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AN INDEPENDENT SET DEGREE CONDITION FOR FRACTIONAL CRITICAL DELETED GRAPHS

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Abstract. Let $i \geq 2$, $\Delta \geq 0$, $1 \leq a \leq b - \Delta$, $n > \frac{(a+b)(ib+2m-2)}{a} + n'$ and $\delta(G) \geq \frac{b^2}{a} + n' + 2m$, and let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$ for each $x \in V(G)$. In this article, it is determined that G is a fractional (g, f, n', m)-critical deleted graph if $\max\{d_1, d_2, \cdots, d_i\} \geq \frac{b(n+n')}{a+b}$ for any independent subset $\{x_1, x_2, \ldots, x_i\} \subseteq V(G)$. The result is tight on independent set degree condition.

1. Introduction.

The fractional factor problem can be regarded as a relaxation problem of the cardinality matching. It has wide applications in fields such as combinatorial polyhedron, network design and scheduling. For instance, in a communication network, several large data packets were to be sent to certain destinations via several channels. We can improve the efficiency of this task by dividing the large data packets into small parcels. The available distribution of data packets can be considered as a fractional flow problem. Moreover, it can be looked upon as a fractional factor problem if the sources and destinations of a network are different.

In theoretical model, the whole network can be expressed as a graph in which each vertex represents a site and each edge denotes a channel between two sites. In this setting, the framework of data transmission problem is the existence of fractional factor in the graph corresponding to a network.

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The graphs discussed in this paper are all simple. Let $G=(V,E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $x\in V(G)$, we use $d_G(x)$ and $N_G(x)$ to express the degree and the neighborhood of x in G, respectively. We denote by $G[S]$ the subgraph of G induced by $S\subseteq V(G)$. Let $e_G(S,T)$ be the number of edges with one end in S and the other end in T, where S and T are two vertex-disjoint subsets of G. Set $\delta(G)=\min\{d(x)|x\in V(G)\}$, $\Delta(G)=\max\{d(x)|x\in V(G)\}$, and $d_G(x,y)=d(x,y)$ as the distance between two vertices x and y(computed by the length of a shortest path between them). More terminologies and notations used but clearly explained in this work can refer to [1].

Suppose that g and f are two integer-valued functions on $V(G)$ such that $0\le g(x)\le f(x)$ for all $x\in V(G)$. A *fractional* (g,f)*-factor* is a function h that assigns to each edge of a graph G a number in [0,1] so that $g(x)\le d^h_G(x)\le f(x)$ for each vertex x, where $d^h_G(x)=\sum\limits_{e\in E(x)}h(e)$. A fractional (g,f)-factor is a fractional f-factor if $g(x)=f(x)$ for all $x\in V(G)$. Furthermore, a fractional (g,f)-factor is just a fractional k-factor if $g(x)=f(x)=k$ ($k\ge 1$ is an integer) for all $x\in V(G)$.

If for each edge subset $H\subseteq E(G)$ with $|H|=m$, there exists a fractional (g,f)-factor h such that $h(e)=0$ for all $e\in H$, then a graph G is called a *fractional* (g,f,m)*-deleted graph*. It means, after deleting any m edges, the resulting graph still has a fractional (g,f)-factor. A graph G is called a *fractional* (g,f,n')*-critical graph* if after removing any n' vertices from G, the resulting graph still has a fractional (g,f)-factor. As extensions of fractional factor, fractional critical graph and Fractional deleted graph measure the existence of fractional factor in communication networks if some sites or channels are not well used.

A graph G is called a *fractional* (g,f,n',m)*-critical deleted graph* if after removing any n' vertices from G, the resulting graph is still a fractional (g,f,m)-deleted graph. If $g(x)=f(x)$ for each $x\in V(G)$, then fractional (g,f,n')-critical graph, fractional (g,f,m)-deleted graph, and fractional (g,f,n',m)-critical deleted graph are fractional (f,n')-critical graph, fractional (f,m)-deleted graph and fractional (f,n',m)-critical deleted graph, respectively. In addition, if $g(x)=f(x)=k$ ($k\ge 1$ is an integer) for any $x\in V(G)$, then fractional (g,f,n')-critical graph, fractional (g,f,m)-deleted graph and fractional (g,f,n',m)-critical deleted graph are just fractional (k,n')-critical graph, fractional (k,m)-deleted graph and fractional (k,n',m)-critical deleted graph, respectively.

