FRACTAL DIMENSION FOR IFS-ATTRACTORS REVISITED

Abstract. One of the milestones in Fractal Geometry is the so-called Moran's Theorem, which allows the calculation of the similarity dimension of any strict self-similar set under the open set condition. In this paper, we contribute a generalized version of the Moran's theorem, which does not require the OSC to be satisfied by the similitudes that give rise to the corresponding attractor. To deal with, two generalized versions for the classical fractal dimensions, namely, the box and the Hausdorff dimensions, are explored in terms of fractal structures, a kind of uniform spaces.

1. Introduction

In this paper, we re-explore a classical problem in Fractal Geometry, namely, how to calculate the Hausdorff dimension of the attractor of an iterated function system (IFS in the sequel). It is worth mentioning that a particular solution for such an awkward problem needs the open set condition (OSC, hereafter) to be satisfied by the similitudes of the corresponding IFS. The OSC allows to control the overlapping among the self-similar copies of the whole IFS-attractor, sometimes called as *pre-fractals*. Equivalently, it is also said that under the OSC, the pieces $f_i(\mathcal{K})$ have only "small overlap", also called "just touching", as pointed out in [16].

We have to trace back to the forties to find out the key result that allows the effective calculation of the Hausdorff dimension for strict self-similar sets from their similarity ratios. It was first proved by the Australian mathematician P.A.P. Moran, a Besicovitch pupil at Cambridge (c.f. [14, Theorem II]). It is worth mentioning that this theorem becomes a particular case of stronger [14, Theorem III], though it could be also deduced from [14, Theorem I], which establishes a connection between the Hausdorff dimension and the existence of finite measures with some metric properties. More specifically, it states that whether there exists a finite nonzero measure ϕ such that $\phi(R) \leq \kappa \cdot \mathrm{diam}\,(R)^p$, where R is a $q-$dimensional cube containing a compact subset $E \subseteq \mathbb{R}^q$ for appropriate constants κ and p, then $\dim_{\mathrm{H}}(E) \geq p$. In addition, he also provided an easy formula to calculate the Hausdorff dimension of attractors whose similitudes have a common similarity ratio.

Key words and phrases. Fractal; iterated function system; IFS-attractor; self-similar set; fractal structure; box dimension; Hausdorff dimension; open set condition; Moran's Theorem.

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That quantity only depends on the number of similitudes in that IFS and that common value, as well.

A new point of view regarding *fractals* arises from the concept of *fractal structure*, which derives from asymmetric topology. A fractal structure is a kind of uniformity which provides better approaches of a given space as deeper stages in its structure, called *levels*, are explored. In fact, the underlying idea is to endow a fractal structure on a (topological) space, which allows to study fractal patterns therein, in contrast to understand such a space as a fractal itself depending on the self-similar properties it presents at a whole range of scales. It is also worth mentioning that fractal structures do provide a novel context where new Hausdorff type measures could be defined. In other words, the classical fractal dimension models, namely, both the box and the Hausdorff dimensions, remain as particular cases from some discrete models of fractal dimension for a fractal structure that are explored along this paper. The main contribution in this paper consists of a generalized Moran's Theorem for IFS-attractors not required to lie under the OSC.

The structure of this paper is as follows. In Section 2, we provide all the mathematical background which makes this article self-contained. It includes the notation from the domain of words as well as the basics on attractors, the OSC, a brief description regarding both the box and the Hausdorff dimensions, and fractal structures, as well. Section 3 explains how the discrete models of fractal dimension explored along this paper allow to generalize the classical fractal dimensions. This has been carried out through both Theorems 3.7 and 3.9. Section 4 contains the main result in this paper, namely, a generalized Moran's Theorem (c.f. Theorem 4.2). Finally, Section 5 summarizes the main results contributed along this paper.

2. Preliminaries

2.1. **General notation.** Along the sequel, we shall use the following notation. Let $\Sigma = \{1, \ldots, k\}$ be a finite (nonempty) set. For each natural number n, let $\Sigma^n = \{\mathbf{i} = i_1 \ldots\ i_n : i_j \in \Sigma, j = 1, \ldots, n\}$ be the set consisting of all the words of length n from Σ. Further, let Σ^∞ denote the collection of either all finite ($\cup_{n\in\mathbb{N}}\Sigma^n$) or infinite ($\Sigma^\mathbb{N}$) words from Σ, i.e., $\Sigma^\infty = \cup_{n\in\mathbb{N}}\Sigma^n \cup \Sigma^\mathbb{N}$.

2.2. **IFS-attractors.** Let $k \geq 2$. By an iterated function system (IFS), we shall understand a finite collection of similitudes on $\mathbb{R}^d$, say $\mathcal{F} = \{f_1, \ldots, f_k\}$, where each self-map $f_i : \mathbb{R}^d \to \mathbb{R}^d$ fulfils the following identity:

$$d(f_i(x), f_i(y)) = c_i \cdot d(x, y) \text{ for all } x, y \in \mathbb{R}^d,$$

where $c_i \in (0, 1)$ is the similarity ratio associated with each $f_i \in \mathcal{F}$, and d denotes the Euclidean distance. In this paper, we shall be focused on Euclidean iterated function systems (EIFS). Then there exists a unique (nonempty) compact subset $\mathcal{K} \subset \mathbb{R}^d$ such that

