THE SPATIAL HILL LUNAR PROBLEM: PERIODIC SOLUTIONS EMERGING FROM EQUILIBRIA

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Abstract. In this paper we provide sufficient conditions for the existence of periodic solutions emerging of the equilibrium points of the spatial Hill lunar problem having equations of motion

$$\begin{aligned}
&\frac{d^2x}{dt^2} - 2\frac{dy}{dt} - 9x = \varepsilon F_1\left(t, x, \frac{dx}{dt}, y, \frac{dy}{dt}\right),\\
&\frac{d^2y}{dt^2} + 2\frac{dx}{dt} + 3y = \varepsilon F_2\left(t, x, \frac{dx}{dt}, y, \frac{dy}{dt}\right),\\
&\frac{d^2z}{dt^2} + 4z = \varepsilon F_3\left(t, x, \frac{dx}{dt}, y, \frac{dy}{dt}\right).
\end{aligned}$$

ε is a small parameter and F_i, $i \in \{1,2,3\}$, are smooth periodic functions in t which define a perturbation in resonance p:q with some of the periodic solutions of the spatial Hill Lunar problem being p and q positive relatively prime integers.

1. Introduction and statement of the main results

We are going to work in a particular case of the restricted three–body problem. The restricted problem is modified into five versions:

(1) The *planar circular restricted three–body problem*, when the primaries revolve in a circular orbit around their common center of mass as well as the third body moves in the same plane.
(2) If the motion of the primaries is not circular and have a conic section. The important case is when the primaries move in elliptical orbits around center of their masses. In this case It is called *elliptical* or *pseudo restricted problem*.
(3) When the third body moves out of the plane of the primaries the problem is called *three–dimensional problem*. This problem is applicable in the study of the orbits for some minor planets with large inclination to the ecliptic.
(4) If the masses of the primaries or third body vary with time. It is called *restricted problem with variable mass* and it has important applications in stellar dynamics and cosmology.
(5) If the mass ratio of the smaller primary to the sum masses of the primaries is very small and may be tends to zero, the problem, is called *Hill problem*. In this case the problems with a very small mass ratio appear as a perturbed problems for the two-body problem.

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We will center our attention in the case of the perturbed spatial Hill lunar problem. This problem is very important from the astrophysics point of view, see for instance [1, 9, 10, 11, 12, 15, 16, 18].

The equations of motion of the spatial Hill Lunar problem are

$$
\begin{aligned}
&\frac{d^2q_1}{dt^2} - 2\frac{dq_2}{dt} - 3q_1 + \frac{q_1}{(q_1^2+q_2^2+q_3^2)^{3/2}} = 0,\\
&\frac{d^2q_2}{dt^2} + 2\frac{dq_1}{dt} + \frac{q_2}{(q_1^2+q_2^2+q_3^2)^{3/2}} = 0,\\
&\frac{d^2q_3}{dt^2} + q_3 + \frac{q_3}{(q_1^2+q_2^2+q_3^2)^{3/2}} = 0
\end{aligned}
\tag{1}
$$

(see [13]). The equilibrium solutions of the previous differential equations are given by $\mathcal{L}_1 = (q_1, q_2, q_3) = \left(3^{-1/3}, 0, 0\right)$ and $\mathcal{L}_2 = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3) = \left(-3^{-1/3}, 0, 0\right)$. In this paper we provide sufficient conditions for the existence of periodic solutions emerging of the equilibrium $\mathcal{L}_1$. The same ones are valid for $\mathcal{L}_2$.

The problem of studying the existence of periodic solutions which come from equilibria solutions of systems of differential equations is a classical problem in the literature, see for more information [13].

It is well known the existence of periodic orbits emerging from eulerian points in the restricted three body problem and in the Hill problem too. For the proofs of these results the main tool used is the Lyapunov Center Theorem or some of its variations, see for details [13] and [2].

But in some case in which the perturbative forces envolved are not conservative the Lyapunov type theorems do not work. This is exactly the case that we are going to treat in the present work.

By means of the change of variable $x = q_1 - 3^{-1/3}$, $y = q_2$ and $z = q_3$ and linearizing, the differential equations of the spatial Hill Lunar problem are

$$
\begin{aligned}
&\frac{d^2x}{dt^2} - 2\frac{dy}{dt} - 9x = 0,\\
&\frac{d^2y}{dt^2} + 2\frac{dx}{dt} + 3y = 0,\\
&\frac{d^2z}{dt^2} + 4z = 0.
\end{aligned}
\tag{2}
$$

The objective of this work is to provide a system of nonlinear equations, by means of the averaging theory, whose simple zeros provide periodic solutions of the perturbed system with equations of motion

$$
\begin{aligned}
&\frac{d^2x}{dt^2} - 2\frac{dy}{dt} - 9x = \varepsilon F_1\left(t, x, \frac{dx}{dt}, y, \frac{dy}{dt}\right),\\
&\frac{d^2y}{dt^2} + 2\frac{dx}{dt} + 3y = \varepsilon F_2\left(t, x, \frac{dx}{dt}, y, \frac{dy}{dt}\right),\\
&\frac{d^2z}{dt^2} + 4z = \varepsilon F_3\left(t, x, \frac{dx}{dt}, y, \frac{dy}{dt}\right),
\end{aligned}
\tag{3}
$$

where ε is a small parameter. Here the smooth functions F_1, F_2 and F_3 define the perturbation. These functions are periodic in t and in resonance p:q with some of the periodic solutions of (2), being p and q positive integers relatively primes. In order to present our results we need some preliminary definitions and notations.