In the whole context, we always assume that $n=|V(G)|$ and G is not complete. For fractional deleted graphs and fractional critical graphs, there are some known results. Zhou and Liu [15] proved that G is a fractional (k,m)-deleted graph if for any $k\ge 2$ and $m\ge 0$, $n\ge 4k^2+2k-6+\frac{(4k^2+6k-2)m-2}{k-1}$, $\delta(G)\ge k+m+\frac{m}{k+1}$ and $\max\{d_G(u),d_G(v)\}\ge\frac{n}{2}$ for any vertices x and y of G with $d_G(x,y)=2$. Zhou [13] determined a sufficient condition for fractional (k,m)-deleted graph: $n\ge 9k-1-\sqrt{2(k-1)^2+2}+2(2k+1)m$, $\delta(G)\ge k+m+\frac{(m+1)^2-1}{4k}$, and $|N_G(x)\cup N_G(y)|\ge\frac{1}{2}(n+k-2)$ for each pair of non-adjacent vertices x, y of G, where $k\ge 2$ and $m\ge 0$. Zhou [12] presented that G is a fractional (k,m)-deleted graph if $n\ge 4k-5+2(2k+1)m$ and $\delta(G)\ge\frac{n}{2}$. Gao and Wang [7] improved the above result and showed that is a fractional (k,m)-deleted graph if $n\ge 4k+4m-3$, $\delta(G)\ge k+m$ and $\max\{d_G(u),d_G(v)\}\ge\frac{n}{2}$ for each pair of non-adjacent vertices u and v of G, where $k\ge 2$ and $m\ge 0$.

More sufficient conditions for graphs to have fractional factors can be found in Gao and Gao [3], Gao and Farahani [6], Gao et al. [4, 5], Gao and Wang [7], [8] [9] and [10], Zhou [13], [14] and [16], and Zhou and Bian [17].

The main contribution of our paper is to determine an independent set degree condition for fractional (g, f, n', m)-critical deleted graph, and the main conclusion can be stated as follows.

Theorem 1.1. *Let G be a graph of order n, and let a, b, n', m, Δ, and i be non-negative integers such that $i \geq 2$, $1 \leq a \leq b - \Delta$, $n > \frac{(a+b)(ib+2m-2)}{a} + n'$ and $\delta(G) \geq \frac{b^2(i-1)}{a} + n' + 2m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$ for each $x \in V(G)$. If G satisfies*

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{b(n+n')}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional (g, f, n', m)-critical deleted graph.

Set $\Delta = 0$ in Theorem 1.1, we get the following result for a graph to be fractional (g, f, n', m)-critical deleted.

Corollary 1. *Let G be a graph of order n, and let a, b, n', m, and i be non-negative integers such that $i \geq 2$, $1 \leq a \leq b$, $n > \frac{(a+b)(ib+2m-2)}{a} + n'$ and $\delta(G) \geq \frac{b^2(i-1)}{a} + n' + 2m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If G satisfies*

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{b(n+n')}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional (g, f, n', m)-critical deleted graph.

Set $a = b = k$ in Theorem 1.1, then $f(x) = g(x) = k$ for any $x \in V(G)$, and $\Delta = 0$. The independent set degree condition for fractional (k, n', m)-critical deleted graph can be described as follows.

Corollary 2. *Let G be a graph of order n. Let k, i, n', m be four non-negative integers with $i \geq 2$, $k \geq 1$ and $m, n' \geq 0$. If $\delta(G) \geq k(i-1) + 2m + n'$, $n > 2ki + n' + 4m - 4$, and*

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{n+n'}{2}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional (k, n', m)-critical deleted graph.

Set $n' = 0$ in Theorem 1.1, then the following becomes necessary for independent set degree condition on fractional (g, f, m)-deleted graph.

Corollary 3. *Let G be a graph of order n, and let a, b, m, Δ, and i be non-negative integers such that $i \geq 2$, $1 \leq a \leq b - \Delta$, $n > \frac{(a+b)(ib+2m-2)}{a}$ and $\delta(G) \geq \frac{b^2(i-1)}{a} + 2m$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$ for each $x \in V(G)$. If G satisfies*

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{bn}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional (g, f, m)-deleted graph.