$$\mathcal{K} = \cup\{f_i(\mathcal{K}) : i = 1, \ldots, k\}. \tag{1}$$

The previous expression is usually known as Hutchinson's equation [12] and $\mathcal{K}$ is named the IFS-attractor (equivalently, the *self-similar set*) generated by $\mathcal{F}$ and consists of (smaller) self-similar copies $\mathcal{K}_i$ of the whole attractor $\mathcal{K}$, also known as *pre-fractals* of $\mathcal{K}$. Thus, $\mathcal{K}_i = f_i(\mathcal{K})$ for all $i = 1, \ldots, k$. Also, we shall denote $\mathcal{K}_{ij} = f_i(f_j(\mathcal{K}))$, and so on. Further, if we write $f_{\mathbf{i}} = f_{i_1} \circ \ldots \circ f_{i_n}$, $c_{\mathbf{i}} = c_{i_1} \ldots c_{i_n}$, and $\mathcal{K}_{\mathbf{i}} = f_{\mathbf{i}}(\mathcal{K})$, then Eq. (1) can be rewritten as follows:

$$\mathcal{K} = \cup\{\mathcal{K}_{\mathbf{i}} : i \in \Sigma^n\}.$$

Letting $n \to \infty$, the *address map* $\pi : \Sigma^\infty \to \mathcal{K}$ stands as a continuous map from Σ^∞ onto the IFS-attractor $\mathcal{K}$.

2.3. **The open set condition.** An IFS $\mathcal{F} = \{f_1, \ldots, f_k\}$ (or $\mathcal{K}$, for short) fulfils the open set condition (OSC) if there exists a nonempty open subset $\mathcal{V} \subseteq \mathbb{R}^d$ such that all the images $f_i(\mathcal{V})$ are contained in $\mathcal{V}$ and remain pairwise disjoint, i.e.,

$$\cup_{i=1}^k f_i(\mathcal{V}) \subseteq \mathcal{V}, \text{with } f_i(\mathcal{V}) \cap f_j(\mathcal{V}) = \varnothing, \text{ if } i \neq j.$$

As it was stated in [2], the open subset $\mathcal{V}$ is named a *feasible open set* of the similitudes $f_i \in \mathcal{F}$ (or of $\mathcal{K}$). The OSC was first contributed by Moran in [14] to show that the Hausdorff measure is positive on the IFS-attractor $\mathcal{K}$. Interestingly, the reciprocal is also true, i.e., a positive Hausdorff measure implies the OSC. This fact was contributed by Schief (c.f. [16]). It is worth noting that if α is the similarity dimension of $\mathcal{K}$ (c.f. Section 4), then we always have $\mathcal{H}^\alpha_{\mathrm{H}}(\mathcal{K}) < \infty$ (without requiring any additional assumption) (c.f. [12, Proposition 4 (i)]).

On the other hand, since the feasible open set $\mathcal{V}$ and the IFS-attractor $\mathcal{K}$ may be disjoint, Lalley strengthened the definition of the OSC in the following terms [13]: the strong open set condition (SOSC) stands iff it is satisfied, additionally, that $\mathcal{K} \cap \mathcal{V} \neq \varnothing$. Schief also proved that both the OSC and the SOSC are equivalent in the Euclidean setting (c.f. [16, Theorem 2.2]).

2.4. **The Hausdorff and box dimensions.** Let (X, ρ) be a metric space. Along the sequel, $\operatorname{diam}(A)$ will denote the diameter of any subset A of X, i.e., $\operatorname{diam}(A) = \sup\{\rho(x,y) : x, y \in A\}$, as usual. In addition, let $\delta > 0$ and F be a subset of X. By a $\delta-$cover of F, we shall understand a countable family of subsets $\{U_i\}_{i \in I}$ such that $F \subseteq \cup_{i \in I} U_i$ with $\operatorname{diam}(U_i) \leq \delta$. Moreover, let $\mathcal{C}_\delta(F)$ be the collection of all $\delta-$covers of F and define

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum \operatorname{diam}(U_i)^s : \{U_i\}_{i \in I} \in \mathcal{C}_\delta(F)\right\}.$$

Then $\mathcal{H}^\alpha_{\mathrm{H}}(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F)$ always exists and is known as the ($s-$dimensional) Hausdorff measure of F. It allows to characterize the Hausdorff dimension of F as the (unique) critical point $s \geq 0$ where $\mathcal{H}^\alpha_{\mathrm{H}}(F)$ "jumps" from ∞ to zero, namely,

$$\dim_{\mathrm{H}}(F) = \sup\{s : \mathcal{H}^\alpha_{\mathrm{H}}(F) = \infty\} = \inf\{s : \mathcal{H}^\alpha_{\mathrm{H}}(F) = 0\}.$$

In particular, for $s = \dim_{\mathrm{H}}(F)$, it holds that $\mathcal{H}^s_{\mathrm{H}}(F) \in \{0, d, \infty : d \in (0, \infty)\}$.

Though the Hausdorff dimension is the most accurate model for fractal dimension, the box dimension becomes more appropriate to tackle with empirical applications. The (lower/upper) box dimension of $F \subseteq \mathbb{R}^d$ is given by the (lower/upper) limit that follows:

$$\dim_{\mathrm{B}}(F) = \lim_{\delta \to 0} \frac{\log \mathcal{N}_\delta(F)}{-\log \delta},$$

where $\mathcal{N}_\delta(F)$ is the number of $\delta-$cubes that intersect F. Recall that a $\delta-$cube in $\mathbb{R}^d$ is a set of the form $\{[k_1\,\delta, (k_1+1)\,\delta] \times \ldots \times [k_d\,\delta, (k_d+1)\,\delta] : k_1, \ldots, k_d \in \mathbb{Z}\}$. In particular, we can discretize $\delta = 2^{-n}$. Also, $\mathcal{N}_\delta(F)$ could be calculated equivalently as the smallest number of sets with diameters $\leq \delta$ that cover F.

2.5. **Fractal structures.** The concept of fractal structure, which naturally appears in several topics related to asymmetric topology, was first sketched by Bandt and Retta in [3]. It is worth pointing out that in [6, 11], such a concept was applied to explore novel definitions of fractal dimension in more general contexts.