The unperturbed system (2) has a unique singular point at the origin with eigenvalues

$$\pm\lambda, \quad \pm\omega_1\, i, \quad \pm\omega_2\, i$$

with $\lambda = \sqrt{2\sqrt{7}+1}$, $\omega_1 = \sqrt{2\sqrt{7}-1}$ and $\omega_2 = 2$. This system in the phase space $\left(x, \dfrac{dx}{dt}, y, \dfrac{dy}{dt}, z, \dfrac{dz}{dt}\right)$ has two planes filled of periodic solutions with the exception of the origin. These periodic solutions have periods

$$T_1 = \frac{2\pi}{\omega_1} \quad \text{or} \quad T_2 = \pi,$$

according they belong to the plane associated to the eigenvectors with eigenvalues $\pm\omega_1\, i$ or $\pm\omega_2\, i$, respectively. We shall study which of these periodic solutions persist for the perturbed system (3) when the parameter ε is sufficiently small and the perturbed functions F_i for $i \in \{1, 2, 3\}$ have period either pT_1/q, or pT_2/q, where p and q are positive integers relatively prime.

Let $Z^0 = (Z_1^0, Z_2^0)$, and consider the Malkin bifurcation function $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$ for the system (3) given by: (see [4], section 2)

$$\begin{aligned}
\mathcal{G}_1(Z^0) &= \frac{1}{pT_2}\int_0^{pT_2} \langle(\cos(\omega_2 t), -\sin(\omega_2 t)), (F_5^*(t), F_6^*(t))\rangle dt, = \\
&= \frac{1}{pT_2}\int_0^{pT_2} \cos(\omega_2 t) F_5^*(t) dt, \\
\mathcal{G}_2(Z^0) &= \frac{1}{pT_2}\int_0^{pT_2} \langle(\sin(\omega_2 t), \cos(\omega_2 t)), (F_5^*(t), F_6^*(t))\rangle dt = \\
&= \frac{1}{pT_2}\int_0^{pT_2} \sin(\omega_2 t) F_5^*(t) dt,
\end{aligned} \tag{4}$$

where $\langle , \rangle$ is the scalar product and

$$F_5^*(t) = \frac{1}{2} F_3,$$

with $F_3 = F_3(\sigma_1^2(t), \ldots, \sigma_6^2(t))$, and $\sigma_j^2(t) = 0, j = 1, \ldots, 4,$

$$\begin{aligned}
\sigma_5^2(t) &= -Z_2^0 \cos(\omega_2 t) - Z_1^0 \sin(\omega_2 t), \\
\sigma_6^2(t) &= 2(Z_1^0 \cos(\omega_2 t) + Z_2^0 \sin(\omega_2 t))
\end{aligned}$$

and $F_6^*(t) = 0$.

A zero $Z^{0*} = (Z_1^{0*}, Z_2^{0*})$ of the nonlinear system $\mathcal{G}(Z^0) = 0$ such that

$$\det\left(\frac{\partial \mathcal{G}}{\partial Z^0}|_{Z^0 = Z^{0*}}\right) \neq 0$$

is called a *simple zero* of system.

If there exists a simple zero Z^{0*} of the Malkin bifurcation function $\mathcal{G}$, then from the T_2-periodic solution of the unperturbed system with initial value Z^{0*} emanates a branch of T_2-periodic solutions of the perturbed system (see [4], section 2).

The statement of our main result on the periodic solutions of the differential system (3) which bifurcate from the periodic solutions of period T_2 of the unperturbed system traveled p times is the following. Note that this result provides sufficient conditions for the existence of families of periodic orbits giving the initial conditions of them. The theorem is a tool for analyzing the periodic structure of the Hill problem when the perturbations up to first order are non conservative forces.

Theorem 1. *Let p and q be positive integers relatively prime and assume that the smooth functions F_1, F_2 and F_3 of the equations of motion of (3) are periodic in the variable t of period pT_2/q. Then, for $\varepsilon \neq 0$ sufficiently small and for every simple zero $Z^{0*} \neq 0$ of the nonlinear system $\mathcal{G}(Z^0) = 0$, the perturbed system (3) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ tending to the periodic solution $(x(t), y(t), z(t)) = (\sigma_1^2(t), \sigma_3^2(t), \sigma_5^2(t))|_{Z^0=Z^{0*}}$ of the unperturbed system (2) traveled p times.*

Theorem 1 is proved in section 2. Its proof is based in the averaging theory for computing periodic solutions (see [4, 5, 6, 7] and [8] where different problems are treated with this or similar tools).

An application of Theorem 1 is presented in the following corollary, which will be proved in section 3.