Set $m = 0$ in Theorem 1.1, then we deduce the following corollary on the independent set degree condition of fractional (g, f, n')-critical graph.

Corollary 4. *Let G be a graph of order n, and let a, b, n', Δ, and i be non-negative integers such that $i \geq 2$, $1 \leq a \leq b - \Delta$, $n > \frac{(a+b)(ib-2)}{a} + n'$ and $\delta(G) \geq \frac{b^2(i-1)}{a} + n'$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) - \Delta \leq b - \Delta$ for each $x \in V(G)$. If G satisfies*

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{b(n+n')}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional (g, f, n')-critical graph.

Set $f(x) = g(x)$ for any $x \in V(G)$ in Corollary 1, we yield the following corollary on the independent set degree condition of fractional (f, n', m)-critical deleted graph.

Corollary 5. *Let G be a graph of order n, and let a, b, n', m, and i be non-negative integers such that $i \geq 2$, $1 \leq a \leq b$, $n > \frac{(a+b)(ib+2m-2)}{a} + n'$ and $\delta(G) \geq \frac{b^2(i-1)}{a} + n' + 2m$. Let f be an integer-valued functions defined on $V(G)$ such that $a \leq f(x) \leq b$ for each $x \in V(G)$. If G satisfies*

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{b(n+n')}{a+b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional (f, n', m)-critical deleted graph.

By setting $n' = 0$ in Corollary 2, then it becomes the following independent set degree condition of fractional (k, m)-deleted graph.

Corollary 6. *Let G be a graph of order n. Let k, i, m be three non-negative integers with $i \geq 2$, $k \geq 1$ and $m \geq 0$. If $\delta(G) \geq k(i-1) + 2m$, $n > 4ki + 4m - 4$, and*

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{n}{2}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional (k, m)-deleted graph.

If setting $m = 0$ in Corollary 2, then we deduce the following independent set degree condition of fractional (k, n')-critical graph.

Corollary 7. *Let G be a graph of order n. Let k, i, n' be three non-negative integers with $i \geq 2$, $k \geq 1$ and $n' \geq 0$. If $\delta(G) \geq k(i-1) + n'$, $n > 4ki + n' - 4$, and*

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{n+n'}{2}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional (k, n')-critical graph.

The prove of Theorem 1.1 is depended heavily on the following Lemma which manifests the necessary and sufficient condition of fractional (g, f, n', m)-critical deleted graph.

Lemma 1.2. (Gao [2]) *Let G be a graph, g, f be two integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$. Let n', m be two non-negative integers. Then G is a fractional (g, f, n', m)-critical deleted graph if and only if*

$$\begin{aligned} & f(S) - g(T) + d_{G-S}(T) \\ \geq & \max_{U \subseteq S, |U|=n', H \subseteq E(G-U), |H|=m} \{f(U) + \sum_{x \in T} d_H(x) - e_H(T, S)\} \end{aligned} \tag{1}$$

for all disjoint subsets S, T of $V(G)$ with $|S| \geq n'$.

2. **Proof of Theorem 1.1.** Assume in the contrary that G satisfies the conditions of the theorem, but is not a fractional (g, f, n', m)-critical deleted graph. By Lemma 1.2 and noting the fact that $\sum_{x \in T} d_H(x) - e_H(T, S) \leq 2m$, there exist disjoint subsets S and T of $V(G)$ with $|S| \geq n'$ satisfying

$$\delta_G(S, T) = f(S - U) + \sum_{x \in T} d_{G-S}(x) - g(T) \leq 2m - 1. \tag{2}$$

We choose subsets S and T such that $|T|$ is minimal. Obviously, $T \neq \emptyset$.

Claim 1. $d_{G-S}(x) \leq g(x) - 1 \leq b - \Delta - 1$ for any $x \in T$.