A family Γ of subsets of (a nonempty set) X is said to be a covering if $X = \cup\{A : A \in \Gamma\}$. Let Γ_1 and Γ_2 be two coverings of X. By $\Gamma_1 \prec \Gamma_2$, we shall understand that Γ_1 is a *refinement* of Γ_2, i.e., for all $A \in \Gamma_1$, there exists $B \in \Gamma_2$ such that $A \subseteq B$. Moreover, $\Gamma_1 \prec\prec \Gamma_2$ means that Γ_1 is a strong refinement of Γ_2, namely, $\Gamma_1 \prec \Gamma_2$ and for all $B \in \Gamma_2$, we can write $B = \cup\{A \in \Gamma_1 : A \subseteq B\}$. A fractal structure on X is a countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n\}_{n\in\mathbb{N}}$, where $\Gamma_{n+1} \prec\prec \Gamma_n$ for all $n \in \mathbb{N}$. The covering Γ_n is called *level* n of $\mathbf{\Gamma}$. Along the sequel, we shall allow that a set can appear twice or more in any level of a fractal structure. A fractal structure is said to be finite provided that all its levels are finite coverings. It is worth mentioning any IFS-attractor can be always endowed with a *natural* fractal structure we define next.

Definition 2.1 (c.f. [1], Definition 4.4)**.** *Let $\mathcal{F}$ be an IFS with associated attractor $\mathcal{K}$. The natural fractal structure on $\mathcal{K}$ (as a self-similar set) is the countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n\}_{n\in\mathbb{N}}$ with levels given by $\Gamma_n = \{f_{\boldsymbol{i}}(\mathcal{K}) : \boldsymbol{i} \in \Sigma^n\}$.*

Remark 2.2. *Equivalently, the levels of the natural fractal structure on any IFS-attractor $\mathcal{K}$ can be described as follows: $\Gamma_1 = \{f_i(\mathcal{K}) : i \in \Sigma\}$, and $\Gamma_{n+1} = \{f_i(A) : A \in \Gamma_n, i \in \Sigma\}$ for all $n \in \mathbb{N}$.*

On the other hand, any subset of $\mathbb{R}^d$ can be endowed with its *natural fractal structure* (as a Euclidean subset).

Definition 2.3 (c.f. [8], Definition 3.1)**.** *Let $F \subseteq \mathbb{R}^d$. The natural fractal structure on F (as a Euclidean subset) is defined as the countable family of coverings $\mathbf{\Gamma} = \{\Gamma_n\}_{n\in\mathbb{N}}$ with levels given by*

$$\Gamma_n = \left\{ \left[\frac{k_1}{2^n}, \frac{k_1+1}{2^n}\right] \times \ldots \times \left[\frac{k_d}{2^n}, \frac{k_d+1}{2^n}\right] : k_1, \ldots, k_d \in \mathbb{Z} \right\}.$$

Such a fractal structure is a tiling consisting of $2^{-n}-$cubes on $\mathbb{R}^d$.

3. Generalizing both the Hausdorff and box dimensions in the Euclidean setting

In this section, we prove that fractal dimensions III and IV do generalize the classical fractal dimensions in the Euclidean setting with respect to the natural fractal structures for Euclidean subsets (c.f. Definition 2.3). More specifically, we show that fractal dimension III generalizes the box dimension for Euclidean subsets (endowed with their natural fractal structures), whereas fractal dimension IV extends the classical Hausdorff dimension for compact Euclidean subsets (endowed with their natural fractal structures, as well). Accordingly, the classical fractal dimensions can be calculated equivalently throughout these discretized models with respect to the natural fractal structure which any Euclidean subset can be endowed with.

Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X, ρ), F be a subset of X, and define $\mathcal{A}_n(F) = \{A \in \Gamma_n : A \cap F \neq \varnothing\}$. In addition, let $\operatorname{diam}(\Gamma_n) = \sup\{\operatorname{diam}(A) : A \in \Gamma_n\}$ and $\operatorname{diam}(F, \Gamma_n) = \sup\{\operatorname{diam}(A) : A \in \mathcal{A}_n(F)\}$.

Definition 3.1 (Fractal dimensions III and IV for a fractal structure)**.** *Assume that* $\operatorname{diam}(F, \Gamma_n) \to 0$ *and define the following expression for* $k = 3, 4$*:*

$$\mathcal{H}^s_{n,k}(F) = \inf\left\{\sum \operatorname{diam}(A_i)^s : \{A_i\}_{i\in I} \in \mathcal{A}_{n,k}(F)\right\}, \textit{where}$$

(i) $\mathcal{A}_{n,3}(F) = \{\{A : A \in \mathcal{A}_l(F)\} : l \geq n\}$.
(ii) $\mathcal{A}_{n,4}(F) = \{\{A_i\}_{i\in I} : A_i \in \cup_{l\geq n}\Gamma_l, F \subseteq \cup_{i\in I} A_i, \operatorname{Card}(I) < \infty\}$*, where* $\operatorname{Card}(A)$ *denotes the cardinal number of* A*, i.e., the number of elements* A *contains.*

Define $\mathcal{H}^s_k(F) = \lim_{n\to\infty} \mathcal{H}^s_{n,k}(F)$ *for* $k = 3, 4$.
The fractal dimension III (resp., the fractal dimension IV) of F *is the (unique) critical point* $s \geq 0$ *satisfying the following equality:*

$$\dim^k_{\mathbf{\Gamma}}(F) = \sup\{s : \mathcal{H}^s_k(F) = \infty\} = \inf\{s : \mathcal{H}^s_k(F) = 0\} : k = 3, 4.$$

It is worth mentioning that fractal dimension III always exists since the sequence $\{\mathcal{H}^s_{n,3}(F)\}_{n\in\mathbb{N}}$ is monotonic in $n \in \mathbb{N}$. **Moreover, it has been assumed that** $\inf \varnothing = \infty$ **in Definition 3.1. For instance, if there exists a subset** F **of** X **such that** $\mathcal{A}_{n,4}(F) = \varnothing$**, then** $\dim^4_{\mathbf{\Gamma}}(F) = \infty$**.**

We would like to point out that the condition $\operatorname{diam}(F, \Gamma_n) \to 0$, though necessary in previous Definition 3.1, is not too restrictive, as the following remark highlights.