Corollary 2. *Assume that $F_1(t,x,\dot{x},y,\dot{y},z,\dot{z}) = f(t)$, $F_2(t,x,\dot{x},y,\dot{y},z,\dot{z}) = g(t)$, $F_3(t,x,\dot{x},y,\dot{y},z,\dot{z}) = z^3+\sin(\omega_2 t)(z^2-\dot{z}^2)$ where $f(t)$ and $g(t)$ are arbitrary smooth functions. Then, the system (3), for $\varepsilon \neq 0$ sufficiently small, has two periodic solutions $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ tending to the periodic solutions $(x(t), y(t), z(t)) = (\sigma_1^2(t), \sigma_3^2(t), \sigma_5^2(t))|_{Z^0=Z^{0*}}$ of (2) when $\varepsilon \to 0$, given by $Z^{0*} = (\frac{-1}{3}, 0)$ and $Z^{0*} = (\frac{-11}{8}, \frac{\sqrt{55}}{8})$.*

Corollary 2 will be proved in section 3.

Let $Y^0 = (Y_1^0, Y_2^0)$, and consider the Malkin bifurcation function $\bar{\mathcal{G}} = (\mathcal{G}_3, \mathcal{G}_4)$ for the system (3) given by:(see [4], section 2).

$$
\begin{aligned}
\mathcal{G}_3(Y^0) &= \frac{1}{pT_1}\int_0^{pT_1} \langle(\cos(\omega_1 t), -\sin(\omega_1 t)), (F_3^*(t), F_4^*(t))\rangle dt = \\
&= \frac{1}{pT_1}\int_0^{pT_1} (\cos(\omega_1 t)F_3^*(t) - \sin(\omega_1 t)F_4^*(t))dt, \\
\mathcal{G}_4(Y^0) &= \frac{1}{pT_1}\int_0^{pT_1} \langle(\sin(\omega_1 t), \cos(\omega_1 t)), (F_3^*(t), F_4^*(t))\rangle dt = \\
&= \frac{1}{pT_1}\int_0^{pT_1} (\cos(\omega_1 t)F_4^*(t) + \sin(\omega_1 t)F_3^*(t))dt,
\end{aligned}
$$

where

$$
F_3^*(t) = \left(\frac{1}{4} + \frac{1}{\sqrt{7}}\right)F_2, \quad F_4^*(t) = -\frac{1}{4}\sqrt{-\frac{1}{7} + \frac{2}{\sqrt{7}}}F_1
$$

with $F_i = F_i(\sigma_1^1(t), \ldots, \sigma_6^1(t)), i = 3, 4$, and

$$\begin{aligned}
\sigma_1^1(t) &= \frac{2}{9}\left(\sqrt{7}-4\right)(Y_1\cos(\omega_1 t) + Y_2\sin(\omega_1 t))\\
\sigma_2^1(t) &= -\frac{\left(4-2\sqrt{7}\right)(Y_2\cos(\omega_1 t) - Y_1\sin(\omega_1 t))}{\omega_1}\\
\sigma_3^1(t) &= -\frac{2\,(Y_2\cos(\omega_1 t) - Y_1\sin(\omega_1 t))}{\omega_1}\\
\sigma_4^1(t) &= 2\,(Y_1\cos(\omega_1 t) + Y_2\sin(\omega_1 t))\\
\sigma_5^1(t) &= \sigma_6^1(t) = 0
\end{aligned}$$

As we said before, if there exists a simple zero Y^{0*} of the Malkin bifurcation function $\bar{\mathcal{G}}$, then from the T_1-periodic solution of the unperturbed system with initial value Y^{0*} emanates a branch of T_1-periodic solutions of the perturbed system (see [4], section 2).

The statement of our main result on the periodic solutions of the differential system (3) which bifurcate from the periodic solutions of period T_1 of the unperturbed system traveled p times is the following.

Theorem 3. *Let p and q be positive integers relatively prime and assume that the smooth functions F_1, F_2 and F_3 of the equations of motion of (3) are periodic in the variable t of period pT_1/q. Then for $\varepsilon \neq 0$ sufficiently small and for every simple zero $Y^{0*} \neq 0$ of the nonlinear system $\bar{\mathcal{G}}(Y^0) = 0$, the perturbed system (3) has a periodic solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ tending to the periodic solution $(x(t), y(t), z(t)) = (\sigma_1^1(t), \sigma_3^1(t), \sigma_5^1(t))|_{Y^0=Y^{0*}}$ of the unperturbed system (2) traveled p times.*

Theorem 3 is proved in section 2.

In the next corollary an application of Theorem 3 is given.