Proof. If $d_{G-S}(x) \geq g(x)$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (2). This contradicts the choice of S and T. □

Let $d_1 = \min\{d_{G-S}(x) | x \in T\}$ and choose $x_1 \in T$ such that $d_{G-S}(x_1) = d_1$. If $z \geq 2$ and $T \setminus (\cup_{j=1}^{z-1} N_T[x_j]) \neq \emptyset$, let

$$d_z = \min\{d_{G-S}(x) | x \in T \setminus (\cup_{j=1}^{z-1} N_T[x_j])\}$$

and choose $x_z \in T \setminus (\cup_{j=1}^{z-1} N_T[x_j])$ such that $d_{G-S}(x_z) = d_z$. So, we get a sequence such that $0 \leq d_1 \leq d_2 \leq \cdots \leq d_\pi \leq g(x) - 1 \leq b - \Delta - 1$ and an independent set $\{x_1, x_2, \ldots, x_\pi\} \subseteq T$.

Claim 2. $|T| \geq (i-1)b + 1$.

Proof. Assume that $|T| \leq (i-1)b$. Then $|S| + d_1 \geq d_G(x_1) \geq \delta(G) \geq \frac{b^2(i-1)}{a} + n' + 2m$. By (2) and $0 \leq d_1 \leq b - \Delta - 1$, we have

$$\begin{aligned} 2m - 1 & \geq f(S - U) - g(T) + d_{G-S}(T) \\ & \geq (a + \Delta)|S - U| + d_1|T| - (b - \Delta)|T| \\ & = (a + \Delta)|S - U| + (d_1 - b + \Delta)|T| \\ & \geq (a + \Delta)(\frac{b^2(i-1)}{a} - d_1 + 2m) + (d_1 - b + \Delta)(i-1)b \\ & = b^2(i-1) + d_1(b(i-1) - a) - b^2(i-1) + 2m(a + \Delta) \\ & \quad + \Delta(\frac{b^2(i-1)}{a} - d_1 + b(i-1)) \\ & \geq 2m. \end{aligned}$$

This produces a contradiction. □

Since $d_{G-S}(x) \leq b - \Delta - 1$ and $|T| \geq (i-1)b + 1$, we get $\pi \geq i$. Thus, we can choose an independent set $\{x_1, x_2, \ldots, x_i\} \subseteq T$.

In view of the condition of the theorem, we get

$$\frac{b(n + n')}{a + b} \leq \max\{d_1, d_2, \cdots, d_i\} \leq |S| + d_i$$

and

$$|S| \geq \frac{b(n+n')}{a+b} - d_i. \tag{3}$$

By means of

$$|N_T[x_j]| - |N_T[x_j] \cap (\cup_{z=1}^{j-1} N_T[x_z])| \geq 1, j = 2, 3, \ldots, i-1$$

and

$$\begin{aligned} |\cup_{z=1}^{j} N_T[x_z]| &\leq \sum_{z=1}^{j} |N_T[x_z]| \\ &\leq \sum_{z=1}^{j} (d_{G-S}(x_z)+1) \\ &= \sum_{z=1}^{j} (d_z+1), j = 1, 2, \ldots, i, \end{aligned}$$

we deduce

$$\begin{aligned} & f(S-U) + d_{G-S}(T) - g(T) \\ \geq\ & (a+\Delta)|S-U| - (b-\Delta)|T| + d_1|N_T[x_1]| + d_2(|N_T[x_2]| - |N_T[x_2] \cap N_T[x_1]|) \\ & + \cdots + d_{i-1}(|N_T[x_{i-1}]| - |N_T[x_{i-1}] \cap (\cup_{j=1}^{i-2} N_T[x_j])|) \\ & + d_i(|T| - |\cup_{j=1}^{i-1} N_T[x_j])||) \\ \geq\ & (a+\Delta)|S-U| + (d_1-d_i)|N_T[x_1]| + \sum_{j=2}^{i-1} d_j + (d_i - b + \Delta)|T| - d_i \sum_{j=2}^{i-1} |N_T[x_j]| \\ =\ & (a+\Delta)|S-U| + (d_1-d_i)(d_1+1) + \sum_{j=2}^{i-1} d_j + (d_i - b + \Delta)|T| - d_i \sum_{j=2}^{i-1} (d_j+1) \\ =\ & (a+\Delta)|S-U| + d_1^2 + \sum_{j=1}^{i-1} d_j + (d_i - b + \Delta)|T| - d_i \sum_{j=1}^{i-1} (d_j+1). \end{aligned}$$