Remark 3.2. *Let* $\mathcal{K}$ *be an IFS-attractor endowed with its natural fractal structure as a self-similar set (c.f. Definition 2.1). Then* $\operatorname{diam}(\mathcal{K}, \Gamma_n) \to 0$*, since the sequence of diameters* $\{\operatorname{diam}(\Gamma_n)\}_{n\in\mathbb{N}}$ *decreases geometrically.*

The next result provides a handy expression for fractal dimension III calculation purposes.

Theorem 3.3 (c.f. [7], Theorem 4.7)**.** *Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ), F be a subset of X, and assume that there exists*

$$\mathcal{H}^s(F) = \lim_{n\to\infty} \mathcal{H}^s_n(F),$$

where $\mathcal{H}^s_n(F) = \sum \{\operatorname{diam}(A)^s : A \in \mathcal{A}_n(F)\}$. *Then*

$$\dim^3_{\mathbf{\Gamma}}(F) = \sup\{s : \mathcal{H}^s(F) = \infty\} = \inf\{s : \mathcal{H}^s(F) = 0\}.$$

The next two results provide connections among Hausdorff dimension and fractal dimensions III and IV.

Lemma 3.4 (c.f. [9], Proposition 3.5 (3))**.** *Let $\mathbf{\Gamma}$ be a finite fractal structure on a metric space (X,ρ) and F be a subset of X. If* $\operatorname{diam}(F,\Gamma_n) \to 0$, *then*

$$\dim_{\mathrm{H}}(F) \leq \dim^4_{\mathbf{\Gamma}}(F) \leq \dim^3_{\mathbf{\Gamma}}(F).$$

Corollary 3.5. *Let $\mathcal{F}$ be an IFS with associated attractor $\mathcal{K}$ and $\mathbf{\Gamma}$ be the natural fractal structure on $\mathcal{K}$ as a self-similar set. Then*

$$\dim_{\mathrm{H}}(\mathcal{K}) \leq \dim^4_{\mathbf{\Gamma}}(\mathcal{K}) \leq \dim^3_{\mathbf{\Gamma}}(\mathcal{K}).$$

Proof. It follows as a consequence of both Remark 3.2 and Lemma 3.4 since the natural fractal structure which any IFS-attractor can be endowed with is finite. □

The two results that follow (both Theorems 3.7 and 3.9) have been proved in detail for the sake of completeness. Firstly, we show that fractal dimension III generalizes the classical box dimension. To deal with, we shall prove a more general result based on the next Euclidean property: for each $\delta > 0$ and all $F \subseteq \mathbb{R}^d$ with $\operatorname{diam}(F) \leq \delta$, there are at most 3^d $\delta-$cubes in $\mathbb{R}^d$ intersected by F. This motivates the following definition.

Definition 3.6. *Let $\mathbf{\Gamma}$ be a fractal structure on a metrizable space X and F be a subset of X. We shall understand that $\mathbf{\Gamma}$ is under the $\kappa-$condition if there exists a natural number κ such that for all $n \in \mathbb{N}$, every subset A of X such that* $\operatorname{diam}(A) \leq \operatorname{diam}(F,\Gamma_n)$ *intersects at most to κ elements in level n of $\mathbf{\Gamma}$.*

Theorem 3.7 (c.f. [7], Theorem 4.17)**.** *Let $\mathbf{\Gamma}$ be a fractal structure on a metric space (X,ρ) lying under the $\kappa-$condition and F be a subset of X. Assume that there exists the box dimension of F. If* $\operatorname{diam}(F,\Gamma_n) \to 0$ *and* $\operatorname{diam}(A) = \operatorname{diam}(F,\Gamma_n)$ *for all $A \in \mathcal{A}_n(F)$, then the fractal dimension III of F equals the box dimension of F, namely,*

$$\dim_{\mathrm{B}}(F) = \dim^3_{\mathbf{\Gamma}}(F).$$

Proof. First of all, we affirm that

$$\dim^3_{\mathbf{\Gamma}}(F) = \varliminf_{n\to\infty} \frac{\log \mathcal{N}_n(F)}{-\log \operatorname{diam}(F,\Gamma_n)}, \tag{2}$$

where $\mathcal{N}_n(F) = \text{Card}\,(\{A \in \Gamma_n : A \cap F \neq \varnothing\}) = \text{Card}\,(\mathcal{A}_n(F))$ for all $n \in \mathbb{N}$. In fact, let $\beta = \underline{\lim}_{n\to\infty} \frac{\log \mathcal{N}_n(F)}{-\log \text{diam}\,(F,\Gamma_n)}$. By definition of lower limit, there exists a subsequence

$$\left\{ \frac{\log \mathcal{N}_{n_k}(F)}{-\log \text{diam}\,(F, \Gamma_{n_k})} \right\}_{n_k \in \mathbb{N}} \subseteq \left\{ \frac{\log \mathcal{N}_n(F)}{-\log \text{diam}\,(F, \Gamma_n)} \right\}_{n \in \mathbb{N}}$$

such that $\beta = \lim_{k\to\infty} \frac{\log \mathcal{N}_{n_k}(F)}{-\log \text{diam}\,(F,\Gamma_{n_k})}$. Let $\varepsilon > 0$ be fixed but arbitrarily chosen. Then there exists $n_1 \in \mathbb{N}$ such that

$$\text{diam}\,(F, \Gamma_{n_k})^{-(\beta-\varepsilon)} \leq \mathcal{N}_{n_k}(F) \leq \text{diam}\,(F, \Gamma_{n_k})^{-(\beta+\varepsilon)} \text{ for all } k \geq n_1. \tag{3}$$

