c^c(O_n^m,x) = {}& \frac{1}{x-1}\Bigg((48x^{4n-1}+48x^{3n}-48x^{4n})(\frac{1}{x})^{(\frac{n}{2})}-48x^{(3n+1)}(\frac{1}{x})^{n}+32x^{4n}\\
&-48x^{(7n/2)}+96x^{4n-1}+16x^{4n+1}\Bigg)\\
&+48\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=s+1}^{\frac{n}{2}+1}x^{(4n-3(s-1)-t)}\right]+48\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=2}^{s}x^{(4(n+1)-s-3t)}\right]\\
&+48\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1}x^{(3(n-t)+s+2)}\right]+48\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=n-s+2}^{\frac{n}{2}+1}x^{(n+3s-t-1)}\right]\\
&+48\sum_{s=2}^{\frac{n}{2}-1}\left[\sum_{t=\frac{n}{2}+2}^{n-s+1}x^{(3(n-s)+t+1)}\right]+48\sum_{s=2}^{\frac{n}{2}}\left[\sum_{t=n+2-s}^{n}x^{(n-s+3t-2)}\right]\\
&+48\sum_{s=\frac{n}{2}+2}^{n}\left[\sum_{t=\frac{n}{2}+2}^{s}x^{(4s+t-n-3)}\right]+48\sum_{s=\frac{n}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n}x^{(s+3t-4)}\right].
\end{aligned}
$$

□

Theorem 10. *For every $n \geq 3$ and $n \equiv 1(\mod 2)$ consider the graph of $G \cong O_n^m$, with $n = m$. Then the modified augmented eccentric connectivity polynomial of G is equal to*

$$
\begin{aligned}
{}^{MA}\xi_c^c(O_n^m,x) = {}& \frac{1}{x-1}\Bigg((-48x^{3n+2}-48x^{4n+1}-48x^{4n+2})(\frac{1}{x})^{(\frac{n}{2}+\frac{3}{2})}-48x^{(3n+1)}(\frac{1}{x})^{n}+32x^{4n}\\
&-48x^{(\frac{7n}{2})-\frac{1}{2}}+96x^{4n-1}+16x^{4n+1}\Bigg)\\
&+108\sum_{s=2}^{\frac{n+1}{2}-1}\left[\sum_{t=s+1}^{\frac{n+1}{2}}x^{(4n-3(s-1)-t)}\right]+108\sum_{s=2}^{\frac{n+1}{2}}\left[\sum_{t=2}^{s}x^{(4(n+1)-s-3t)}\right]\\
&+108\sum_{s=\frac{n+1}{2}+1}^{n-1}\left[\sum_{t=2}^{n-s+1}x^{(3(n-t)+s+2)}\right]+108\sum_{s=\frac{n+1}{2}+1}^{n}\left[\sum_{t=n-s+2}^{\frac{n+1}{2}}x^{(n+3s-t-1)}\right]\\
&+108\sum_{s=1}^{\frac{n+1}{2}-1}\left[\sum_{t=\frac{n+1}{2}+1}^{n-s+1}x^{(3(n-s)+t+1)}\right]+108\sum_{s=2}^{\frac{n+1}{2}}\left[\sum_{t=n+2-s}^{n}x^{(n-s+3t-2)}\right]
\end{aligned}
$$

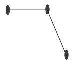
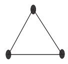
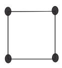
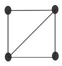
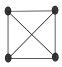
$$+\sum_{s=\frac{n+1}{2}+1}^{n}\left[\sum_{t=\frac{n+1}{2}+1}^{s}x^{(4s+t-n-3)}\right]+108\sum_{s=\frac{n+1}{2}+1}^{n-1}\left[\sum_{t=s+1}^{n}x^{(s+3t-4)}\right].$$

Proof. As in Theorem 9, by using Table 4 and the equation (8) the result follows.

□

4 Conclusions and comparison between the indices

Table 5 Values of eccentric connectivity index, modified eccentric connectivity index and modified augmented eccentric connectivity index for all possible structures with three and four.