It implies

$$\begin{aligned} & (n - |S| - |T|)(b - \Delta - d_i) \\ \geq\ & f(S-U) + d_{G-S}(T) - g(T) - 2m + 1 \\ \geq\ & (a+\Delta)|S| + d_1^2 + \sum_{j=1}^{i-1} d_j + (d_i - b + \Delta)|T| - d_i \sum_{j=1}^{i-1} (d_j+1) - (a+\Delta)n' - 2m + 1. \end{aligned}$$

Equivalently,

$$0 \leq n(b-\Delta-d_i) - (a+b-d_i)|S| + d_i \sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j + d_i(i-1) - d_1^2 + (a+\Delta)n' + 2m - 1. \tag{4}$$

By (3), (4), $d_1 \le d_2 \le \cdots \le d_i \le b-\Delta-1$ and $n > \frac{(a+b)(ib+2m-2)}{a} + n'$, we infer

$$\begin{aligned}
0 &\le n(b-\Delta-d_i) - (a+b-d_i)(\frac{b(n+n')}{a+b} - d_i) + d_i\sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j \\
&\quad + d_i(i-1) - d_1^2 + (a+\Delta)n' + 2m - 1 \\
&= -\frac{an}{a+b}d_i + (a+b)d_i - d_i^2 + d_i\sum_{j=1}^{i-1} d_j - \sum_{j=1}^{i-1} d_j \\
&\quad + d_i(i-1) - d_1^2 + (a+\Delta-b+\frac{bd_i}{a+b})n' + 2m - 1 - n\Delta \\
&= -\frac{an}{a+b}d_i + (d_id_1 - d_1 - d_1^2) + (d_i-1)\sum_{j=2}^{i-1} d_j \\
&\quad + d_i(a+b+i-1) - d_i^2 + (a+\Delta-b+\frac{bd_i}{a+b})n' + 2m - 1 - n\Delta \\
&\le -\frac{an}{a+b}d_i + (d_i\frac{d_i-1}{2} - \frac{d_i-1}{2} - (\frac{d_i-1}{2})^2) + (d_i-1)\sum_{j=2}^{i-1} d_i \\
&\quad + d_i(a+b+i-1) - d_i^2 + (a+\Delta-b+\frac{bd_i}{a+b})n' + 2m - 1 - n\Delta \\
&= -\frac{an}{a+b}d_i + (i-\frac{11}{4})d_i^2 + d_i(a+b+\frac{1}{2}) + \frac{1}{4} + (a+\Delta-b+\frac{bd_i}{a+b})n' \\
&\quad + 2m - 1 - n\Delta \\
&< -(ib+2m-2)d_i + (i-\frac{11}{4})d_i^2 + d_i(a+b+\frac{1}{2}) \\
&\quad + ((b-a)(\frac{d_i}{a+b} - 1) + \Delta)n' + 2m - \frac{3}{4} - n\Delta \\
&= (i-\frac{11}{4})d_i^2 + d_i(a+b+\frac{5}{2} - ib) + ((b-a)(\frac{d_i}{a+b} - 1) + \Delta)n' \\
&\quad + 2m(1-d_i) - \frac{3}{4} - n\Delta \\
&\le (i-\frac{11}{4})d_i^2 + d_i(a+b+\frac{5}{2} - ib) + (\Delta(\frac{d_i}{a+b} - 1) + \Delta)n' \\
&\quad + 2m(1-d_i) - \frac{3}{4} - n\Delta \\
&= (i-\frac{11}{4})d_i^2 + d_i(a+b+\frac{5}{2} - ib) + \Delta(\frac{d_i}{a+b}n' - n) + 2m(1-d_i) - \frac{3}{4} \\
&\le (i-\frac{11}{4})d_i^2 + d_i(a+b+\frac{5}{2} - ib) + 2m(1-d_i) - \frac{3}{4}.
\end{aligned}$$