On the other hand, by definition of $\mathcal{H}^s_{n,3}$, it holds that

$$\mathcal{H}^s_{n_k,3}(F) \leq \mathcal{N}_m(F)\,\text{diam}\,(F, \Gamma_m)^s \leq \text{diam}\,(F, \Gamma_m)^{s-(\beta+\varepsilon)} \tag{4}$$

for all $m \geq k \geq n_1$ since $\text{diam}\,(A) = \text{diam}\,(F, \Gamma_n)$ for all $A \in \mathcal{A}_n(F)$, and also due to Eq. (3). Letting $k \to \infty$ in Eq. (4), we have

$$\mathcal{H}^s_3(F) = \lim_{k\to\infty} \mathcal{H}^s_{n_k,3}(F) \leq \lim_{m\to\infty} \text{diam}\,(F, \Gamma_m)^{s-(\beta+\varepsilon)} = \begin{cases} \infty & \text{if } s < \beta + \varepsilon \\ 0 & \text{if } s > \beta + \varepsilon \end{cases},$$

where the condition $\text{diam}\,(F, \Gamma_n) \to 0$ has been applied in the last equality. Hence,

$$\dim^3_{\mathbf{\Gamma}}(F) \leq \underline{\lim}_{n\to\infty} \frac{\log \mathcal{N}_n(F)}{-\log \text{diam}\,(F, \Gamma_n)} + \varepsilon. \tag{5}$$

Next, we shall focus on the opposite inequality. Let $\delta > 0$ be fixed but arbitrarily chosen. For all $k \in \mathbb{N}$, there exists a natural number $m_k \geq n_k$ satisfying the following expression:

$$\text{diam}\,(F, \Gamma_{m_k})^{s-(\beta-\varepsilon)} \leq \mathcal{N}_{m_k}(F)\,\text{diam}\,(F, \Gamma_{m_k})^s \leq \delta + \mathcal{H}^s_{n_k,3}(F), \tag{6}$$

where Eq. (3) has been applied. Letting $k \to \infty$, we have

$$\lim_{k\to\infty} \text{diam}\,(F, \Gamma_{m_k})^{s-(\beta-\varepsilon)} \leq \lim_{k\to\infty} \mathcal{N}_{m_k}(F)\,\text{diam}\,(F, \Gamma_{m_k})^s \leq \delta + \mathcal{H}^s_3(F).$$

Observe that

$$\lim_{k\to\infty} \text{diam}\,(F, \Gamma_{m_k})^{s-(\beta-\varepsilon)} = \begin{cases} \infty & \text{if } s < \beta - \varepsilon \\ 0 & \text{if } s > \beta - \varepsilon \end{cases}.$$

Thus, for all $s < \beta - \varepsilon$, it holds that $\delta + \mathcal{H}^s_3(F) = \infty$. The arbitrariness of δ leads to $\mathcal{H}^s_3(F) = \infty$ for all $s < \beta - \varepsilon$. In other words,

$$\underline{\lim}_{n\to\infty} \frac{\log \mathcal{N}_n(F)}{-\log \text{diam}\,(F, \Gamma_n)} - \varepsilon \leq \dim^3_{\mathbf{\Gamma}}(F). \tag{7}$$

Hence, Eq. (2) follows from both Eqs. (5) and (7). To conclude the proof, let $\mathcal{N}_\delta(F)$ be the smallest number of sets of diameters $\leq \delta$ that cover F

(c.f. Subsection 2.4). This quantity will be applied for box dimension calculation purposes. Since $F \subseteq \cup\{A \in \Gamma_n : A \cap F \neq \varnothing\} = \cup\{A : A \in \mathcal{A}_n(F)\}$, then F can be covered by $\mathcal{N}_n(F)$ sets with diameters $\leq \operatorname{diam}(F, \Gamma_n)$, so

$$\underline{\dim}_{\mathrm{B}}(F) \leq \varliminf_{n\to\infty} \frac{\log \mathcal{N}_n(F)}{-\log \operatorname{diam}(F, \Gamma_n)}.$$

Finally, the $\kappa-$condition leads to $\mathcal{N}_n(F) \leq \kappa \mathcal{N}_{\delta(F,\Gamma_n)}(F)$. Thus,

$$\overline{\lim}_{n\to\infty} \frac{\log \mathcal{N}_n(F)}{-\log \operatorname{diam}(F, \Gamma_n)} \leq \overline{\lim}_{n\to\infty} \frac{\log \mathcal{N}_{\delta(F,\Gamma_n)}(F)}{-\log \delta(F, \Gamma_n)} = \overline{\dim}_{\mathrm{B}}(F).$$

□

The following result stands immediately from Theorem 3.7.

Corollary 3.8 (c.f. [7], Theorem 4.15)**.** *Let F be a subset of $\mathbb{R}^d$ and $\mathbf{\Gamma}$ be the natural fractal structure on F as a Euclidean subset. Then*

$$\underline{\dim}_{\mathrm{B}}(F) = \dim_{\mathbf{\Gamma}}^{3}(F).$$

In fact, notice that the natural fractal structure on any Euclidean subset (c.f. Definition 2.3) consists of elements with diameters equal to 2^{-n} in level n of $\mathbf{\Gamma}$. In addition, observe that the $\kappa-$condition stands for such a fractal structure.