Corollary 4. *Assume that $F_1(t,x,\dot{x},y,\dot{y},z,\dot{z}) = \sin(\omega_1 t) - x^2 - xy + y^2 + f_1(z) + g_1(\dot{z})$, $F_2(t,x,\dot{x},y,\dot{y},z,\dot{z}) = \cos(\omega_1)ty - \dot{y}^2 + x + f_2(z) + g_2(\dot{z})$, $F_3(t,x,\dot{x},y,\dot{y},z,\dot{z}) = h(t)$ where $f_1(z), f_2(z), g_1(\dot{z}), g_2(\dot{z})$ and $h(t)$ are arbitrary functions. Then the system* (3) *for $\varepsilon \neq 0$ sufficiently small has one periodic solution $(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon))$ tending to the periodic solutions $(x(t), y(t), z(t)) = (\sigma_1^1(t), \sigma_3^1(t), \sigma_5^1(t))|_{Y^0=Y^{0*}}$ of* (2) *when $\varepsilon \to 0$, given by $Y^{0*} = \left(\frac{1}{2}\omega_1, -\frac{1}{2}\omega_1\right)$.*

Corollary 4 will be proved in section 3.

Remark 1. *We note that the ratio of the frequencies ω_2/ω_1 are not resonant with π. Indeed, to prove our results we will need that a certain determinant be nonzero and this will happened if only $\sin^2(\pi p\omega_1/\omega_2) \neq 0$. However, if the determinant were zero, we would have $\pi p\omega_1/\omega_2 = k\pi$, k integer. But this would imply that $\sqrt{7}$ would be a rational number, what is not the case. Thus, the nonresonant condition on the ratio ω_2/ω_1 is automatically satisfied and need not to be mentioned in the statements of the results.*

Remark 2. *Unfortunately, the Theorem 6 of Appendix does not provide sufficient conditions for the linear stability of the periodic orbits studied. The reader interested at this point, which is not treat in the present work, could consider the application of the Floquet theory to obtain information on the stability of the found orbits.*

2. Proof of the theorems 1 and 3

Introducing the variables $(x_1, x_2, y_1, y_2, z_1, z_2) = (x, \frac{dx}{dt}, y, \frac{dy}{dt}, z, \frac{dz}{dt})$ we can write the differential system (3) as a first-order differential system defined in $\mathbb{R}^6$ in the following form

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_2\\
\frac{dx_2}{dt} &= 9x_1 + 2y_2 + \varepsilon F_1(x_1, x_2, y_1, y_2, z_1, z_2)\\
\frac{dy_1}{dt} &= y_2\\
\frac{dy_2}{dt} &= -2x_2 - 3y_1 + \varepsilon F_2(x_1, x_2, y_1, y_2, z_1, z_2)\\
\frac{dz_1}{dt} &= z_2\\
\frac{dz_2}{dt} &= -4z_1 + \varepsilon F_3(x_1, x_2, y_1, y_2, z_1, z_2)
\end{aligned}
\tag{5}
$$

Note that the differential system (5) when $\varepsilon = 0$ is equivalent to the differential system (2), called simply in what follows the *unperturbed system*. When $\varepsilon \neq 0$ we called it the *perturbed system*.

The change of variables $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{z}_1, \bar{z}_2) \to (x_1, x_2, y_1, y_2, z_1, z_2)$ given by

$$
\begin{pmatrix} x_1\\ x_2\\ y_1\\ y_2\\ z_1\\ z_2 \end{pmatrix} =
\begin{pmatrix}
\frac{1}{\lambda} & -\frac{1}{\lambda} & -\frac{2}{9}\Phi_2 & 0 & 0 & 0\\
1 & 1 & 0 & \frac{2\Phi_1}{\omega_1} & 0 & 0\\
\frac{1}{3}\Phi_1 & \frac{1}{3}\Phi_1 & 0 & -\frac{2}{\omega_1} & 0 & 0\\
-\frac{\Phi_2}{\lambda} & \frac{\Phi_2}{\lambda} & 2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & 0 & 2 & 0
\end{pmatrix}
\begin{pmatrix} \bar{x}_1\\ \bar{x}_2\\ \bar{y}_1\\ \bar{y}_2\\ \bar{z}_1\\ \bar{z}_2 \end{pmatrix}
\tag{6}
$$

where $\Phi_1 = 2 - \sqrt{7}$ and $\Phi_2 = 4 - \sqrt{7}$ writes the linear part of the differential system (5) in its real Jordan normal form, and this system in the new variables $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{z}_1, \bar{z}_2)$ becomes

$$
\begin{aligned}
\dot{\bar{x}}_1 &= \lambda \bar{x}_1 + \varepsilon F_1^*\\
\dot{\bar{x}}_2 &= -\lambda \bar{x}_2 + \varepsilon F_2^*\\
\dot{\bar{y}}_1 &= \omega_1 \bar{y}_2 + \varepsilon F_3^*\\
\dot{\bar{y}}_2 &= -\omega_1 \bar{y}_1 + \varepsilon F_4^*\\
\dot{\bar{z}}_1 &= \omega_2 \bar{z}_2 + \varepsilon F_5^*\\
\dot{\bar{z}}_2 &= -\omega_2 \bar{z}_1 + \varepsilon F_6^*
\end{aligned}
\tag{7}
$$

where

$$
\begin{aligned}
F_1^* &= \frac{1}{28}\left(\left(7+2\sqrt{7}\right)F_1+\sqrt{7+14\sqrt{7}}F_2\right)\\
F_2^* &= \frac{1}{28}\left(\left(7+2\sqrt{7}\right)F_1-\sqrt{7+14\sqrt{7}}F_2\right)\\
F_3^* &= \left(\frac{1}{4}+\frac{1}{\sqrt{7}}\right)F_2\\
F_4^* &= -\frac{1}{4}\sqrt{\frac{2}{\sqrt{7}}-\frac{1}{7}}F_1\\
F_5^* &= \frac{F_3}{2}\\
F_6^* &= 0
\end{aligned}
$$