Now, we consider the following two cases:
Case 1. If $d_i > 0$, then by $2m(1-d_i) \le 0$, we infer

$$0 < (i-\frac{11}{4})d_i^2 + d_i(a+b+\frac{5}{2} - ib) - \frac{3}{4}. \tag{5}$$

Set

$$h(d_i) = (i-\frac{11}{4})d_i^2 + d_i(a+b+\frac{5}{2} - ib) - \frac{3}{4}. \tag{6}$$

- If $i \leq \frac{11}{4}$, then i must equal to 2. Thus,

$$h(d_i) = -\frac{3}{4}d_i^2 + d_i(a - b + \frac{5}{2}) - \frac{3}{4}.$$

If $b \geq a + \frac{5}{2}$, clearly $h(d_i) < 0$. If $b = a + 2$, then $h(d_i) = -\frac{3}{4}d_i^2 + \frac{1}{2}d_i - \frac{3}{4} < 0$. If $b = a + 1$, then $h(d_i) = -\frac{3}{4}d_i^2 + \frac{3}{2}d_i - \frac{3}{4}$ and $\max\{h(d_i)\} = d(1) = 0$. If $a = b = k$, then $n > \frac{(a+b)(ib+2m-2)}{a} + n'$ can be rewritten as $n \geq 4k + 4m + n' - 3$. In terms of Theorem 7 in Gao and Yan [11] (it determined that G is a fractional (k, n', m)-critical deleted graph if $n \geq 4k + 4m + n' - 3$, $\delta(G) \geq k + m + n'$ and $\max\{d(u), d(v)\} \geq \frac{n+n'}{2}$ for each pair of non-adjacent vertices u and v of G, where $k \geq 2$, $m, n' \geq 0$), we ensure that G is a fractional (k, n', m)-critical deleted graph unless $a = b = 1$. If $a = b = 1$, then $d_2 = 0$ by $d_i \leq k - 1$, but it is contradictory with the assumption that $d_i > 0$.

- If $i > \frac{11}{4}$ (it means $i \geq 3$ since i is an integer), then $\max\{h(d_i)\} = \max\{h(0), h(b-1)\}$. Since $h(0) = -\frac{3}{4} < 0$ and (if $b - 1 \neq 0$, then $b \geq 2$)

$$h(b-1) = (i-\frac{11}{4})(b-1)^2 + (b-1)(a+b+\frac{5}{2}-ib) - \frac{3}{4} = (b-1)(a-\frac{7}{4}b-i+\frac{21}{4}) - \frac{3}{4}.$$

If $b \geq 3$, then $h(b-1) < 0$ by $b \geq a$ and $a - \frac{7}{4}b - i + \frac{21}{4} \leq 0$. If $b = 2$, then using $i \geq 3$, we yield $h(b-1) = (b-1)(a - \frac{7}{4}b - i + \frac{21}{4}) - \frac{3}{4} \leq 0$. If $a = b = 1$, then $h(b-1) = -\frac{3}{4} < 0$. Therefore, we conclude that $\max\{h(d_i)\} \leq 0$ in the case $i > \frac{11}{4}$.

Case 2. If $d_i = 0$, then $d_1 = \cdots = d_i = 0$. By (2), we have $|S| \geq \frac{b(n+n')}{a+b}$ and $|T| \leq n - |S| \leq \frac{an-bn'}{a+b}$. Since $d_{G-S}(T) \geq \sum_{x\in T} d_H(x) - e_H(T, S)$, we have

$$\begin{aligned} & f(S-U) + d_{G-S}(T) - g(T) - (\sum_{x\in T} d_H(x) - e_H(T,S)) \\ \geq\ & a\cdot(\frac{b(n+n')}{a+b} - n') - b\cdot\frac{an-bn'}{a+b} + (d_{G-S}(T) - \sum_{x\in T} d_H(x) + e_H(T,S)) \\ \geq\ & (b-a)n' \geq 0, \end{aligned}$$

also a contradiction. This completes the proof of the theorem. □

3. **Sharpness.** Theorem 1.1 is best possible, to some extent, under the condition. Actually, we can construct some graphs such that the independent set degree condition in Theorem 1.1 can't be replaced by $\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{b(n+n')}{a+b} - 1$.