Next, we prove that the Hausdorff dimension of any compact Euclidean subset can be calculated in terms of finite coverings. It is worth mentioning that such a theoretical result allows the effective calculation of Hausdorff dimension in applications [9]. Also, the fractal dimension IV will be applied to calculate the fractal dimension of attractors not lying under the OSC (c.f. Section 4). In other words, both Theorems 3.7 and 3.9 will lead to a generalized version of the classical Moran's Theorem.

Theorem 3.9 (c.f. [9], Theorem 3.12)**.** *Let F be a compact Euclidean subset of $\mathbb{R}^d$ and $\mathbf{\Gamma}$ be the natural fractal structure on $\mathbb{R}^d$ as a Euclidean subset. Then the fractal dimension IV of F equals the Hausdorff dimension of F, i.e.,*

$$\dim_{\mathrm{H}}(F) = \dim_{\mathbf{\Gamma}}^{4}(F).$$

Proof. Firstly, it is clear that $\mathcal{A}_{n,4}(F) \subseteq \mathcal{C}_\delta(F)$, so $\dim_{\mathrm{H}}(F) \leq \dim_{\mathbf{\Gamma}}^{4}(F)$. Thus, we shall focus on the opposite inequality. Let $s \geq 0$ be such that $\mathcal{H}_{\mathrm{H}}^{\alpha}(F) = 0$ and let $\varepsilon > 0$ be fixed but arbitrarily chosen. Then there exists $\xi > 0$ such that $\mathcal{H}_\delta^s(F) < \lambda$ for all $\delta < \xi$. Specifically, λ can be chosen to be equal to $\frac{\varepsilon}{3^d\, d^{\frac{s}{2}}}$. Let $m \in \mathbb{N}$ be such that $\xi > 2^{-m}$. Thus, $\mathcal{H}_{2^{-m}}^s(F) < \lambda$. Hence, for I countable, there exists a $2^{-m}-$covering of F, say $\mathcal{D} = \{D_i : i \in I\}$, by open balls of $\mathbb{R}^d$ (c.f. [5, Section 2.4]). The three following hold:

(i) $\operatorname{diam}(D_i) < 2^{-m}$ for all $i \in I$.
(ii) $F \subseteq \cup\{D_i : i \in I\}$.
(iii) $\sum\{\operatorname{diam}(D_i)^s : i \in I\} < \lambda$.

Observe that Eq. (ii) can be rewritten in the following terms, due to the compactness of F. There exists $J \subseteq I$ with $\mathrm{Card}\,(J) < \infty$ such that $F \subseteq \cup\,\{D_j : j \in J\}$. On the other hand, for all $D_j \in \mathcal{D}$, let $n_j \in \mathbb{N}$ be such that

$$2^{-n_j} \leq \mathrm{diam}\,(D_j) \leq 2^{1-n_j}. \tag{8}$$

By Eq. (i), it holds that $m < n_j$ for all $j \in J$. Thus, for each level n_j of $\mathbf{\Gamma}$, we can consider a covering of F consisting of all the elements in that level which intersect D_j. In other words, let $\mathcal{C}_j = \{A \in \Gamma_{n_j} : A \cap D_j \neq \varnothing\}$ and denote $\mathcal{C} = \cup\{\mathcal{C}_j : j \in J\}$. Further, since $\mathrm{diam}\,(D_j) < 2^{1-n_j}$, then there are at most 3^d elements in each covering $\mathcal{C}_j$ of D_j. Hence,

(1) For each $A \in \mathcal{C}$, there exists $n_j \in \mathbb{N} : m < n_j$ such that $A \in \Gamma_{n_j}$. It is clear by definition of $\mathcal{C}$.

(2) $\mathcal{C}$ is a covering of F. In fact,

$$\begin{aligned} F &\subseteq \cup\{D_j : j \in J\} \subseteq \cup_{j\in J} \cup \{A : A \in \mathcal{C}_j\} \\ &= \cup\{A : A \in \cup_{j\in J}\mathcal{C}_j\} = \cup\{A : A \in \mathcal{C}\}. \end{aligned}$$

(3) $\sum\{\mathrm{diam}\,(A)^s : A \in \mathcal{C}\} < \varepsilon$. To prove that, recall that $\mathrm{diam}\,(A) = 2^{-n_j}\sqrt{d}$ for all $A \in \Gamma_{n_j}$. Thus,

$$\begin{aligned} \sum\{\mathrm{diam}\,(A)^s : A \in \mathcal{C}\} &= \sum\{\mathrm{diam}\,(A)^s : A \in \cup_{j\in J}\mathcal{C}_j\} \\ &= \sum_{j\in J}\sum_{A\in\mathcal{C}_j} \mathrm{diam}\,(A)^s = \sum_{j\in J}\sum_{A\in\mathcal{C}_j} 2^{-n_j s} d^{\frac{s}{2}} \\ &\leq 3^d\, d^{\frac{s}{2}} \sum\{\mathrm{diam}\,(D_j)^s : j \in J\} < \varepsilon, \end{aligned}$$

since each covering $\mathcal{C}_j$ contains 3^d elements at most.