with $F_i = F_i(\sigma_1, \ldots, \sigma_6)$, and

$$
\begin{aligned}
\sigma_1 &= \frac{x_1-x_2}{\sqrt{1+2\sqrt{7}}}+\frac{2}{9}\left(\sqrt{7}-4\right)y_1\\
\sigma_2 &= x_1+x_2-2\sqrt{\frac{1}{3}\left(2\sqrt{7}-5\right)}y_2\\
\sigma_3 &= \frac{1}{9}\left(-3\left(\sqrt{7}-2\right)x_1-3\left(\sqrt{7}-2\right)x_2-2\sqrt{3+6\sqrt{7}}y_2\right)\\
\sigma_4 &= \frac{\left(\sqrt{7}-4\right)x_1-\left(\sqrt{7}-4\right)x_2+2\sqrt{1+2\sqrt{7}}y_1}{\sqrt{1+2\sqrt{7}}}\\
\sigma_5 &= -z_2\\
\sigma_6 &= 2z_1
\end{aligned}
$$

Now, in the following lemma we characterize the periodic orbits of the unperturbed system as a first step for proving Theorems 1 and 3.

Lemma 5. *The periodic solutions* $(\bar{x}_1(t), \bar{x}_2(t), \bar{y}_1(t), \bar{y}_2(t), \bar{z}_1(t), \bar{z}_2(t))$ *of the differential system (7) with* $\varepsilon = 0$ *are*

$$(0, 0, Y_1^0\cos(\omega_1 t) + Y_2^0\sin(\omega_1 t), Y_2^0\cos(\omega_1 t) - Y_1^0\sin(\omega_1 t), 0, 0), \tag{8}$$

of period T_1*, and*

$$(0, 0, 0, 0, Z_1^0\cos(\omega_2 t) + Z_2^0\sin(\omega_2 t), Z_2^0\cos(\omega_2 t) - Z_1^0\sin(\omega_2 t)), \tag{9}$$

of period T_2*.*

Proof. Since (7) for $\varepsilon = 0$ is a linear differential system the proof follows easily. □

Proof of Theorem 1. Assume that the functions F_1, F_2 and F_3 of (3) are periodic in t of period pT_1/q with p and q positive integers relatively prime. Then, we can consider that the differential system (7) and the periodic solutions (9) have the same period pT_2.

We apply Theorem 6 of Appendix I to the differential system (7), and we use the notation introduced there. Note that system (7) can be written in the form of system (10) taking

$$\mathrm{x} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{y}_1 \\ \bar{y}_2 \\ \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}, G_0(t,\mathrm{x}) = \begin{pmatrix} \lambda\bar{x}_1 \\ -\lambda\bar{x}_2 \\ \omega_1\bar{y}_2 \\ -\omega_1\bar{y}_1 \\ \omega_2\bar{z}_2 \\ -\omega_2\bar{z}_1 \end{pmatrix}, G_1(t,\mathrm{x}) = \begin{pmatrix} F_1^* \\ F_2^* \\ F_3^* \\ F_4^* \\ F_5^* \\ F_6^* \end{pmatrix}, G_2(t,\mathrm{x},\varepsilon) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now we shall study what periodic solutions of the unperturbed system (7) with $\varepsilon = 0$ of the type (9) persist as periodic solutions for the perturbed one for $\varepsilon \neq 0$ sufficiently small.

We start with the description of the different elements which appear in the statement of Theorem 6 for the particular case of the differential system (7). Thus, we have that $\Omega = \mathbb{R}^6$, $k = 2$ and $n = 6$. Now, let $r_1 > 0$ be arbitrarily small and let $r_2 > 0$ be arbitrarily large. Let V be the open and bounded subset of the plane $\bar{x}_1 = \bar{x}_2 = \bar{y}_1 = \bar{y}_2 = 0$ of the form $V = \{(0,0,0,0,Z_1^0,Z_2^0) \in \mathbb{R}^6 \ : \ r_1 < \sqrt{(Z_1^0)^2 + (Z_2^0)^2} < r_2\}$. As usual $\mathrm{Cl}(V)$ denotes the closure of V. If $\alpha = (Z_1^0, Z_2^0)$, then we identify V with the set $\{\alpha \in \mathbb{R}^2 \ : \ r_1 < ||\alpha|| < r_2\}$, being $||\cdot||$ the Euclidean norm in $\mathbb{R}^2$. The function $\beta\colon \mathrm{Cl(V)} \to \mathbb{R}^4$ is $\beta(\alpha) = (0,0,0,0)$. Therefore, for our system we have

$$\begin{aligned} \mathcal{Z} &= \{\mathrm{z}_\alpha = (\beta(\alpha),\alpha),\ \alpha \in \mathrm{Cl}(V)\} = \\ &= \{(0,0,0,0,Z_1^0,Z_2^0) \in \mathbb{R}^6 \ : \ r_1 \leq \sqrt{(Z_1^0)^2 + (Z_2^0)^2} \leq r_2\} \end{aligned}$$

We consider for each $\mathrm{z}_\alpha \in \mathcal{Z}$ the periodic solution

$$\mathrm{x}(t,\mathrm{z}_\alpha) = (0,0,0,0,Z_1(t),Z_2(t))$$

given by (9) of period pT_2.