Let $G_1 = K_{bt+n'}$ be a complete graph, $G_2 = (at+1)K_1$ be a graph consisting of $at + 1$ isolated vertices, and $G = G_1 \vee G_2$, where t is sufficiently large and $n' < \frac{a}{b-a}$. Then $n = |G_1| + |G_2| = (a + b)t + n' + 1$, and for any independent set $\{x_1, x_2, \ldots, x_i\} \subseteq V(G_2)$, we have

$$\frac{b(n+n')}{a+b} > \max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} = bt + n' > \frac{b(n+n')}{a+b} - 1.$$

Let $S = V(G_1)$, and $g(x) = f(x) = a$ for any $x \in V(G_1)$; $T = V(G_2)$, and $g(x) = f(x) = b$ for any $x \in V(G_2)$. Then

$$\begin{aligned} & f(S) - g(T) + d_{G-S}(T) \\ & - \max_{U\subseteq S, |U|=n', H\subseteq E(G-U), |H|=m} \{f(U) + \sum_{x\in T} d_H(x) - e_H(T,S)\} \\ =\ & a|S-U| - b|T| = a(bt) - b(at+1) = -b < 0. \end{aligned}$$

Hence, G is not a fractional (g, f, n', m)-critical deleted graph.

4. **An independent set degree condition for fractional (a, b, n', m)-critical deleted graphs.** Let $g(x) = a$, $f(x) = b$ for each $x \in V(G)$. The sufficient and necessary condition for a graph to be fractional (a, b, n', m)-critical deleted is derived from Lemma 1.2.

Lemma 4.1. *Let G be a graph. Let a, b, n', m be non-negative integers such that $a \leq b$. Then G is a fractional (a, b, n', m)-critical deleted graph if and only if*

$$b|S| - a|T| + d_{G-S}(T) \geq \max_{|H|=m} \{bn' + \sum_{x \in T} d_H(x) - e_H(T, S)\} \tag{7}$$

for all disjoint subsets S, T of $V(G)$ with $|S| \geq n'$.

In view of Lemma 4.1, if we assume that G is not a fractional (a, b, n', m)-critical deleted graph, then $T \neq \emptyset$ and there exist disjoint subsets S and T of $V(G)$ satisfying

$$b|S| - a|T| + d_{G-S}(T) \leq bn' + 2m - 1, \tag{8}$$

where $|S| \geq n'$. We choose S and T such that $|T|$ is minimum. Thus, $d_{G-S}(x) \leq a - 1$ for each $x \in T$.

Let $d_1 = \min\{d_{G-S}(x) : x \in T\}$. If $T - N_T[x_1] \neq \emptyset$, let $d_2 = \min\{d_{G-S}(x) : x \in T - N_T[x_1]\}$ and choose $x_2 \in T - N_T[x_1]$ such that $d_{G-S}(x_2) = d_2$. Then, using the similar way described in Section 2, we get a sequence such that $0 \leq d_1 \leq d_2 \leq \cdots \leq d_i \leq a - 1$ and an independent set $\{x_1, x_2, \ldots, x_i\} \subseteq T$.

Finally, according to similar fashion presented in Section 2, we get the following conclusion on independent set degree condition for fractional (a, b, n', m)-critical deleted graphs, and the detailed proofs is skipped.

Theorem 4.2. *Let G be a graph of order n, and let a, b, n', m, and i be non-negative integers such that $i \geq 2$, $1 \leq a \leq b$, $n > \frac{(a+b)(ib+2m-2)}{b} + n'$ and $\delta(G) \geq a + m + n'$. If G satisfies*

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_i)\} \geq \frac{an + bn'}{a + b}$$

for any independent subset $\{x_1, x_2, \ldots, x_i\}$ of $V(G)$, then G is a fractional (a, b, n', m)-critical deleted graph.

Remark 1. *Although fractional (a, b, n', m)-critical deleted graph is a special kind of fractional (g, f, n', m)-critical deleted graph when $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, Theorem 4.2 can't be derived directly from corollary 1. Hence, clues for proving Theorem 4.2 which we present above are necessary.*

The example $G = K_{bt+n'} \vee G_2 = (at + 1)K_1$ similar as in Section 3 reveals that the independent set degree condition in Theorem 4.2 is also sharp in some sense.

5. **Conflict of Interests.** The authors declare that there is no conflict of interests regarding the publication of this paper.

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