Accordingly, $\mathcal{H}^s_4(F) = 0$ for all $s > \dim_{\mathrm{H}}(F)$. Hence, $\dim^4_{\mathbf{\Gamma}}(F) \leq s$ for all $s > \dim_{\mathrm{H}}(F)$. □

4. The Theorem

The Moran's Theorem constitutes one of the milestones in Fractal Geometry. It was first contributed by P.A.P. Moran (1946) (c.f. [14]), who required an IFS $\mathcal{F}$ to be under the OSC to reach the equality between the Hausdorff dimension of its attractor $\mathcal{K}$ and the similarity dimension of $\mathcal{K}$. In other words, the OSC was applied therein to control the overlap among the pieces $\mathcal{K}_{\mathbf{i}}$. However, that result still remains quite powerful, since without a wide amount of effort, the Hausdorff dimension of a wide class of self-similar sets can be calculated with easiness. For instance, both the box and the Hausdorff dimensions of the standard middle third Cantor set equal the value $\frac{\log 2}{\log 3}$, since two similarities can be applied to construct it, each of them having a similarity ratio equal to a half. Next, we recall the concept of similarity dimension for IFS−attractors.

Definition 4.1. *Let $\mathcal{F}$ be an* IFS *and $\mathcal{K}$ its attractor. By the similarity dimension of $\mathcal{K}$, we shall understand the unique solution $\alpha > 0$ of the equation*

$\sum_{i=1}^k c_i^s = 1$. *In other words, the similarity dimension of* $\mathcal{K}$ *is the unique* $\alpha > 0$ *such that* $P(\alpha) = 0$, *where* $P(s) = \sum_{i=1}^k c_i^s - 1$.

Along the sequel, α will denote the similarity dimension of an attractor. The classical Moran's Theorem is stated in the Euclidean setting.

Moran's Theorem (1946). *Let* $\mathcal{F}$ *be a* EIFS *with associated attractor* $\mathcal{K}$. *Let* c_i *be the similarity ratio associated with* $f_i \in \mathcal{F}$ *and assume that* $\mathcal{F}$ *is under the* OSC. *Then* $\dim_{\mathrm{H}}(\mathcal{K}) = \dim_{\mathrm{B}}(\mathcal{K}) = \alpha$. *In addition,* $\mathcal{H}_{\mathrm{H}}^\alpha(\mathcal{K}) \in (0, \infty)$.

A proof for Moran's Theorem can be found out in Falconer's book (c.f. [5, Subsection 9.2]), though the reader may check that the proof for a lower bound for the Hausdorff dimension, $s \leq \dim_{\mathrm{H}}(\mathcal{K})$, becomes quite awkward. However, whether the OSC is not fulfilled by $\mathcal{F}$, then the calculation of the Hausdorff dimension of $\mathcal{K}$ becomes harder and only some partial results are known in that context (c.f., e.g., [4, 15]). However, even in such a situation, it holds that both the box and the Hausdorff dimensions of $\mathcal{K}$ can be approximated by fractal dimension III, which still equals the similarity dimension of $\mathcal{K}$. Next, we provide the main theoretical result in this paper, a generalized version of the classical Moran's Theorem which also stands in the Euclidean setting.

Theorem 4.2. *Let* $\mathcal{F}$ *be a* EIFS *with* $\mathcal{K}$ *being its associated attractor. Assume that* c_i *is the similarity ratio associated with* $f_i \in \mathcal{F}$ *and let* $\mathbf{\Gamma}$ *be the natural fractal structure on* $\mathcal{K}$ *as a self-similar set. Then*

(i) [c.f. [7], Theorem 4.20] $\dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \alpha$. *Further,* $\mathcal{H}_3^\alpha(\mathcal{K}) \in (0, \infty)$.
(ii) Additionally, if $\mathcal{F}$ *is under the* OSC, *then*

$$\dim_{\mathrm{H}}(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \dim_{\mathrm{B}}(\mathcal{K}) = \alpha.$$

Moreover, $0 < \mathcal{H}_{\mathrm{H}}^\alpha(\mathcal{K}) \leq \mathcal{H}_4^\alpha(\mathcal{K}) \leq \mathcal{H}_3^\alpha(\mathcal{K}) < \infty$.

Proof. First of all, recall that $\mathcal{K}$ is the unique nonempty compact subset of $\mathbb{R}^d$ satisfying the Hutchinson's equation:

$$\mathcal{K} = \cup\{\mathcal{K}_{\mathbf{i}} : \mathbf{i} \in \Sigma^n\}.$$

(i) Notice that $\mathcal{A}_{n,3}(\mathcal{K}) = \{\Gamma_m : m \geq n\}$. Also, it holds that $c_{\mathbf{i}}$ is the similarity ratio associated with $f_{\mathbf{i}}$. Thus, $\mathrm{diam}\,(\mathcal{K}_{\mathbf{i}}) = c_{\mathbf{i}} \cdot \mathrm{diam}\,(\mathcal{K})$ for all $\mathbf{i} \in \Sigma^n$. Further,

$$\sum_{\mathbf{i}\in\Sigma^n} c_{\mathbf{i}}^\alpha = \sum_{i_1=1}^k c_{i_1}^\alpha \cdot \ldots \cdot \sum_{i_n=1}^k c_{i_n}^\alpha = 1$$

for all $\mathbf{i} = i_1 \dots i_n \in \Sigma^n$. Hence, for all natural number n, we have

$$\begin{aligned}\mathcal{H}^\alpha_{n,3}(\mathcal{K}) &= \inf\left\{\sum \operatorname{diam}(A)^\alpha : A \in \Gamma_m, m \geq n\right\}\\ &= \inf\left\{\sum \operatorname{diam}(\mathcal{K}_{\mathbf{i}})^\alpha : \mathbf{i} = i_1 \dots i_m \in \Sigma^m, m \geq n\right\}\\ &= \inf\left\{\sum c_{\mathbf{i}}^\alpha \operatorname{diam}(\mathcal{K})^\alpha : \mathbf{i} \in \Sigma^m, m \geq n\right\}.\end{aligned}$$

Since $\mathcal{H}^\alpha_3(\mathcal{K}) = \operatorname{diam}(\mathcal{K})^\alpha \notin \{0,\infty\}$, then we conclude $\dim^3_{\mathbf{\Gamma}}(\mathcal{K}) = \alpha$.