Computing the fundamental matrix $M_{\mathrm{z}_\alpha}(t)$ of the linear differential system (7) with $\varepsilon = 0$ associated to the pT_2 periodic solution $\mathrm{z}_\alpha = (0,0,0,0,Z_1^0,Z_2^0)$ such that $M_{\mathrm{z}_\alpha}(0)$ be the identity of $\mathbb{R}^6$, we get

$$M_{\mathrm{z}_\alpha}(t) = M(t) =$$

$$= \begin{pmatrix} e^{\lambda t} & 0 & 0 & 0 & 0 & 0 \\ 0 & -e^{-\lambda t} & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\omega_1 t) & \sin(\omega_1 t) & 0 & 0 \\ 0 & 0 & -\sin(\omega_1 t) & \cos(\omega_1 t) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\omega_2 t) & \sin(\omega_2 t) \\ 0 & 0 & 0 & 0 & -\sin(\omega_2 t) & \cos(\omega_2 t) \end{pmatrix}$$

Note that the matrix $M_{\mathrm{z}_\alpha}(t)$ does not depend on the particular periodic solution $\mathrm{x}(t,\mathrm{z}_\alpha,\ 0)$. The matrix

$$M^{-1}(0) - M^{-1}(pT_1) =$$

$$= \begin{pmatrix} 1-e^{-\frac{2\pi\lambda p}{\omega_2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\frac{2\pi\lambda p}{\omega_2}}-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sin^2\left(\frac{\pi p\omega_1}{\omega_2}\right) & \sin\left(\frac{2\pi p\omega_1}{\omega_2}\right) & 0 & 0 \\ 0 & 0 & -\sin\left(\frac{2\pi p\omega_1}{\omega_2}\right) & 2\sin^2\left(\frac{\pi p\omega_1}{\omega_2}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\sin^2(\pi p) & \sin(2\pi p) \\ 0 & 0 & 0 & 0 & -\sin(2\pi p) & 2\sin^2(\pi p) \end{pmatrix}$$

satisfies the assumptions of statement (ii) of Theorem 6 because the determinant

$$\begin{vmatrix} 1-e^{-\frac{2\pi\lambda p}{\omega_2}} & 0 & 0 & 0 \\ 0 & e^{\frac{2\pi\lambda p}{\omega_2}}-1 & 0 & 0 \\ 0 & 0 & 2\sin^2\left(\frac{\pi p\omega_1}{\omega_2}\right) & \sin\left(\frac{2\pi p\omega_1}{\omega_2}\right) \\ 0 & 0 & -\sin\left(\frac{2\pi p\omega_1}{\omega_2}\right) & 2\sin^2\left(\frac{\pi p\omega_1}{\omega_2}\right) \end{vmatrix} =$$

$$= 16\sinh^2\left(\frac{\pi\lambda p}{\omega_2}\right)\sin^2\left(\frac{\pi p\omega_1}{\omega_2}\right) \neq 0,$$

because the ratio of the frequencies is non-resonant with π, see Remark 1. In short, all the assumptions of Theorem 6 are satisfied by the system (7).

For our system the map $\xi : \mathbb{R}^6 \to \mathbb{R}^2$ has the form $\xi(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{z}_1, \bar{z}_2) = (\bar{z}_1, \bar{z}_2)$. Calculating the function

$$\mathcal{G}(Z_1^0, Z_2^0) = \mathcal{G}(\alpha) = \xi\Big(\frac{1}{pT_2}\int_0^{pT_2} M_{z_\alpha}^{-1}(t) G_1(t, \mathrm{x}(t,\ \mathrm{z}_\alpha,\ 0))dt\Big),$$

we obtain that $\mathcal{G}(Z^0) = (\mathcal{G}_1(Z^0), \mathcal{G}_2(Z^0))$, where the functions $\mathcal{G}_k$ for $k = 1, 2$ are the ones given in (4). Then, by Theorem 6 we have that for every simple zero $Z^{0*} \in V$ of the system of nonlinear functions $\mathcal{G}(Z^0) = 0$ we have a periodic solution $(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{z}_1, \bar{z}_2)(t, \varepsilon)$ of system (7) such that

$$(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{z}_1, \bar{z}_2)(0, \varepsilon) \to (0, 0, 0, 0, Z_1^{0*}, Z_2^{0*}) \text{ when } \varepsilon \to 0.$$