S.N	Structure	$\xi(G)$	${}^{MA}\xi^{c}(G)$	$\xi_c(G)$
1		6	9	10
2		6	12	12
3		14	16	24
4		9	19	21
5		13	32	32
6		16	32	32
7		14	60	29
8		12	108	36

High discriminating power and extremely low degeneracy are desirable properties of an ideal topological index, which researchers in theoretical chemistry are striving to achieve. The values of ${}^{MA}\xi^{c}(G)$ were computed for all the possible structure of three and four vertices. The values and the structures have been presented in Table 5 and their comparison is presented in Table 6. Modified augmented eccentric connectivity index demonstrate exceptionally high discriminating power, defined as the ratio of the highest to lowest value for all possible structures with the same number of vertices. This is evident from the fact that the ratio of the highest to lowest value for all possible structure containing three and four vertices is very high in contrast to $\xi(G)$ and $\xi_c(G)$. The ratio of the highest to lowest value for all possible structures containing four vertices for ${}^{MA}\xi^{c}(G)$ is 6.75 in comparison to 1.78 and 1.7 for $\xi(G)$ and $\xi_c(G)$, respectively. The exceptionally high discriminating power of the proposed indices makes them extremely sensitive towards minor change(s) in molecular structure. This extreme sensitivity towards branching and the discriminating power of proposed indices are clearly evident from the respective index values of all the possible structures with four vertices.

Degeneracy: the number of compounds having identical values/the total number of compounds with the same number of vertices.

Degeneracy is a measure of the ability of an index to differentiate between the relative positions of atom in

Table 6 Comparison of the discriminating power and degeneracy of eccentric connectivity index, modified eccentric connectivity index and modified augmented eccentric connectivity index using all possible structures with three and four vertices.

	$\xi(G)$	${}^{MA}\xi^{c}(G)$	$\xi_c(G)$
• For three vertices			
Minimum value	6	9	10
Maximum value	6	12	12
Ratio	1:1	1:1.34	1:1.2
Degeneracy	1/2	0/2	0/2
• For four vertices			
Minimum value	9	16	21
Maximum value	16	108	36
Ratio	1:1.78	1:6.75	1:1.7
Degeneracy	1/6	1/6	1/6

a molecule. ${}^{MA}\xi^{c}(G)$ did not exhibit any degeneracy for all possible structures with three vertices whereas ${}^{MA}\xi^{c}(G)$ had a very low degeneracy of one in the case of all possible structures with four vertices (Table 6). $\xi(G)$ had one identical values out of 6 structures with only four vertices. Extremely low degeneracy indicates the enhanced capability of these indices to differentiate and demonstrate slight variations in the molecular structure, which clearly reveals the remote chance of different structures having the same value.

The Table 7 shows a comparison between the eccentric connectivity index, modified eccentric connectivity index and modified augmented eccentric connectivity index for octagonal grid O_m^n for finite $n = 3, \ldots, 10$.

Table 7 comparison of $\xi_c(O_m^n)$, ${}^{E}\xi^{c}(O_m^n)$ and ${}^{MA}\xi^{c}(O_m^n)$ for O_m^n, when $m = n$.

$[n,\ m]$	$\xi_c(O_m^n)$	${}^{E}\xi^{c}(O_m^n)$	${}^{MA}\xi^{c}(O_m^n)$
[3,3]	2888	$\frac{5369}{165}$	5880
[4,4]	6564	$\frac{616039}{15015}$	14456
[5,5]	13460	$\frac{74609462}{1322685}$	32260
[6,6]	22972	$\frac{563513878}{8580495}$	56868
[7,7]	37280	$\frac{803471620793}{10039179150}$	95576
[8,8]	55364	$\frac{1361165885969}{15168440430}$	144424
[9,9]	79784	$\frac{1929246726361}{18627909300}$	212136
[10,10]	109176	$\frac{1149037176620287}{10119188365650}$	293432

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