(ii) Since the natural fractal structure on $\mathcal{K}$ as a self-similar set (c.f. Definition 2.1) is finite, then $\mathcal{A}_{n,3}(\mathcal{K}) \subseteq \mathcal{A}_{n,4}(\mathcal{K})$ for all $n \in \mathbb{N}$. Hence, $\mathcal{H}^\alpha_{n,4}(\mathcal{K}) \leq \mathcal{H}^\alpha_{n,3}(\mathcal{K})$. Letting $n \to \infty$, it follows that

$$\dim^4_{\mathbf{\Gamma}}(\mathcal{K}) \leq \dim^3_{\mathbf{\Gamma}}(\mathcal{K}). \tag{9}$$

Moreover, since $\mathbf{\Gamma}$ also satisfies that $\operatorname{diam}(\mathcal{K},\Gamma_n) \to 0$ (c.f. Remark 3.2), then we can state that

$$\dim_{\mathrm{H}}(\mathcal{K}) \leq \dim^4_{\mathbf{\Gamma}}(\mathcal{K}), \tag{10}$$

since all the coverings in $\mathcal{A}_{n,4}(\mathcal{K})$ are $\delta-$coverings for appropiate scales δ. The next chain of inequalities stands due to both Eqs. (9) and (10):

$$\dim_{\mathrm{H}}(\mathcal{K}) \leq \dim^4_{\mathbf{\Gamma}}(\mathcal{K}) \leq \dim^3_{\mathbf{\Gamma}}(\mathcal{K}) = \alpha, \tag{11}$$

where the last equality is due to Theorem 4.2 (i). In addition, if $\mathcal{F}$ is under the OSC, then

$$\dim_{\mathrm{H}}(\mathcal{K}) = \dim^4_{\mathbf{\Gamma}}(\mathcal{K}) = \dim^3_{\mathbf{\Gamma}}(\mathcal{K}) = \dim_{\mathrm{B}}(\mathcal{K}) = \alpha,$$

as a consequence of Eq. (11) and Moran's Theorem. Finally, observe that $\mathcal{H}^\alpha_{\mathrm{H}}(\mathcal{K}) \leq \mathcal{H}^\alpha_4(\mathcal{K}) \leq \mathcal{H}^\alpha_3(\mathcal{K})$. To conclude the proof, recall that $\mathcal{H}^\alpha_{\mathrm{H}}(\mathcal{K}) > 0$ (by Moran's Theorem) and $\mathcal{H}^\alpha_3(\mathcal{K}) < \infty$ (c.f. Theorem 4.2 (i)).

□

It is worth mentioning that Theorem 4.2 (i) can be extended to the general setting, i.e., for IFS-attractors on complete metric spaces.

Notice that Theorem 4.2 becomes quite useful to deal with the effective calculation of the fractal dimension of attractors. For illustration purposes, the next example highlights that it is easy to calculate the fractal dimension III for attractors unlike their Hausdorff dimensions.

Example 4.3. *Let $\mathcal{K}$ be the unique IFS-attractor on the closed unit interval satisfying the Hutchinson's equation $\mathcal{K} = \cup\{\mathcal{K}_i : i = 1,\dots,k\}$, where its* IFS, *$\mathcal{F}$, consists of the three similitudes $f_i : [0,1] \to [0,1]$ given by*

$$f_i(x) = \begin{cases} \frac{3}{10}x & \text{if } i = 1\\ \frac{1}{10}(1+3x) & \text{if } i = 2\\ \frac{1}{10}(7+3x) & \text{if } i = 3.\end{cases} \tag{12}$$

It is worth pointing out that $\mathcal{F}$ does not satisfy the OSC *since the three pre-fractals of $\mathcal{K}$ do overlap among them. Thus, the Moran's Theorem cannot be applied to calculate the Hausdorff dimension of $\mathcal{K}$. However, Theorem 4.2 can be applied for fractal dimension calculation purposes. In fact, $\dim_{\mathbf{\Gamma}}^3(\mathcal{K})$ can be calculated as the solution of the equation $3\left(\frac{3}{10}\right)^s = 1$. In fact,*

$$\dim_{\mathbf{\Gamma}}^3(\mathcal{K}) = \frac{\log \frac{1}{3}}{\log \frac{3}{10}} \simeq 0.912.$$

5. Conclusion

In this section, we summarize all the results contributed along this paper.

Theorem 5.1. *Let $\mathcal{F} = \{f_1, \ldots, f_k\}$ be a* EIFS, *$\mathcal{K}$ its attractor, $\mathbf{\Gamma}$ the natural fractal structure on $\mathcal{K}$ as a self-similar set, c_i the similarity ratio associated with $f_i \in \mathcal{F}$, and α the similarity dimension of $\mathcal{K}$. Consider the following statements:*

(i) SOSC.
(ii) OSC.
(iii) $\mathcal{H}^\alpha(\mathcal{K}) > 0$.
(iv) $\dim_{\mathrm{H}}(\mathcal{K}) = \dim_{\mathrm{B}}(\mathcal{K}) = \alpha$.
(v) $\dim_{\mathbf{\Gamma}}^4(\mathcal{K}) = \dim_{\mathbf{\Gamma}}^3(\mathcal{K})$.

The next chain of implications and equivalences stands:

$$(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \Rightarrow (v).$$

The implication $(iv) \Rightarrow (iii)$ is not true due to a counterexample provided by Mattila (c.f. [16, 17]). However, $(v) \Rightarrow (iv)$ remains as an open question.

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