Going back through the change of coordinates (6) we get a periodic solution $(x_1, x_2, y_1, y_2, z_1, z_2)(t, \varepsilon)$ of system (7) such that

$$\begin{pmatrix} x_1(t,\varepsilon) \\ x_2(t,\varepsilon) \\ y_1(t,\varepsilon) \\ y_2(t,\varepsilon) \\ z_1(t,\varepsilon) \\ z_2(t,\varepsilon) \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -Z_2^{0*}\cos(\omega_2 t) + Z_1^{0*}\sin(\omega_2 t)) \\ \omega_2(Z_1^{0*}\cos(\omega_2 t) + Z_2^{0*}\sin(\omega_2 t)) \end{pmatrix} \text{ when } \varepsilon \to 0.$$

Consequently we obtain a periodic solution $(x, y, z)(t, \varepsilon)$ of system (3) such that

$$(x, y, z)(t, \varepsilon) \to \begin{pmatrix} 0 \\ 0 \\ -Z_2^{0*}\cos(\omega_2 t) + Z_1^{0*}\sin(\omega_2 t)) \end{pmatrix} \text{ when } \varepsilon \to 0.$$

This completes the proof of Theorem 1. □

Proof of Theorem 3. The proof is analogous to the proof of Theorem 1 changing the roles of T_2 for T_1, and we obtain the periodic solution

$$\begin{pmatrix} \frac{2}{9}\left(\sqrt{7}-4\right)\left(Y_1\cos\left(\omega_1 t\right)+Y_2\sin\left(\omega_1 t\right)\right) \\ \left(\left(4-2\sqrt{7}\right)\left(Y_2\cos\left(\omega_1 t\right)-Y_1\sin\left(\omega_1 t\right)\right)\right)/\omega_1 \\ -\left(2\left(Y_2\cos\left(\omega_1 t\right)-Y_1\sin\left(\omega_1 t\right)\right)\right)/\omega_1 \\ 2\left(Y_1\cos\left(\omega_1 t\right)+Y_2\sin\left(\omega_1 t\right)\right) \\ 0 \\ 0 \end{pmatrix} \text{ when } \varepsilon\to 0.$$

Consequently we obtain a periodic solution $(x,y,z)(t,\varepsilon)$ of system (3) such that

$$(x,y,z)(t,\varepsilon)\to\begin{pmatrix} \frac{2}{9}\left(\sqrt{7}-4\right)\left(Y_1\cos\left(\omega_1 t\right)+Y_2\sin\left(\omega_1 t\right)\right) \\ -\left(2\left(Y_2\cos\left(\omega_1 t\right)-Y_1\sin\left(\omega_1 t\right)\right)\right)/\omega_1 \\ 0 \end{pmatrix} \text{ when } \varepsilon\to 0.$$

□

3. Proof of Corollaries 2 and 4

Proof of Corollary 2. Under the assumptions of Corollary 2, the nonlinear system (4) becomes

$$\begin{aligned}\mathcal{G}_1(Z_1^0,Z_2^0)&=\frac{-3}{16}Z_2^0\left(2Z_1^0+(Z_1^0)^2+(Z_2^0)^2\right)=0,\\ \mathcal{G}_2(Z_1^0,Z_2^0)&=\frac{1}{16}\left(-(Z_1^0)^2-3(Z_1^0)^3-11(Z_2^0)^2-3Z_1^0(Z_2^0)^2\right)=0.\end{aligned}$$

This system has the following solutions

$$Z^{0*}=(\frac{-1}{3},0) \text{ and } Z^{0*}=(\frac{-11}{8},\pm\frac{\sqrt{55}}{8}).$$

Note that the solutions which differs in a sign are different initial conditions of the same periodic solution of the system. Moreover, since

$$\det\left(\frac{\partial\mathcal{G}}{\partial Z^0}\right)\bigg|_{Z^{0*}=\left(-\frac{1}{3},0\right)}=\frac{5}{2304}\neq 0,$$

and

$$\det\left(\frac{\partial\mathcal{G}}{\partial Z^0}\right)\bigg|_{Z^{0*}=\left(-\frac{11}{8},\frac{\sqrt{55}}{8}\right)}=-\frac{1815}{4096}\neq 0,$$

these solutions are simple. Finally, by Theorem 1 we have two periodic solutions for the system, ending the proof. □

Proof of Corollary 4. Under the assumptions of Corollary 4, the nonlinear system $\bar{\mathcal{G}}(Y^0)=0$ becomes

$$\begin{aligned}\mathcal{G}_3(Y_1^0,Y_2^0)&=\frac{1}{8\sqrt{7}\omega_1}\left(1-\frac{2Y_1^0}{\omega_1}\right)=0,\\ \mathcal{G}_4(Y_1^0,Y_2^0)&=\frac{1}{8\sqrt{7}\omega_1}\left(-1-\frac{2Y_2^0}{\omega_1}\right)=0.\end{aligned}$$

This system has the following solution

$$Y^{0*}=\left(\frac{1}{2}\omega_1,-\frac{1}{2}\omega_1\right).$$

Moreover, since

$$\det\left(\frac{\partial \bar{\mathcal{G}}}{\partial Y^0}\right)\Big|_{Y^{0*}=\left(\frac{1}{2}\omega_1,-\frac{1}{2}\omega_1\right)}=\frac{1}{112}\neq 0$$

this solution is simple. Finally, by Theorem 3 we only have one periodic solution for this system and the proof is over. □

4. Conclusions

In this work we have presented several results for obtaining sufficient conditions for the existence of periodic solutions emerging from the equilibrium points of the spatial Hill lunar problem having equations of motion perturbed by small perturbations of periodic type. The technics that we have used are based on the averaging theory of dynamical systems.

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5. Appendix: Basic results on averaging theory

In this appendix we present the basic result from the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of $T-$periodic solutions from a differential system of the form

$$\dot{\mathrm{x}}(t)=G_0(t,\mathrm{x})+\varepsilon G_1(t,\mathrm{x})+\varepsilon^2 G_2(t,x,\varepsilon), \tag{10}$$

with $\varepsilon\neq 0$ sufficiently small. Here the functions $G_0, G_1\colon \mathbb{R}\times\Omega\to\mathbb{R}^n$ and $G_2\colon\mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^n$ are $\mathcal{C}^2$ functions and $T-$periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathrm{x}}(t)=G_0(t,\mathrm{x}) \tag{11}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Malkin (see [4] and references therein) studied the bifurcation of T–periodic solutions in the T–periodic system $\dot{x}=G_0(t,x)+\varepsilon G_1(t,x,\varepsilon)$, whose unperturbed system has a family of T–periodic solutions with initial conditions given by a smooth function $\beta\colon\mathbb{R}^k\to\mathbb{R}^n$, and proved that if the bifurcation function

$$\mathcal{G}(\alpha)=\int_0^T\begin{pmatrix}< u_1(t,\alpha),g(t,x(t,\beta(\alpha)),0)>\\ \cdots\\ < u_k(t,\alpha),g(t,x(t,\beta(\alpha)),0)>\end{pmatrix}dt,$$

where u_i, $i=1,\dots,k$, are k linearly independent T-periodic solutions of the adjoint linearized differential system, has a simple zero α_0 such that $\det(\mathcal{G})|_{\alpha=\alpha_0}\neq 0$, then for any $\varepsilon>0$ sufficiently small, the system $\dot{x}=G_0(t,x)+\varepsilon G_1(t,x,\varepsilon)$ has an unique T–periodic solution x_ε such that $x_\varepsilon(0)\to\beta(\alpha_0)$ as $\varepsilon\to 0$.

This can be rephrased as follows: Let $\mathrm{x}(t,\mathrm{z},\varepsilon)$ be the solution of the system (11) such that $\mathrm{x}(0,\mathrm{z},\varepsilon)=\mathrm{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathrm{x}(t,\mathrm{z},0)$ as

$$\dot{\mathrm{y}}=D_{\mathrm{x}}G_0(t,\mathrm{x}(t,\mathrm{z},0))\mathrm{y}. \tag{12}$$

In what follows we denote by $M_{\mathrm{z}}(t)$ some fundamental matrix of the linear differential system (12), and by $\xi\colon \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e. $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

We assume that there exists a k-dimensional submanifold $\mathcal{Z}$ of Ω filled with $T-$periodic solutions of (11). Then an answer to the problem of bifurcation of $T-$periodic solutions from the periodic solutions contained in $\mathcal{Z}$ for system (10) is given in the following result.

Theorem 6. *Let V be an open and bounded subset of $\mathbb{R}^k$, and let $\beta\colon \mathrm{Cl}(V) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that*

(i) $\mathcal{Z} = \{\mathrm{z}_\alpha = (\alpha, \beta(\alpha)), \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ *and that for each* $\mathrm{z}_\alpha \in \mathcal{Z}$ *the solution* $\mathrm{x}(t, \mathrm{z}_\alpha)$ *of (11) is T-periodic;*

(ii) *for each* $\mathrm{z}_\alpha \in \mathcal{Z}$ *there is a fundamental matrix* $M_{\mathrm{z}_\alpha}(t)$ *of (12) such that the matrix* $M_{\mathrm{z}_\alpha}^{-1}(0) - M_{\mathrm{z}_\alpha}^{-1}(T)$ *has in the upper right corner the* $k \times (n-k)$ *zero matrix, and in the lower right corner a* $(n-k) \times (n-k)$ *matrix* Δ_α *with* $\det(\Delta_\alpha) \neq 0$.

We consider the function $\mathcal{G}\colon \mathrm{Cl}(V) \to \mathbb{R}^k$

$$\mathcal{G}(\alpha) = \xi\Big(\frac{1}{T}\int_0^T M_{\mathrm{z}_\alpha}^{-1}(t) G_1(t, \mathrm{x}(t, \mathrm{z}_\alpha, 0))dt\Big). \tag{13}$$

If there exists $a \in V$ *with* $\mathcal{G}(a) = 0$ *and* $\det((d\mathcal{G}/d\alpha)(a)) \neq 0$, *then there is a $T-$periodic solution* $\mathrm{x}(t, \varepsilon)$ *of system (10) such that* $\mathrm{x}(0, \varepsilon) \to \mathrm{z}_a$ *as* $\varepsilon \to 0$.

For a proof of Theorem 6 see Malkin [14] and Roseau [17], or [3] for an alternative proof.

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